



On the Algebraic Properties of the Multiplication Operation of 4-Cyclic Refined Neutrosophic Real Matrices

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ABSTRACT

The objective of this paper is to study some of the elementary algebraic properties of 4-cyclic refined neutrosophic matrices with real entries, where we study the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrate many examples that explain the validity of our work.

Keywords: 4-cyclic refined neutrosophic matrix ▪ eigenvalue ▪ eigenvector ▪ 4-cyclic refined neutrosophic ring

1. INTRODUCTION

The algebraic structures based on n -cyclic refined neutrosophic sets were studied by many different authors, where it can find some interesting generalizations such as n -cyclic refined neutrosophic rings, 3-cyclic refined neutrosophic units, 4-cyclic refined neutrosophic units, 4-cyclic refined neutrosophic Diophantine equations, and n -cyclic refined neutrosophic groups and complex numbers [1–13]. In [14], 2-cyclic refined neutrosophic matrices have been studied and formulated, where the computations of eigenvectors and eigenvalues were introduced, as well as the computation of determinants, inverses, and the diagonalizing properties.

This has motivated us to study the general case of 4-cyclic refined neutrosophic matrices for the first time, where some of the elementary algebraic properties of 4-cyclic refined neutrosophic matrices with real entries will be discussed, and we study the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrate many examples that explain the validity of our work.

2. MAIN DISCUSSION

Definition. Let X_i , $0 \leq i \leq 4$, be five $n \times n$ matrices with real entries. Then the corresponding 4-cyclic refined neutrosophic matrix is defined as follows:

$$X = X_0 + \sum_{i=1}^4 X_i I_i. \quad (1)$$

Example. Consider

$$\begin{aligned} X_0 &= \begin{pmatrix} 1 & 2 \\ -1 & 7 \end{pmatrix}, & X_1 &= \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}, \\ X_2 &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, & X_3 &= \begin{pmatrix} -4 & 2 \\ 6 & -9 \end{pmatrix}, \\ X_4 &= \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}. \end{aligned}$$

Then

$$X = X_0 + \sum_{i=1}^4 X_i I_i$$

$$= \begin{pmatrix} 1+3I_1+I_2-4I_3+I_4 & 2+3I_1-I_2+2I_3+3I_4 \\ -1+6I_3-I_4 & 7+I_1+2I_2-9I_3+2I_4 \end{pmatrix}.$$

Remark. We denote the set of all $n \times n$ 4-cyclic refined neutrosophic matrices with real entries by $R_{n \times n}^{(4)}(I)$.

Remark. Let

$$X = X_0 + \sum_{i=1}^4 X_i I_i, \quad Y = Y_0 + \sum_{i=1}^4 Y_i I_i.$$

Then

$$X + Y = (X_0 + Y_0) + \sum_{i=1}^4 (X_i + Y_i) I_i, \quad (2)$$

and

$$X \cdot Y = X_0 Y_0$$

$$+ I_1 (X_0 Y_1 + X_1 Y_0 + X_1 Y_4 + X_4 Y_1$$

$$+ X_2 Y_3 + X_3 Y_2)$$

$$+ I_2 (X_0 Y_2 + X_2 Y_0 + X_1 Y_1 + X_2 Y_4$$

$$+ X_4 Y_2 + X_3 Y_3)$$

$$+ I_3 (X_0 Y_3 + X_3 Y_0 + X_1 Y_2 + X_2 Y_1$$

$$+ X_3 Y_4 + X_4 Y_3)$$

$$+ I_4 (X_0 Y_4 + X_4 Y_0 + X_1 Y_3 + X_3 Y_1$$

$$+ X_4 Y_4 + X_2 Y_2). \quad (3)$$

Example. Take

$$X = \begin{pmatrix} 1+I_1-I_3 & I_2+I_4 \\ 2I_1+I_3 & 1-I_2 \end{pmatrix},$$

$$Y = \begin{pmatrix} 2+I_2 & I_4 \\ 1+I_2+I_3 & 1+I_1-I_4 \end{pmatrix}.$$

Then

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$Y_0 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$Y_4 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Moreover,

$$X + Y = \begin{pmatrix} 3+I_1+I_2-I_3 & I_2+2I_4 \\ 1+2I_1+I_2+2I_3 & 2+I_1-I_2-I_4 \end{pmatrix}.$$

For the coefficient of I_1 , the required products are

$$X_0 Y_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},$$

$$X_0 Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X_1 Y_4 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix},$$

$$X_1 Y_0 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix},$$

$$X_4 Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$X_2 Y_3 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix},$$

$$X_3 Y_2 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus,

$$X_0 Y_1 + X_1 Y_0 + X_1 Y_4 + X_4 Y_1 + X_2 Y_3 + X_3 Y_2 = \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}.$$

For the coefficient of I_2 ,

$$X_0 Y_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$X_2 Y_0 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

$$X_1 Y_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_2 Y_4 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},$$

$$X_4 Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_3 Y_3 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus,

$$X_0 Y_2 + X_2 Y_0 + X_1 Y_1 + X_2 Y_4 + X_4 Y_2 + X_3 Y_3 = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}.$$

For the coefficient of I_3 ,

$$X_0 Y_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$X_3Y_0 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & 0 \end{pmatrix},$$

$$X_1Y_2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},$$

$$X_2Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

$$X_3Y_4 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},$$

$$X_4Y_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus,

$$X_0Y_3 + X_3Y_0 + X_1Y_2 + X_2Y_1 + X_3Y_4 + X_4Y_3 = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}.$$

For the coefficient of I_4 ,

$$X_0Y_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

$$X_4Y_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$X_1Y_3 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_3Y_1 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_4Y_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

$$X_2Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

Thus,

$$X_0Y_4 + X_4Y_0 + X_1Y_3 + X_3Y_1 + X_4Y_4 + X_2Y_2 = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

Consequently,

$$\begin{aligned} X \cdot Y &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + I_1 \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix} + I_2 \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + I_3 \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} + I_4 \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2+2I_1+3I_2+2I_4 & 2I_1+I_4 \\ 1+4I_1+5I_3-I_4 & 1+3I_1-I_4 \end{pmatrix}. \end{aligned}$$

Example. Let us try to find the inverse of

$$X = \begin{pmatrix} 1+I_4 & 3 \\ I_4 & 1 \end{pmatrix}$$

$$\begin{aligned} &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_2 \\ &\quad + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_3 + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} I_4. \end{aligned}$$

Let

$$Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = X^{-1},$$

where

$$\begin{cases} A = a_0 + \sum_{i=1}^4 a_i I_i, & B = b_0 + \sum_{i=1}^4 b_i I_i, \\ C = c_0 + \sum_{i=1}^4 c_i I_i, & D = d_0 + \sum_{i=1}^4 d_i I_i. \end{cases}$$

$$\text{Since } XY = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{cases} A(1+I_4) + BI_4 = 1, \\ 3A + B = 0, \\ C(1+I_4) + DI_4 = 0, \\ 3C + D = 1. \end{cases}$$

The first equation expands as

$$\begin{aligned} A(1+I_4) + BI_4 &= a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4 \\ &\quad + a_0 I_4 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4 \\ &\quad + b_0 I_4 + b_1 I_1 + b_2 I_2 + b_3 I_3 + b_4 I_4 = 1. \end{aligned}$$

Thus,

$$\begin{aligned} &a_0 + I_1(2a_1 + b_1) + I_2(2a_2 + b_2) \\ &\quad + I_3(2a_3 + b_3) + I_4(a_0 + 2a_4 + b_0 + b_4) = 1, \end{aligned}$$

so that

$$\begin{cases} a_0 = 1, & 2a_1 + b_1 = 0, \\ 2a_2 + b_2 = 0, & 2a_3 + b_3 = 0, \\ 1 + 2a_4 + b_0 + b_4 = 0. \end{cases}$$

From the equation $3A + B = 0$, we get

$$\begin{cases} 3a_0 + b_0 = 0 & \implies b_0 = -3, \\ 3a_1 + b_1 = 0, \\ 3a_2 + b_2 = 0, \\ 3a_3 + b_3 = 0, \\ 3a_4 + b_4 = 0. \end{cases}$$

Thus,

$$\begin{cases} a_1 = 0, & b_1 = 0, \\ a_2 = 0, & b_2 = 0, \\ a_3 = 0, & b_3 = 0, \\ a_4 = -2, & b_4 = 6, \end{cases} \quad \begin{cases} A = 1 - 2I_4, \\ B = -3 + 6I_4. \end{cases}$$

The equation $C(1 + I_4) + DI_4 = 0$ implies that

$$\begin{cases} c_0 = 0, & 2c_1 + d_1 = 0, \\ 2c_2 + d_2 = 0, & 2c_3 + d_3 = 0, \\ d_0 + d_4 + 2c_4 = 0. \end{cases}$$

The equation $3C + D = 1$ implies

$$\begin{cases} 3c_0 + d_0 = 1 & \implies d_0 = 1, \\ 3c_1 + d_1 = 0, \\ 3c_2 + d_2 = 0, \\ 3c_3 + d_3 = 0, \\ 3c_4 + d_4 = 0. \end{cases}$$

Hence,

$$\begin{cases} c_1 = d_1 = 0, \\ c_2 = d_2 = 0, \\ c_3 = d_3 = 0, \\ c_4 = 1, & d_4 = -3. \end{cases}$$

This implies that $C = I_4$ and $D = 1 - 3I_4$, and therefore

$$Y = \begin{pmatrix} 1 - 2I_4 & -3 + 6I_4 \\ I_4 & 1 - 3I_4 \end{pmatrix}.$$

Example. Consider

$$A = \begin{pmatrix} 1 + I_3 & I_2 \\ I_4 & 1 - I_3 \end{pmatrix}.$$

We will try to find its eigenvalues. The equation is

$$AX = a \cdot X,$$

where

$$\begin{aligned} X &= x_0 + \sum_{i=1}^4 x_i I_i, & x_i &\in \mathbb{R}^2, \\ a &= a_0 + \sum_{i=1}^4 a_i I_i, & X_i &\in R_4(I). \end{aligned}$$

Write

$$A \cdot X = \begin{pmatrix} 1 + I_3 & I_2 \\ I_4 & 1 - I_3 \end{pmatrix} \begin{pmatrix} X' \\ X'' \end{pmatrix},$$

where

$$\begin{cases} X' = x'_0 + \sum_{i=1}^4 x'_i I_i, \\ X'' = x''_0 + \sum_{i=1}^4 x''_i I_i, \end{cases} \quad x'_i, x''_i \in \mathbb{R}.$$

Then

$$A \cdot X = \begin{pmatrix} (1 + I_3)X' + I_2X'' \\ I_4X' + (1 - I_3)X'' \end{pmatrix} = \begin{pmatrix} T \\ L \end{pmatrix}.$$

The first component is

$$\begin{aligned} T &= (1 + I_3)(x'_0 + x'_1 I_1 + x'_2 I_2 + x'_3 I_3 + x'_4 I_4) \\ &\quad + I_2(x''_0 + x''_1 I_1 + x''_2 I_2 + x''_3 I_3 + x''_4 I_4) \\ &= x'_0 + I_1(x'_1 + x'_2) + I_2(x'_2 + x'_3) + I_3(x'_3 + x'_4) \end{aligned}$$

$$\begin{aligned} &+ I_4(x'_4 + x'_1) + x''_0 I_2 + x''_1 I_3 + x''_2 I_4 + x''_3 I_1 + x''_4 I_2 \\ &= x'_0 + I_1(x'_1 + x'_2 + x'_3) + I_2(x'_2 + x'_3 + x''_0 + x''_4) \\ &\quad + I_3(x'_3 + x'_4 + x''_1) + I_4(x'_4 + x'_1 + x''_2). \end{aligned}$$

The second component is

$$\begin{aligned} L &= I_4(x'_0 + x'_1 I_1 + x'_2 I_2 + x'_3 I_3 + x'_4 I_4) \\ &\quad + (1 - I_3)(x''_0 + x''_1 I_1 + x''_2 I_2 + x''_3 I_3 + x''_4 I_4) \\ &= x''_0 + I_1(x'_1 + x''_1 - x''_2) + I_2(x'_2 + x''_2 - x''_3) \\ &\quad + I_3(x'_3 + x''_3 - x''_0 - x''_4) + I_4(x'_0 + x'_4 + x''_4 - x''_1). \end{aligned}$$

By a similar argument, we can compute $a \cdot X$. The equation $AX = aX$ is equivalent to

$$\begin{cases} T = a \cdot X', \\ L = a \cdot X''. \end{cases}$$

Hence,

$$\begin{cases} x'_0 = a_0 x'_0, \\ x'_1 + x'_2 + x'_3 = a_0 x'_1 + a_1 x'_0 + a_2 x'_3 + a_3 x'_2 \\ \quad + a_1 x'_4 + a_4 x'_1, \\ x'_2 + x'_3 + x''_0 + x''_4 = a_0 x'_2 + a_2 x'_0 + a_1 x'_1 \\ \quad + a_2 x'_4 + a_4 x'_2, \\ x'_3 + x'_4 + x''_1 = a_0 x'_3 + a_3 x'_0 + a_1 x'_2 + a_2 x'_1 \\ \quad + a_3 x'_4 + a_4 x'_3, \\ x'_4 + x'_1 + x''_2 = a_0 x'_4 + a_4 x'_0 + a_2 x'_2 + a_3 x'_1 \\ \quad + a_1 x'_3 + a_4 x'_4. \end{cases}$$

Also,

$$\begin{cases} x''_0 = a_0 x''_0, \\ x'_1 + x''_1 - x''_2 = a_0 x''_1 + a_1 x''_0 + a_2 x''_3 + a_3 x''_2 \\ \quad + a_1 x''_4 + a_4 x''_1, \\ x'_2 + x''_2 - x''_3 = a_0 x''_2 + a_2 x''_0 + a_1 x''_1 \\ \quad + a_2 x''_4 + a_4 x''_2, \\ x'_3 + x''_3 - x''_0 - x''_4 = a_0 x''_3 + a_3 x''_0 + a_1 x''_2 + a_2 x''_1 \\ \quad + a_3 x''_4 + a_4 x''_3, \\ x'_0 + x'_4 + x''_4 - x''_1 = a_0 x''_4 + a_4 x''_0 + a_1 x''_3 + a_3 x''_1 \\ \quad + a_2 x''_2 + a_4 x''_4. \end{cases}$$

It is clear that $x'_0 = x''_0 = 0$ and $a_0 = 1$. Solving the other equations is still a hard problem. It can be solved by using computers.

3. CONCLUSION

In this paper we studied some of the elementary algebraic properties of 4-cyclic refined neutrosophic matrices with real entries, where we studied the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrated many examples that explain the validity of our work.

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