



On The Algebraic Properties of 3-Cyclic Refined Neutrosophic Matrices with Real Entries

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Abstract

The objective of this paper is to study some of the elementary algebraic properties of 3-cyclic refined neutrosophic matrices with real entries, where we study the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrate many examples that explain the validity of our work.

Keywords: 3-cyclic refined neutrosophic matrix; Inverse; Associative; Indeterminacy

1. Introduction

The algebraic structures based on n-cyclic refined neutrosophic sets were studied by many different authors, where it can find some interesting generalizations such as n-cyclic refined neutrosophic rings, 3-cyclic refined neutrosophic units, 4-cyclic refined neutrosophic units, 4-cyclic refined neutrosophic Diophantine equations, and n-cyclic refined neutrosophic groups and complex numbers [1-13].

In [14], 2-cyclic refined neutrosophic matrices have been studied and formulated, where the computations of eigenvectors and eigenvalues were introduced, as well as the computation of determinants, inverses, and the diagonal zing properties.

This has motivated us to study the general case of 3-cyclic refined neutrosophic matrices for the first time, where some of the elementary algebraic properties of 3-cyclic refined neutrosophic matrices with real entries will be discussed, and we study the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrate many examples that explain the validity of our work.

2. Main Discussion

Definition:

Let X_0, X_1, X_2, X_3 be four $n \times n$ matrices with real entries, the corresponding 3-cyclic refined neutrosophic matrix is defined as follows:

$$X = X_0 + \sum_{i=1}^3 X_i I_i.$$

Example:

Take: $X_0 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ -1 & 7 & 4 \end{pmatrix}, X_1 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix},$

$$X_2 = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & 4 \\ -5 & -2 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \text{ then:}$$

$$X = X_0 + \sum_{i=1}^3 X_i I_i = \begin{pmatrix} 1 - I_1 + 4I_2 + I_3 & 2 + I_1 - I_2 + I_3 & 1 + I_1 + I_3 \\ I_1 + 3I_2 + I_3 & -I_1 + I_2 + I_3 & 3 + I_1 + 4I_2 + I_3 \\ -1 - 5I_2 + I_3 & 7 - 2I_2 + I_3 & 4 + 2I_1 - 2I_2 \end{pmatrix}$$

Definition:

Let $X = X_0 + \sum_{i=1}^3 X_i I_i$, $Y = Y_0 + \sum_{i=1}^3 Y_i I_i$ be two 3-cyclic refined neutrosophic square matrices with real entries, then:

$$X + Y = (X_0 + Y_0) + \sum_{i=1}^3 (X_i + Y_i) I_i,$$

$$X \cdot Y = X_0 Y_0 + I_1 (X_0 Y_1 + X_1 Y_0 + X_1 Y_3 + X_3 Y_1 + X_2 Y_2) + I_2 (X_0 Y_2 + X_2 Y_0 + X_2 Y_3 + X_3 Y_2 + X_1 Y_1) + I_3 (X_1 Y_2 + X_2 Y_1 + X_0 Y_3 + X_3 Y_0 + X_3 Y_3).$$

Example:

Take: $X = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} I_3,$

$$Y = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} I_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_3,$$

We have:

$$\begin{cases} X_0 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, & Y_0 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \\ X_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ X_2 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, & Y_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ X_3 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, & Y_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{cases}$$

$$\text{And: } \begin{cases} X_0 + Y_0 = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} \\ X_1 + Y_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \\ X_2 + Y_2 = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \\ X_3 + Y_3 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \end{cases}$$

Hence $X + Y = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} I_3$

$$X_0 Y_0 = \begin{pmatrix} -1 & 3 \\ -1 & 1 \end{pmatrix},$$

$$X_0 Y_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

$$X_1 Y_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix},$$

$$X_1 Y_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_3 Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_2 Y_2 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow X_0 Y_1 + X_1 Y_0 + X_1 Y_3 + X_3 Y_1 + X_2 Y_2 = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}.$$

$$X_0 Y_2 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

$$X_2 Y_0 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix},$$

$$X_1 Y_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

$$X_2 Y_3 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_3 Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix},$$

$$\Rightarrow X_0 Y_2 + X_2 Y_0 + X_1 Y_1 + X_2 Y_3 + X_3 Y_2 = \begin{pmatrix} -1 & 6 \\ 1 & 3 \end{pmatrix}.$$

$$\begin{aligned}
 X_0Y_3 &= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
 X_3Y_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \\
 X_1Y_2 &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\
 X_2Y_1 &= \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \\
 X_3Y_3 &= \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
 \Rightarrow X_0Y_3 + X_3Y_0 + X_1Y_2 + X_2Y_1 + X_3Y_3 &= \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}.
 \end{aligned}$$

Therefore

$$X.Y = \begin{pmatrix} -1 & 3 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} -1 & 6 \\ 1 & 3 \end{pmatrix} I_2 + \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} I_3.$$

Remark:

The set of all $n \times n$ square 3-cyclic refined neutrosophic matrices $(R_{n \times n}(I), +, \times)$ is a ring.

The unitary matrix in $R_{n \times n}(I)$ is:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_3.$$

Example:

Let's try to find the inverse of:

$$X = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I_1 + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} I_2 + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} I_3 = \begin{pmatrix} 1 + I_3 & I_1 + I_2 + I_3 \\ 2 + I_1 + I_3 & 1 + I_2 \end{pmatrix},$$

Let $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = X^{-1}$; $A = a_0 + \sum_{i=1}^3 a_i I_i$,

$$B = b_0 + \sum_{i=1}^3 b_i I_i, C = c_0 + \sum_{i=1}^3 c_i I_i,$$

$D = d_0 + \sum_{i=1}^3 d_i I_i$, $a_i, b_i, c_i, d_i \in \mathbb{R}$.

$$X.Y = U \Leftrightarrow \begin{cases} A(1 + I_3) + C(I_1 + I_2 + I_3) = 1 & (1) \\ B(1 + I_3) + D(I_1 + I_2 + I_3) = 0 & (2) \\ A(2 + I_1 + I_3) + C(1 + I_2) = 0 & (3) \\ B(2 + I_1 + I_3) + D(1 + I_2) = 1 & (4) \end{cases}$$

Form (1):

$A(1 + I_3) = (a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3)(1 + I_3) = a_0 + I_1(a_1 + a_1) + I_2(a_2 + a_2) + I_3(2a_3 + a_0) = a_0 + 2a_1 I_1 + 2a_2 I_2 + (2a_3 + a_0) I_3,$

$C(I_1 + I_2 + I_3) = (c_0 + c_1 I_1 + c_2 I_2 + c_3 I_3)(I_1 + I_2 + I_3) = c_0 I_1 + c_0 I_2 + c_0 I_3 + c_1 I_2 + c_1 I_3 + c_1 I_1 + c_2 I_3 + c_2 I_1 + c_2 I_2 + c_3 I_1 + c_3 I_2 + c_3 I_3 = I_1(c_0 + c_1 + c_2 + c_3) + I_2(c_0 + c_1 + c_2 + c_3) + I_3(c_0 + c_1 + c_2 + c_3),$

Hence:
$$\begin{cases} a_0 = 1 \\ 2a_1 + c_0 + c_1 + c_2 + c_3 = 0 \\ 2a_2 + c_0 + c_1 + c_2 + c_3 = 0 \\ 2a_3 + 1 + c_0 + c_1 + c_2 + c_3 = 0 \end{cases} \text{ system (I)}$$

Form (2):

$A(2 + I_1 + I_3) = (a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3)(2 + I_1 + I_3) = 2a_0 + I_1(a_0 + 3a_1 + a_3) + I_2(a_1 + 3a_2) + I_3(a_0 + 3a_3 + a_2),$

$C(1 + I_2) = (c_0 + c_1 I_1 + c_2 I_2 + c_3 I_3)(1 + I_2) = c_0 + c_0 I_2 + c_1 I_1 + c_1 I_3 + c_2 I_2 + c_2 I_1 + c_3 I_3 + c_3 I_2 = c_0 + I_1(c_1 + c_2) + I_2(c_0 + c_2 + c_3) + I_3(c_1 + c_3).$

Therefore
$$\begin{cases} 2 + c_0 = 0 \\ 1 + 3a_1 + a_3 + c_1 + c_2 = 0 \\ a_1 + 3a_2 + c_0 + c_2 + c_3 = 0 \\ 1 + 3a_3 + a_2 + c_1 + c_3 = 0 \end{cases} \text{ system (II)}$$

By using the equations in system (I) and (II), we get:

$a_0 = 1, c_0 = -2, a_1 = a_2, a_3 = a_1 - \frac{1}{2}.$

$$\text{Hence: } \begin{cases} c_0 + c_2 + c_3 = -4a_1 \Rightarrow c_2 + c_3 = -2 - 4a_1 \\ c_1 + c_2 = -1 - 3a_1 - a_3 \Rightarrow c_1 + c_2 = -\frac{1}{2} - 4a_1 \\ c_1 + c_3 = -1 - 3a_3 - a_2 \Rightarrow c_1 + c_3 = \frac{1}{2} - 4a_1 \end{cases} \Rightarrow \begin{cases} c_1 - c_3 = \frac{3}{2} \\ c_1 + c_3 = \frac{1}{2} - 4a_1 \end{cases} \Rightarrow \begin{cases} c_1 = 1 - 2a_1 \\ c_3 = \frac{-1}{2} - 2a_1 \end{cases}$$

$$\Rightarrow c_2 = -2 - 4a_1 - \left(\frac{-1}{2} - 2a_1\right) \Rightarrow c_2 = \frac{-3}{2} - 2a_1,$$

So that:

$$2a_1 + c_0 + c_1 + c_2 + c_3 = 0 \Rightarrow 2a_1 - 2 - \frac{1}{2} - 4a_1 - \frac{1}{2} - 2a_1 = 0 \Rightarrow -3 - 4a_1 = 0 \Rightarrow a_1 = \frac{-3}{4}, a_2 = \frac{-3}{4}, a_3 = \frac{-5}{4},$$

$$c_0 = -2, c_1 = \frac{7}{2}, c_2 = \frac{-3}{2} + \frac{5}{2} = 1, c_3 = \frac{-1}{2} + \frac{5}{2} = 2,$$

$$\text{Thus: } A = 1 - \frac{3}{4}I_1 - \frac{3}{4}I_2 - \frac{5}{4}I_3, C = -2 + \frac{7}{2}I_1 + I_2 + 2I_3.$$

From equation (2), we get:

$$\begin{cases} b_0 = 0 \\ 2b_1 + d_0 + d_1 + d_2 + d_3 = 0 \\ 2b_2 + d_0 + d_1 + d_2 + d_3 = 0 \\ 2b_3 + d_0 + d_1 + d_2 + d_3 = 0 \end{cases} \text{ system (III)}$$

Thus: $b_1 = b_2 = b_3$.

From (3):

$$\begin{cases} d_0 = 1 \\ 3b_1 + b_2 + d_1 + d_2 = 0 \\ b_1 + 3b_2 + d_0 + d_2 + d_3 = 0 \\ 3b_3 + b_2 + d_1 + d_3 = 0 \end{cases} \text{ system (IV)}$$

$$\text{We get: } \begin{cases} d_1 + d_2 = -4b_1 \\ d_2 + d_3 = -1 - 4b_1 \\ d_1 + d_3 = -4b_1 \end{cases}$$

$$\text{Thus: } \begin{cases} d_1 - d_3 = 1 \\ d_1 + d_3 = -4b_1 \end{cases} \Rightarrow \begin{cases} d_1 = \frac{1}{2} - 2b_1 \\ d_3 = \frac{-1}{2} - 2b_1 \end{cases} \text{ and } d_2 = -1 - 4b_1 + \frac{1}{2} + 2b_1 = \frac{-1}{2} - 2b_1,$$

$$\text{Thus: } 2b_1 + d_0 + d_1 + d_2 + d_3 = 0 \Rightarrow 2b_1 + 1 - 4b_1 - \frac{1}{2} - 2b_1 = 0 \Rightarrow \frac{1}{2} - 4b_1 = 0 \Rightarrow b_1 = \frac{1}{8}, b_2 = \frac{1}{8}, b_3 = \frac{1}{8}$$

$$d_0 = 1, d_1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, d_2 = \frac{-1}{2} - \frac{1}{4} = \frac{-3}{4}, d_3 = \frac{-3}{4},$$

$$\text{And } B = \frac{1}{8}I_1 + \frac{1}{8}I_2 + \frac{1}{8}I_3, D = 1 + \frac{1}{4}I_1 - \frac{3}{4}I_2 - \frac{3}{4}I_3.$$

$$\text{Hence: } Y = \begin{pmatrix} 1 - \frac{3}{4}I_1 - \frac{3}{4}I_2 - \frac{5}{4}I_3 & \frac{1}{8}I_1 + \frac{1}{8}I_2 + \frac{1}{8}I_3 \\ -2 + \frac{7}{2}I_1 + I_2 + 2I_3 & 1 + \frac{1}{4}I_1 - \frac{3}{4}I_2 - \frac{3}{4}I_3 \end{pmatrix}.$$

Definition:

Let $A = A_0 + \sum_{i=1}^3 A_i I_i$ be a 3-cyclic refined neutrosophic matrix with real entries, then $a = a_0 + \sum_{i=1}^3 a_i I_i \in R_3(I)$ is called an eigen value with $X = X_0 + \sum_{i=1}^3 X_i I_i$ as an eigen vector if and only if $A.X = a.X$; $x_i \in \mathbb{R}^{n \times n}$.

Example:

Consider $A = \begin{pmatrix} 1 + I_2 & I_1 + I_2 + I_3 \\ 2 + I_1 + I_3 & 1 + I_2 \end{pmatrix}$, we will try to find some eigen values and vectors.

According to the definition, we have:

$$A.X = a.X \ ; \ X = X_0 + \sum_{i=1}^3 X_i I_i, a = a_0 + \sum_{i=1}^3 a_i I_i,$$

We get:

$$\begin{cases} A_0 X_0 = a_0 X_0, & (1) \\ A_0 X_1 + A_1 X_0 + A_1 X_3 + A_3 X_1 + A_2 X_2 = a_0 X_1 + a_1 X_0 + a_1 X_3 + a_3 X_1 + a_2 X_2, & (2) \\ A_0 X_2 + A_2 X_0 + A_1 X_1 + A_2 X_3 + A_3 X_2 = a_0 X_2 + a_2 X_0 + a_1 X_1 + a_2 X_3 + a_3 X_2, & (3) \\ A_0 X_3 + A_3 X_0 + A_1 X_2 + A_2 X_1 + A_3 X_3 = a_0 X_3 + a_3 X_0 + a_1 X_2 + a_2 X_1 + a_3 X_3, & (4) \end{cases}$$

From the first equation, we can see that a_0 is an eigen value of A_0 with X_0 as eigen vector.

By adding all equations, we get:

$$(A_0 + A_1 + A_2 + A_3)(X_0 + X_1 + X_2 + X_3) = (a_0 + a_1 + a_2 + a_3)(X_0 + X_1 + X_2 + X_3)$$

Hence, $\sum_{i=1}^3 a_i$ is an eigen value of $\sum_{i=1}^3 A_i$ with $\sum_{i=1}^3 X_i$ as an eigen vector.

$$A_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sum_{i=1}^3 A_i = \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}$$

The eigen value of A_0 is $a_0 = 1$.

$$(A_0 - I) \cdot X_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} X_0 \\ X_0' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow X_0 = 0, \text{ hence } X_0 = (0,1).$$

The eigen values of $\sum_{i=1}^3 A_i$:

$$\det \left[\begin{pmatrix} 2-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix} \right] = 0 \Leftrightarrow (2-\lambda)^2 - 12 = 0 \Leftrightarrow \lambda^2 - 4\lambda - 8 = 0$$

$$\Delta = 16 + 32 = 48, \sqrt{\Delta} = 4\sqrt{3}, \lambda_1 = \frac{4+4\sqrt{3}}{2} = 2 + 2\sqrt{3}, \lambda_2 = 2 - 2\sqrt{3}, \text{ thus:}$$

$$\sum_{i=1}^3 a_i \in \{2 + 2\sqrt{3}, 2 - 2\sqrt{3}\}.$$

For $\lambda_1 = 2 + 2\sqrt{3}$, we can write:

$$\begin{pmatrix} -2\sqrt{3} & 3 \\ 4 & -2\sqrt{3} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2\sqrt{3}X + 3Y = 0 \\ 4X - 2\sqrt{3}Y = 0 \end{cases} \Rightarrow Y = \frac{2}{\sqrt{3}}X.$$

And the corresponding eigenvector is: $(1, \frac{2}{\sqrt{3}})$.

For $\lambda_2 = 2 - 2\sqrt{3}$, we can write:

$$\begin{pmatrix} 2\sqrt{3} & 3 \\ 4 & 2\sqrt{3} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2\sqrt{3}X + 3Y = 0 \\ 4X + 2\sqrt{3}Y = 0 \end{cases} \Rightarrow Y = \frac{-2}{\sqrt{3}}X.$$

The corresponding eigenvector is: $(1, \frac{-2}{\sqrt{3}})$.

Now, for $\sum_{i=1}^3 a_i = 2 + 2\sqrt{3}$, $\sum_{i=1}^3 X_i = (1, \frac{2}{\sqrt{3}})$, we get:

$$\begin{cases} a_1 + a_2 + a_3 = 1 + 2\sqrt{3} \\ X_1 + X_2 + X_3 = (1, \frac{2}{\sqrt{3}} - 1) \end{cases} \Leftrightarrow \begin{cases} a_3 = 1 + 2\sqrt{3} - a_1 - a_2 \\ X_3 = \left(1, \frac{2 - \sqrt{3}}{\sqrt{3}}\right) - X_1 - X_2 \end{cases}$$

Consider the equations (2),(3):

$$A_0(X_1 + X_2) + A_1(X_0 + X_3 + X_1) + A_2(X_2 + X_0 + X_3) + A_3(X_1 + X_2) \\ = a_0(X_1 + X_2) + a_1(X_0 + X_3 + X_1) + a_2(X_2 + X_0 + X_3) + a_3(X_1 + X_2)$$

Hence:

$$A_0(X_1 + X_2) + A_1 \left(\left(1, \frac{2}{\sqrt{3}}\right) - X_2 \right) + A_2 \left(\left(1, \frac{2}{\sqrt{3}}\right) - X_1 \right) + A_3(X_1 + X_2) = a_0(X_1 + X_2) + a_1 \left(\left(1, \frac{2}{\sqrt{3}}\right) - X_2 \right) + a_2 \left(\left(1, \frac{2}{\sqrt{3}}\right) - X_1 \right) + a_3(X_1 + X_2),$$

To find a suitable solution, we can put:

$$\begin{cases} (A_0 + A_3)(X_1 + X_2) = (a_0 + a_3)(X_1 + X_2) \\ A_1 X_2 = a_1 X_2 \\ A_2 X_1 = a_2 X_1 \end{cases}$$

This implies that:

a_1 is eigenvalue of A_1 with X_2 as eigenvector,

a_2 is eigenvalue of A_2 with X_1 as eigenvector,

$a_0 + a_3$ is eigenvalue of $A_0 + A_3$ with $X_1 + X_2$ as eigen vector.

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, a_1 = 1, X_2 = (1,1),$$

$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, a_2 = 1, X_1 = (1,0)$$

$$A_0 + A_3 = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}, a_0 + a_3 \in \{1 + \sqrt{3}, 1 - \sqrt{3}\}, \text{ thus:}$$

$$a_3 \in \{\sqrt{3}, -\sqrt{3}\}, X_1 + X_2 \in \{(1, -\sqrt{3}), (1, \sqrt{3})\} \text{ which is impossible.}$$

It remains a hard question, that how can we find acceptable solutions of the previous system.

3. Conclusion

In this paper we studied some of the elementary algebraic properties of 3-cyclic refined neutrosophic matrices with real entries, where we studied the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrated many examples that explain the validity of our work.

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