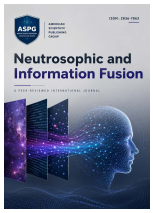
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FULL LENGTH ARTICLE

On the Algebraic Properties of 3-Cyclic Refined Neutrosophic Matrices with Real Entries

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ABSTRACT

The objective of this paper is to study some of the elementary algebraic properties of 3-cyclic refined neutrosophic matrices with real entries, where we study the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrate many examples that explain the validity of our work.

Keywords: 3-cyclic refined neutrosophic matrix ▪ Inverse ▪ Associative ▪ Indeterminacy

1. INTRODUCTION

The algebraic structures based on n-cyclic refined neutrosophic sets were studied by many different authors, where one can find some interesting generalizations such as n-cyclic refined neutrosophic rings, 3-cyclic refined neutrosophic units, 4-cyclic refined neutrosophic units, 4-cyclic refined neutrosophic Diophantine equations, and n-cyclic refined neutrosophic groups and complex numbers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

In [14], refined neutrosophic matrices have been studied through their representation by refined neutrosophic linear functions, together with related computational and structural properties.

This has motivated us to study the general case of 3-cyclic refined neutrosophic matrices for the first time, where some of the elementary algebraic properties of 3-cyclic refined neutrosophic matrices with real entries will be discussed, and we study the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrate many examples that explain the validity of our work.

2. MAIN DISCUSSION

Definition. Let X_0, X_1, X_2, X_3 be four $n \times n$ matrices with real entries. The corresponding 3-cyclic refined neutrosophic matrix is defined as follows:

$$X = X_0 + \sum_{i=1}^{3} X_i I_i.$$

Example. Take

$$X_0 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ -1 & 7 & 4 \end{pmatrix}, \quad X_1 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & 4 \\ -5 & -2 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

$$X = X_0 + \sum_{i=1}^{3} X_i I_i = \begin{pmatrix} 1 - I_1 + 4I_2 + I_3 & 2 + I_1 - I_2 + I_3 & 1 + I_1 + I_3 \\ I_1 + 3I_2 + I_3 & -I_1 + I_2 + I_3 & 3 + I_1 + 4I_2 + I_3 \\ -1 - 5I_2 + I_3 & 7 - 2I_2 + I_3 & 4 + 2I_1 - 2I_2 \end{pmatrix}.$$

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Definition. Let

$$X = X_0 + \sum_{i=1}^{3} X_i I_i, \qquad Y = Y_0 + \sum_{i=1}^{3} Y_i I_i$$

be two 3-cyclic refined neutrosophic square matrices with real entries. Then

$$X + Y = (X_0 + Y_0) + \sum_{i=1}^{3} (X_i + Y_i) I_i,$$

and

$$\begin{aligned} XY = {} & X_0 Y_0 \\ & + I_1(X_0Y_1 + X_1Y_0 + X_1Y_3 + X_3Y_1 + X_2Y_2) \\ & + I_2(X_0Y_2 + X_2Y_0 + X_2Y_3 + X_3Y_2 + X_1Y_1) \\ & + I_3(X_1Y_2 + X_2Y_1 + X_0Y_3 + X_3Y_0 + X_3Y_3). \end{aligned}$$

Example. Take

$$X = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} I_3,$$
$$Y = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} I_1 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_3.$$

We have

$$\begin{cases} X_0 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, & Y_0 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \\ X_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ X_2 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, & Y_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ X_3 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, & Y_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

Also,

$$\begin{cases} X_0 + Y_0 = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}, \\ X_1 + Y_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \\ X_2 + Y_2 = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}, \\ X_3 + Y_3 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}. \end{cases}$$

Hence

$$X + Y = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} I_3.$$

For the product terms,

$$X_0Y_0 = \begin{pmatrix} -1 & 3 \\ -1 & 1 \end{pmatrix},$$

$$X_0Y_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$
$$X_1Y_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix},$$
$$X_1Y_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$X_3Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$X_2Y_2 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Therefore,

$$X_0Y_1 + X_1Y_0 + X_1Y_3 + X_3Y_1 + X_2Y_2 = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}.$$

Similarly,

$$X_0Y_2 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$
$$X_2Y_0 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix},$$
$$X_1Y_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$
$$X_2Y_3 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$X_3Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}.$$

Thus,

$$X_0Y_2 + X_2Y_0 + X_1Y_1 + X_2Y_3 + X_3Y_2 = \begin{pmatrix} -1 & 6 \\ 1 & 3 \end{pmatrix}.$$

Finally,

$$X_0Y_3 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$X_3Y_0 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix},$$
$$X_1Y_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$
$$X_2Y_1 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$
$$X_3Y_3 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$X_0Y_3 + X_3Y_0 + X_1Y_2 + X_2Y_1 + X_3Y_3 = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}.$$

Consequently,

$$XY = \begin{pmatrix} -1 & 3 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} -1 & 6 \\ 1 & 3 \end{pmatrix} I_2 + \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} I_3.$$

Remark. The set of all $n \times n$ square 3-cyclic refined neutrosophic matrices $(R_{n\times n}(I), +, \times)$ is a ring. The unitary matrix in $R_{n\times n}(I)$ is

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} I_3.$$

Example. Let us try to find the inverse of

$$X = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I_1 + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} I_2 + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} I_3 = \begin{pmatrix} 1+I_3 & I_1+I_2+I_3 \\ 2+I_1+I_3 & 1+I_2 \end{pmatrix}.$$

Let

$$Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = X^{-1}, \qquad A = a_0 + \sum_{i=1}^{3} a_i I_i,$$

$$B = b_0 + \sum_{i=1}^{3} b_i I_i, \quad C = c_0 + \sum_{i=1}^{3} c_i I_i, \quad D = d_0 + \sum_{i=1}^{3} d_i I_i,$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}$. Then

$$XY = U \iff \begin{cases} A(1+I_3) + C(I_1+I_2+I_3) = 1, & (1) \\ B(1+I_3) + D(I_1+I_2+I_3) = 0, & (2) \\ A(2+I_1+I_3) + C(1+I_2) = 0, & (3) \\ B(2+I_1+I_3) + D(1+I_2) = 1. & (4) \end{cases}$$

From (1).

$$\begin{aligned} A(1+I_3) &= (a_0 + a_1I_1 + a_2I_2 + a_3I_3)(1+I_3) \\ &= a_0 + 2a_1I_1 + 2a_2I_2 + (2a_3 + a_0)I_3, \end{aligned}$$

and

$$\begin{aligned} C(I_1+I_2+I_3) &= (c_0 + c_1I_1 + c_2I_2 + c_3I_3)(I_1+I_2+I_3) \\ &= I_1(c_0+c_1+c_2+c_3) \\ &\quad + I_2(c_0+c_1+c_2+c_3) \\ &\quad + I_3(c_0+c_1+c_2+c_3). \end{aligned}$$

Hence,

$$\begin{cases} a_0 = 1, \\ 2a_1 + c_0 + c_1 + c_2 + c_3 = 0, \\ 2a_2 + c_0 + c_1 + c_2 + c_3 = 0, \\ 2a_3 + 1 + c_0 + c_1 + c_2 + c_3 = 0 \end{cases} \quad \text{system (I).}$$

From (3).

$$\begin{aligned} A(2+I_1+I_3) &= (a_0 + a_1I_1 + a_2I_2 + a_3I_3)(2+I_1+I_3) \\ &= 2a_0 + I_1(a_0 + 3a_1 + a_3) \\ &\quad + I_2(a_1 + 3a_2) + I_3(a_0 + 3a_3 + a_2), \end{aligned}$$

and

$$\begin{aligned} C(1+I_2) &= (c_0 + c_1I_1 + c_2I_2 + c_3I_3)(1+I_2) \\ &= c_0 + I_1(c_1+c_2) + I_2(c_0+c_2+c_3) \\ &\quad + I_3(c_1+c_3). \end{aligned}$$

Therefore,

$$\begin{cases} 2 + c_0 = 0, \\ 1 + 3a_1 + a_3 + c_1 + c_2 = 0, \\ a_1 + 3a_2 + c_0 + c_2 + c_3 = 0, \\ 1 + 3a_3 + a_2 + c_1 + c_3 = 0 \end{cases} \quad \text{system (II).}$$

Using systems (I) and (II), we obtain

$$a_0 = 1, \qquad c_0 = -2, \qquad a_1 = a_2, \qquad a_3 = a_1 - \frac{1}{2}.$$

Hence,

$$\begin{aligned} c_0 + c_2 + c_3 = -4a_1 &\implies c_2 + c_3 = -2 - 4a_1, \\ c_1 + c_2 = -1 - 3a_1 - a_3 &\implies c_1 + c_2 = -\tfrac{1}{2} - 4a_1, \\ c_1 + c_3 = -1 - 3a_3 - a_2 &\implies c_1 + c_3 = \tfrac{1}{2} - 4a_1. \end{aligned}$$

Thus,

$$\begin{cases} c_1 - c_3 = \frac{3}{2}, \\ c_1 + c_3 = \frac{1}{2} - 4a_1 \end{cases} \implies \begin{cases} c_1 = 1 - 2a_1, \\ c_3 = -\frac{1}{2} - 2a_1, \end{cases}$$

and

$$c_2 = -2 - 4a_1 - \left(-\tfrac{1}{2} - 2a_1\right) = -\tfrac{3}{2} - 2a_1.$$

It follows that

$$\begin{aligned} 2a_1 + c_0 + c_1 + c_2 + c_3 = 0 \\ \implies \quad -3 - 4a_1 = 0, \end{aligned}$$

so

$$a_1 = -\tfrac{3}{4}, \qquad a_2 = -\tfrac{3}{4}, \qquad a_3 = -\tfrac{5}{4},$$

and

$$c_0 = -2, \qquad c_1 = \tfrac{7}{2}, \qquad c_2 = 1, \qquad c_3 = 2.$$

Therefore,

$$A = 1 - \tfrac{3}{4}I_1 - \tfrac{3}{4}I_2 - \tfrac{5}{4}I_3, \qquad C = -2 + \tfrac{7}{2}I_1 + I_2 + 2I_3.$$

From equation (2), we get

$$\begin{cases} b_0 = 0, \\ 2b_1 + d_0 + d_1 + d_2 + d_3 = 0, \\ 2b_2 + d_0 + d_1 + d_2 + d_3 = 0, \\ 2b_3 + d_0 + d_1 + d_2 + d_3 = 0 \end{cases} \quad \text{system (III).}$$

Thus $b_1 = b_2 = b_3$. From equation (4),

$$\left.\begin{array}{r} d_0 = 1, \\ 3b_1 + b_2 + d_1 + d_2 = 0, \\ b_1 + 3b_2 + d_0 + d_2 + d_3 = 0, \\ 3b_3 + b_2 + d_1 + d_3 = 0 \end{array}\right\} \quad \text{system (IV).}$$

We obtain

$$\begin{cases} d_1 + d_2 = -4b_1, \\ d_2 + d_3 = -1 - 4b_1, \\ d_1 + d_3 = -4b_1, \end{cases}$$

and hence

$$\begin{cases} d_1 - d_3 = 1, \\ d_1 + d_3 = -4b_1 \end{cases} \Longrightarrow \begin{cases} d_1 = \frac{1}{2} - 2b_1, \\ d_3 = -\frac{1}{2} - 2b_1. \end{cases}$$

Also,

$$d_2 = -1 - 4b_1 + \tfrac{1}{2} + 2b_1 = -\tfrac{1}{2} - 2b_1.$$

Consequently,

$$2b_1 + d_0 + d_1 + d_2 + d_3 = 0 \Longrightarrow \tfrac{1}{2} - 4b_1 = 0,$$

so

$$b_1 = b_2 = b_3 = \tfrac{1}{8},$$
$$d_0 = 1, \quad d_1 = \tfrac{1}{4}, \quad d_2 = -\tfrac{3}{4}, \quad d_3 = -\tfrac{3}{4}.$$

Therefore,

$$B = \tfrac{1}{8}I_1 + \tfrac{1}{8}I_2 + \tfrac{1}{8}I_3, \qquad D = 1 + \tfrac{1}{4}I_1 - \tfrac{3}{4}I_2 - \tfrac{3}{4}I_3.$$

Hence,

$$Y = \begin{pmatrix} 1 - \frac{3}{4}I_1 - \frac{3}{4}I_2 - \frac{5}{4}I_3 & \frac{1}{8}I_1 + \frac{1}{8}I_2 + \frac{1}{8}I_3 \\ -2 + \frac{7}{2}I_1 + I_2 + 2I_3 & 1 + \frac{1}{4}I_1 - \frac{3}{4}I_2 - \frac{3}{4}I_3 \end{pmatrix}.$$

Definition. Let $A = A_0 + \sum_{i=1}^{3} A_i I_i$ be a 3-cyclic refined neutrosophic matrix with real entries. Then $a = a_0 + \sum_{i=1}^{3} a_i I_i \in R_3(I)$ is called an eigenvalue, with $X = X_0 + \sum_{i=1}^{3} X_i I_i$ as an eigenvector, if and only if

$$AX = aX, \qquad X_i \in \mathbb{R}^{n\times n}.$$

Example. Consider

$$A = \begin{pmatrix} 1 + I_2 & I_1 + I_2 + I_3 \\ 2 + I_1 + I_3 & 1 + I_2 \end{pmatrix}.$$

We will try to find some eigenvalues and eigenvectors. According to the definition,

$$AX = aX, \qquad X = X_0 + \sum_{i=1}^{3} X_i I_i, \qquad a = a_0 + \sum_{i=1}^{3} a_i I_i.$$

We get

$$\begin{cases} A_0X_0 = a_0X_0, & (1) \\ A_0X_1 + A_1X_0 + A_1X_3 + A_3X_1 + A_2X_2 = a_0X_1 + a_1X_0 + a_1X_3 + a_3X_1 + a_2X_2, & (2) \\ A_0X_2 + A_2X_0 + A_1X_1 + A_2X_3 + A_3X_2 = a_0X_2 + a_2X_0 + a_1X_1 + a_2X_3 + a_3X_2, & (3) \\ A_0X_3 + A_3X_0 + A_1X_2 + A_2X_1 + A_3X_3 = a_0X_3 + a_3X_0 + a_1X_2 + a_2X_1 + a_3X_3. & (4) \end{cases}$$

From the first equation, we can see that a_0 is an eigenvalue of A_0 with X_0 as eigenvector. By adding all equations, we get

$$(A_0 + A_1 + A_2 + A_3)(X_0 + X_1 + X_2 + X_3) = (a_0 + a_1 + a_2 + a_3)(X_0 + X_1 + X_2 + X_3).$$

Hence, $\sum_{i=1}^{3} a_i$ is an eigenvalue of $\sum_{i=1}^{3} A_i$, with $\sum_{i=1}^{3} X_i$ as an eigenvector.

Here,

$$A_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sum_{i=1}^{3} A_i = \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}.$$

The eigenvalue of A_0 is $a_0 = 1$. Moreover,

$$(A_0 - I)X_0 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} X_0 \\ X_0' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow X_0 = 0,$$

hence $X_0 = (0, 1)$.

The eigenvalues of $\sum_{i=1}^{3} A_i$ are determined by

$$\det \begin{pmatrix} 2 - \lambda & 3 \\ 4 & 2 - \lambda \end{pmatrix} = 0 \Longleftrightarrow (2 - \lambda)^2 - 12 = 0$$
$$\Longleftrightarrow \lambda^2 - 4\lambda - 8 = 0.$$

Since

$$\Delta = 16 + 32 = 48, \qquad \sqrt{\Delta} = 4\sqrt{3},$$

we obtain

$$\lambda_1 = 2 + 2\sqrt{3}, \qquad \lambda_2 = 2 - 2\sqrt{3},$$

and therefore

$$\sum_{i=1}^{3} a_i \in \{2 + 2\sqrt{3},\ 2 - 2\sqrt{3}\}.$$

For $\lambda_1 = 2 + 2\sqrt{3}$,

$$\begin{pmatrix} -2\sqrt{3} & 3 \\ 4 & -2\sqrt{3} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$-2\sqrt{3}X + 3Y = 0, \qquad 4X - 2\sqrt{3}Y = 0,$$
$$\Longrightarrow \quad Y = \frac{2}{\sqrt{3}}X.$$

The corresponding eigenvector is $(1, 2/\sqrt{3})$. For $\lambda_2 = 2 - 2\sqrt{3}$,

$$\begin{pmatrix} 2\sqrt{3} & 3 \\ 4 & 2\sqrt{3} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$2\sqrt{3}X + 3Y = 0, \qquad 4X + 2\sqrt{3}Y = 0,$$
$$\Longrightarrow \quad Y = -\frac{2}{\sqrt{3}}X.$$

The corresponding eigenvector is $(1, -2/\sqrt{3})$.

Now, for

$$\sum_{i=1}^{3} a_i = 2+2\sqrt{3}, \qquad \sum_{i=1}^{3} X_i = \left(1, \frac{2}{\sqrt{3}}\right),$$

we get

$$\begin{cases} a_1+a_2+a_3 = 1+2\sqrt{3}, \\ X_1+X_2+X_3 = \left(1, \dfrac{2}{\sqrt{3}}-1\right) \end{cases}$$
$$\iff \begin{cases} a_3 = 1+2\sqrt{3}-a_1-a_2, \\ X_3 = \left(1, \dfrac{2-\sqrt{3}}{\sqrt{3}}\right) - X_1 - X_2. \end{cases}$$

Consider equations (2) and (3):

$$\begin{aligned} &A_0(X_1+X_2)+A_1(X_0+X_3+X_1) \\ &\quad +A_2(X_2+X_0+X_3)+A_3(X_1+X_2) \\ &= a_0(X_1+X_2)+a_1(X_0+X_3+X_1) \\ &\quad +a_2(X_2+X_0+X_3)+a_3(X_1+X_2). \end{aligned}$$

Hence,

$$\begin{aligned} &A_0(X_1+X_2)+A_1\left(\left(1,\frac{2}{\sqrt{3}}\right)-X_2\right) \\ &\quad +A_2\left(\left(1,\frac{2}{\sqrt{3}}\right)-X_1\right)+A_3(X_1+X_2) \\ &= a_0(X_1+X_2)+a_1\left(\left(1,\frac{2}{\sqrt{3}}\right)-X_2\right) \\ &\quad +a_2\left(\left(1,\frac{2}{\sqrt{3}}\right)-X_1\right)+a_3(X_1+X_2). \end{aligned}$$

To find a suitable solution, we can put

$$\begin{cases} (A_0+A_3)(X_1+X_2) = (a_0+a_3)(X_1+X_2), \\ A_1X_2 = a_1X_2, \\ A_2X_1 = a_2X_1. \end{cases}$$

This implies that a_1 is an eigenvalue of A_1 with X_2 as eigenvector, a_2 is an eigenvalue of A_2 with X_1 as eigenvector, and a_0+a_3 is an eigenvalue of A_0+A_3 with X_1+X_2 as eigenvector. In particular,

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_1 = 1, \quad X_2 = (1,1),$$

$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad a_2 = 1, \quad X_1 = (1,0),$$

and

$$A_0+A_3 = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}, \qquad a_0+a_3 \in \{1+\sqrt{3},\ 1-\sqrt{3}\}.$$

Thus,

$$a_3 \in \{\sqrt{3}, -\sqrt{3}\}, \qquad X_1+X_2 \in \{(1,-\sqrt{3}),(1,\sqrt{3})\},$$

which is impossible. It remains a hard question how we can find acceptable solutions of the previous system.

3. CONCLUSION

In this paper we studied some of the elementary algebraic properties of 3-cyclic refined neutrosophic matrices with real entries, where we studied the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrated many examples that explain the validity of our work.

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