



Secondary Partial Ordering of Neutrosophic Fuzzy Matrices

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Abstract

In this article, we define secondary generalized inverse of a neutrosophic fuzzy matrices whenever exists. Also, the S-ordering for the set of neutrosophic fuzzy matrices are defined and characterized. A necessary and sufficient condition for the existence of secondary generalized inverse of neutrosophic fuzzy matrices with the help of S-ordering is obtained.

Keywords: Neutrosophic fuzzy matrix; secondary generalized inverse; Matrix Partial order; S-ordering

1 introduction

The neutrosophic set was introduced by F. Smarandache.¹⁹ It is a generalization of fuzzy sets and intuitionistic fuzzy sets defined by L. Zadeh²¹ and Atanassov² respectively. A fuzzy set deals with the uncertainty using membership grade while the neutrosophic sets are characterized by three membership functions respectively, truth membership function (T), indeterminacy membership function (I) and falsity membership function (F). Das et al.⁴ introduced the notion of neutrosophic fuzzy set by combining fuzzy set with neutrosophic set. Neutrosophic fuzzy sets have several applications in the field of decision making^{4,5,20} medical diagnosis,¹² imaging techniques¹³ etc. In⁶ Mamouni et al. defined the concept of neutrosophic fuzzy matrices. Anandhkumar et al.¹ has studied various inverses of neutrosophic fuzzy matrices.

Secondary generalized inverse is the unique generalized inverse of a matrix, whenever exists. It is the Moore Penrose inverse with respect to secondary transpose involution. The outer inverses as well as generalized inverses defined using secondary transpose^{7,8,17} is an emerging area of research. In¹¹ the matrix partial order called the \dagger_s using the s-g inverse is studied extensively. For more details about the secondary generalized inverse and secondary transpose, one can refer.¹⁴⁻¹⁶

In this work, we define the secondary generalized inverse of neutrosophic fuzzy matrices and some of its characterizations. Also, the S-partial order and its properties on the set of neutrosophic fuzzy matrices are studied. Before proceeding further, some preliminaries are given below:

1.1 Preliminaries

In,⁹ W. B. V. Kandaswamy and F. Smarandache proposed the neutrosophic sets and neutrosophic matrices.

Definition 1.1. (Neutrosophic Sets)⁹ Let U be an universe of discourse then the neutrosophic set A is an object having the form $A = \{ \langle x : T_A(x), I_A(x), F_A(x) \rangle, x \in U \}$ where the functions $T, I, F : U \rightarrow]-0, 1 + [$ define respectively the degree of membership (or truthness), the degree of indeterminacy, and the degree of non-membership (or falsehood) of the set $x \in U$ to the set A with the condition

$$-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$$

Definition 1.2. (Neutrosophic Matrices)⁹ Let $M_{m \times n} = \{a_{ij}/a_{ij} \in K(I)\}$, where $K(I)$ is a neutrosophic matrix. We call $M_{m \times n}$ to be the neutrosophic matrix.

A fuzzy matrix is a matrix in which the elements are from the interval $[0, 1]$.

Definition 1.3. (Neutrosophic fuzzy matrix)⁹ Let A be a neutrosophic fuzzy matrix, whose entries is of the form $a+Ib$ (neutrosophic number), where a, b are elements of $[0, 1]$ and I is an indeterminate such that $I^n = I$, n being a positive integer.

For more properties of neutrosophic sets and neutrosophic matrices refer to.¹⁸

Anna Lee² established a relation between the classical transpose A^T and secondary transpose A^S such that $A^S = VA^TV$ where V is a permutation matrix with units in its secondary diagonal. Savitha et.al., used the idea of secondary tranpose to obtain a unique generalized inverse known as secondary generalized inverse for a matrix A .

Definition 1.4. (Secondary generalized inverse)¹⁴ An $m \times n$ matrix G denoted by $A^{\dagger S}$, whenever exists, is called the secondary generalized inverse if it satisfies the system of matrix equations given below:

$$\begin{array}{ll} (1) AXA = A & (2) XAX = X \\ (3) (AX)^S = AX & (4) (XA)^S = XA \end{array}$$

2 Results

Let N_s stands for Fuzzy neutrosophic set. Denote $M_{m \times n}$ be the matrices with entries form N_s to be the fuzzy neutrosophic matrices.

Definition 2.1. For an $M \in M_{m \times n}$ is secondary symmetric if $M^S = M$, i.e., if $M^S = VM^TV$.

Here V is a permutation matrix with unity in the secondary diagonal of the form $V = \begin{pmatrix} 0 & 1+I \\ 1+I & 0 \end{pmatrix}$

For an arbitrary rectangular matrix $M \in M_{m \times n}$, the secondary generalized inverse X of M , whenever exists, can be defined as

$$(1) MXM = M, \quad (2) XMX = X \quad (3^S) (MX)^S = MX \quad (4^S) (XM)^S = XM$$

2.1 Existence of secondary generalized inverse

Proposition 2.2. For any $M \in M_{m \times n}$, the following statements are equivalent:

- (i) $M\{1, 3^S\} \neq \varphi$
- (ii) The condition $XM^SM = M$ has solutions
- (iii) M is left cancellable and $M^SM\{1\} \neq \varphi$

Proof. (i) \implies (ii) : Let $M^{(1,3^S)} \in M\{1, 3^S\}$, we have $MM^{(1,3^S)}M = (MM^{(1,3^S)})^SM = (M^{(1,3^S)})^SM^SM$. Thus, $(M^{(1,3^S)})^S$ is a solution of $XM^SM = M$.

(ii) \implies (iii) Let G be a solution of $XM^SM = M$. For any $G_1, G_2 \in M_{m \times n}$, if $M^SMG_1 = M^SMG_2$, then $MG_1 = GM^SMG_1 = GM^SMG_2 = MG_2$. Hence, M is left cancellable, and $M^SM = M^SMG^SGM^SM$. Then, $G^SG \in M^SM\{1\}$

(ii) \implies (i): Let G be a solution of $GA^SM = M$. Then we have $G \in M\{1, 3^S\}$. \square

In a similar way we have the following result.

Proposition 2.3. For any $M \in M_{m \times n}$, the given statements are equivalent.

- (i) $M\{1, 4^S\} \neq \varphi$.
- (ii) The relation $MM^SY = M$ has solutions;
- (iii) M is right cancellable and $MM^S\{1\} \neq \varphi$.

The following theorem gives the necessary and sufficient conditions for the existence of a neutrosophic fuzzy generalized inverse.

Theorem 2.4. For each $M \in M_{m \times n}$, the following statements are equivalent:

- (i) $M^{\dagger S}$ exists;
- (ii) The relations $XM^SM = M$, $MM^SY = M$ have solutions;
- (iii) $M\{1, 3^S\} \neq \varphi$ and $M\{1, 4^S\} \neq \varphi$
- (iv) M is cancellable, $M^SM\{1\} \neq \varphi$ and $MM^S\{1\} \neq \varphi$.

Proof. (ii) \implies (i) can be proved by Proposition 2.2 and 2.3. Let $P, Q \in M_{m \times n}$ such that $M = PM^SM$ and $M = MM^SQ$. Now, $M^S = M^SMP^S = Q^SMM^S$. Set $G = M^SQP^S$. Then G is secondary generalized inverse of M . □

Corollary 2.5. Let $M \in M_{m \times n}$ and P, Q be a solution of $XM^SM = M, MM^SY = A$ respectively. Then

$$M^{\dagger S} = M^SQP^S = Q^SMP^S = Q^SPM^S$$

Theorem 2.6. Consider $M, N \in M_{m \times n}$, such that $M \stackrel{S}{\leq} N$. If $N^{\dagger S}$ exists and M is cancellable, then $M^{\dagger S}$ exists and $M^{\dagger S} = N^{\dagger S}A(B^{\dagger S})^S$.

Proof. Since $M^SM = M^SN, M^SM = M^SNN^{\dagger S}N = M^SMN^{\dagger S}N, M = MN^{\dagger S}N$ and $M^S = (N^{\dagger S}N)^SM^S = N^{\dagger S}NM^S = N^{\dagger S}MM^S$. Similarly, we have $M = NN^{\dagger S}M$ and $M^S = M^SMN^{\dagger S}$ by $MM^S = NM^S$. Hence, $MN^{\dagger S}M = (N^{\dagger S})^SM^SMN^{\dagger S}M = (N^{\dagger S})^SM^SM = M, (MN^{\dagger S})^S = ((N^{\dagger S})^SM^SMN^{\dagger S})^S = (N^{\dagger S})^SM^SMN^{\dagger S} = MN^{\dagger S}$. Thus, $N^{\dagger S} \in M\{1, 3^S\}$. Similarly, we have $N^{\dagger S} \in M\{1, 4^S\}$. Therefore, $M^{\dagger S}$ exists and $M^{\dagger} = N^{\dagger S}M(N^{\dagger S})^S$ by Theorem 2.4. □

Corollary 2.7. For any $M, N \in M_{m \times n}$ and $M \stackrel{S}{\leq} N$. If $N^{\dagger S}$ exists and M is cancellable, then $N^{\dagger S} \in M\{1, 3^S\} \cap M\{1, 4^S\} = M\{1, 3^S, 4^S\}$

3 S-ordering and its characterization

The concept of T-ordering in Fuzzy matrices is introduced by Jian Miao Cen.³ Here, we use the same techniques in defining and characterizing the S-ordering in Neutrosophic Fuzzy matrices.

Definition 3.1. The S-ordering $M \stackrel{S}{\leq} N$ in $M_{m \times n}$ is defined as

$$M \stackrel{S}{\leq} N \text{ iff } M^SM = M^SN, MM^S = NM^S$$

Now, we can see some characterizations of S-ordering.

Theorem 3.2. Let $M, N \in M_{m \times n}$ and $M^{\dagger s}$ exists. Then the following conditions are equivalent:

1. $M \leq^S N$;
2. $M^{\dagger s} M = M^{\dagger s} N$ and $MM^{\dagger s} = NM^{\dagger s}$;
3. $MM^{\dagger s} N = M = NM^{\dagger s} M$.

Proof. (1) \implies (2). By (1) we have $M^S M = M^S N$ and $MM^S = NM^S$. Then $M^{\dagger s} M = M^{\dagger s} MM^{\dagger s} M = M^{\dagger s} (M^{\dagger s})^S M^S M = M^{\dagger s} (M^{\dagger s})^S M^S N = M^{\dagger s} MM^{\dagger s} N = M^{\dagger s} N$. Similarly, $MM^{\dagger s} = NM^{\dagger s}$.
 (2) \implies (3): $M^{\dagger s} M = M^{\dagger s} N$ implies $M = MM^{\dagger s} M = MM^{\dagger s} N$ and $MM^{\dagger s} = NM^{\dagger s}$ implies $M = MM^{\dagger s} M = NM^{\dagger s} M$.
 (3) \implies (1): By $M = MM^{\dagger s} N$, $(MM^{\dagger s})^S M = (MM^{\dagger s})^S N$. Then $M^S (M^{\dagger s})^S M^S M = M^S (M^{\dagger s})^S M^S N$. Hence $M^S M = M^S N$. Similarly, $MM^S = NM^S$ by $M = NM^{\dagger s} M$. \square

Corollary 3.3. If $M \leq^S N$ and $M^{\dagger s}$ exists, then $N \in M^{\dagger s} \{1\}$.

Proof. By Theorem 3.2, we have $M^{\dagger s} = M^{\dagger s} N M^{\dagger s}$. \square

Theorem 3.4. Let $M, N \in M_{m \times n}$. If $M^{\dagger s}$ and $N^{\dagger s}$ both exists, then the following conditions are equivalent:

1. $M \leq^S N$;
2. $M^{\dagger s} M = N^{\dagger s} M$ and $MM^{\dagger s} = MN^{\dagger s}$;
3. $N^{\dagger s} M M^{\dagger s} = M^{\dagger s} = M^{\dagger s} M N^{\dagger s}$;
4. $M^S M N^{\dagger s} = M^S = N^{\dagger s} M M^S$.

Proof. (1) \implies (4): $M^S M = M^S N$ implies $M^S M = M^S N N^{\dagger s} N$. Then $M^S M = (M^S M)^S = (N^{\dagger s} N)^S (M^S N)^S = N^{\dagger s} N M^S M$. Hence, $M^S M M^{\dagger s} = N^{\dagger s} N M^S M M^{\dagger s}$ and $M^S (M M^{\dagger s})^S = N^{\dagger s} N M^S (M M^{\dagger s})^S$. Therefore, $M^S = N^{\dagger s} N M^S$.
 Similarly, $M^S = M^S N N^{\dagger s}$ by $MM^S = NM^S$.
 (4) \implies (2): By $M^S = N^{\dagger s} M M^S$, $M^S (M^{\dagger s})^S = N^{\dagger s} M M^S (M^{\dagger s})^S$. Then $M^{\dagger s} M = N^{\dagger s} M M^{\dagger s} M = N^{\dagger s} N$.
 Similarly $MM^{\dagger s} = MN^{\dagger s}$ by $M^S = M^S M N^{\dagger s}$.
 (2) \implies (4): By $M^{\dagger s} M = N^{\dagger s} M$, $M^S (M^{\dagger s})^S = N^{\dagger s} M$. Then, $M^S (M^{\dagger s})^S M^S = N^{\dagger s} M M^S$, i.e., $M^S = N^{\dagger s} M M^S$.
 Similarly, $M^S = M^S M N^{\dagger s}$ by $MM^{\dagger s} = MN^{\dagger s}$.
 (2) \implies (3): By $M^{\dagger s} M = N^{\dagger s} M$, $M^{\dagger s} = M^{\dagger s} M M^{\dagger s} = N^{\dagger s} M M^{\dagger s}$. Similarly, $MM^{\dagger s} = MN^{\dagger s}$ implies $M^{\dagger s} = M^{\dagger s} M N^{\dagger s}$.
 (3) \implies (2): $N^{\dagger s} M M^{\dagger s} = M^{\dagger s} = M^{\dagger s} = M^{\dagger s} M N^{\dagger s}$ implies $M^{\dagger s} M = N^{\dagger s} M M^{\dagger s} M = N^{\dagger s} M$ and $MM^{\dagger s} = M M^{\dagger s} M N^{\dagger s} = MN^{\dagger s}$.
 (2) \implies (1): $M^{\dagger s} M = (M^{\dagger s} M)^S = (N^{\dagger s} M)^S = (N^{\dagger s} N N^{\dagger s} M)^S = (M^{\dagger s} M)^S (M^{\dagger s} N)^S = (M^{\dagger s} M)^S N^{\dagger s} N = M^{\dagger s} M N^{\dagger s} N = M^{\dagger s} M M^{\dagger s} N = M^{\dagger s} N$.
 Similarly, we have $MM^{\dagger s} = NM^{\dagger s}$. Thus (1) holds by Theorem 3.2. \square

Corollary 3.5. If $M \leq^S N$ and $M^{\dagger s}, N^{\dagger s}$ both exists, then $N^{\dagger s} \in M \{1\}$.

Let us denote $M_{m \times n}^{\dagger s}$ be the set of all $m \times n$ neutrosophic fuzzy marices having secondary generalized inverse.

Theorem 3.6. In $M_{m \times n}^{\dagger s}, \leq^S$ is a partial ordering.

Proof. $M \leq^S M$ is obvious. If $M \leq^S N, N \leq^S P$, then $M = N M^{\dagger s} M, N = P N^{\dagger s} P$ by Theorem 3.2(3). Thus, by Theorem 3.2(2), $N = P N^{\dagger s} M = N M^{\dagger s} M = M$.
 If $M \leq^S N, N \leq^S P$, then $M = N M^{\dagger s} M$ and $N = P N^{\dagger s} P$, by Theorem 3.2(3). By Theorem 3.4(2), we have $M = N M^{\dagger s} M = P N^{\dagger s} N M^{\dagger s} M = P N^{\dagger s} M = P M^{\dagger s} M$. Similarly, we have $M = M M^{\dagger s} P$. Thus $M \leq^S P$ by Theorem 3.2(3). \square

4 Properties of S-partial ordering

Theorem 4.1. Assume $M \stackrel{S}{\leq} N$, then we have

1. $N^{\dagger s} M = M^{\dagger s} N$ and $MN^{\dagger s} = NM^{\dagger s}$;
2. $N^S M = M^S N$ and $NM^S = MN^S$, i.e., $N^S M$ and NM^S are secondary symmetric;
3. $MN^{\dagger s} M = M = NN^{\dagger s} M = NMN^{\dagger s} = N^{\dagger s} MN, NM^{\dagger s} M = M = MM^{\dagger s} N = MNM^{\dagger s} = M^{\dagger s} NM$.
4. $NM^S M = MM^S N = M^S NM = MNM^S, MN^S N = NN^S M = N^S MN = NMM^S$.

Theorem 4.2. If $M \stackrel{S}{\leq} N$, then we have

1. $M^S \stackrel{S}{\leq} N^S$;
2. $M^{\dagger s} \stackrel{S}{\leq} N^{\dagger s}$;
3. $N^S M \stackrel{S}{\leq} N^S N, MN^S \stackrel{S}{\leq} NN^S$;
4. $N^{\dagger s} M \stackrel{S}{\leq} N^{\dagger s} N, MN^{\dagger s} \stackrel{S}{\leq} NN^{\dagger s}$;
5. $M^S M \stackrel{S}{\leq} N^S N, MM^S \stackrel{S}{\leq} NN^S$;
6. if $N^S N^{\dagger s} = N^{\dagger s} N^S, M^S M^{\dagger s} = M^{\dagger s} M^S$;
7. if $N^{\dagger s} = N^S, M^{\dagger s} = M^S$;
8. if $N^2 = 0, M^2 = 0$;
9. if $N = N^2, M = M^2$;
10. if $N = NN^S, M = MM^S$;
11. if $N = N^S = N^3$ and $M = M^S$, then $M = M^3$.

Proof. (1) and (2) hold clearly.

(3) $(N^S M)^S N^S M = M^S N N^S M = M^S M M^S M = M^S M M^S N = M^S M N^S N = M^S N N^S N$. Similarly, $N^S M (M^S M)^S = N^S N (N^S M)^S$; thus $N^S M \stackrel{S}{\leq} N^S N$.

Similarly, we have $MN^S \stackrel{S}{\leq} NN^S$.

(4) $(N^{\dagger s} M)^S N^{\dagger s} M = (M^{\dagger s} M)^S M^{\dagger s} M = M^S (M^{\dagger s})^S M^{\dagger s} M = M^S (M^{\dagger s})^S M^{\dagger s} N = M^S (M^{\dagger s})^S N^{\dagger s} N = (M^{\dagger s} M)^S N^{\dagger s} N = (N^{\dagger s} M)^S N^{\dagger s} N$, and $N^{\dagger s} M (N^{\dagger s} M)^S = N^{\dagger s} N (N^{\dagger s} M)^S$. Thus, $N^{\dagger s} M \stackrel{S}{\leq} N^{\dagger s} N$.

Similarly, we have $MN^{\dagger s} \stackrel{S}{\leq} NN^{\dagger s}$.

(5) $(M^S M)^S M^S M = M^S M M^S N = M^S M N^S N = (M^S M)^S N^S N$, and $N^S M (M^S M)^S = N^S N (M^S M)^S$.

Thus, $M^S M \stackrel{S}{\leq} N^S N$.

Similarly, we have $MM^S \stackrel{S}{\leq} NN^S$.

(6) $(M^{\dagger s} M)^S N^S N = M^S (M^{\dagger s})^S N^{\dagger s} N = M^S (M^{\dagger s})^S M^{\dagger s} N = M^S (M^{\dagger s})^S M^{\dagger s} M = (M^{\dagger s} M)^S M^{\dagger s} M$ and $N^{\dagger s} N (M^{\dagger s} M)^S = M^{\dagger s} M (M^{\dagger s} M)^S$. Then $M^{\dagger s} M \stackrel{S}{\leq} N^{\dagger s} M$.

Similarly, $MM^{\dagger s} \stackrel{S}{\leq} NN^{\dagger s}$.

(7) $M^S M^{\dagger s} = M^{\dagger s} M N^S N^{\dagger s} M M^{\dagger s} = M^{\dagger s} M N^{\dagger s} N^S M M^{\dagger s} = M^{\dagger s} M^S$.

(8) $M^{\dagger s} = M^{\dagger s} M M^{\dagger s} = N^{\dagger s} M M^{\dagger s} = N^{\dagger s} M N^{\dagger s} = N^{\dagger s} M N^S = N^{\dagger s} M M^S = M^S$.

(9) $M^2 = M N^{\dagger s} N N N^{\dagger s} M = M N^{\dagger s} N^2 N^{\dagger s} M = 0$.

(10) $M^2 = M N^{\dagger s} N N N^{\dagger s} = M N^{\dagger s} N N^{\dagger s} M = M N^{\dagger s} M = M M^{\dagger s} M = M$.

(11) By $N = NN^S$, we have $N^S = N$ and $N^{\dagger s}N$. Then $MM^S = NM^S = NN^S M = NNM^S = NMM^S = N^{\dagger s}MM^S = M^S = M$.
 (12) $M^3 = MM^S M = NM^S MM^{\dagger s} N = NN^S MM^{\dagger s} N = NN^S NM^{\dagger s} N = NM^{\dagger s} N = NN^{\dagger s} M = M$. □

Lemma 4.3. If $MM^S \in M_{m \times n}^{\dagger s}$, then we have

1. $MM^S \in M_{m \times n}^{\dagger s}, M^S M \in M_{m \times n}^{\dagger s}$ and

$$(MM^S)^{\dagger s} = (AM^{\dagger s})^S M^{\dagger s}, \quad (M^S M)^{\dagger s} = M^{\dagger s} (M^{\dagger s})^S \tag{1}$$

2. $MM^{\dagger s} \in M_{m \times m}^{\dagger s}, M^{\dagger s} M \in M_{n \times n}^{\dagger s}$ and

$$(MM^{\dagger s})^{\dagger s} = MM^{\dagger s}, (M^{\dagger s} M)^{\dagger s} = M^{\dagger s} M \tag{2}$$

Proof. (1) $MM^S = MM^{\dagger s} MM^S = (M^{\dagger s})^S M^S MM^S = (M^{\dagger s})^S (MM^{\dagger s} M)^S MM^S = (M^{\dagger s})^S (MM^S) (M^{\dagger s})^S MM^S = (M^{\dagger s})^S M^{\dagger s} MM^S MM^S$. Then, $(M^{\dagger s})^S M^{\dagger s}$ is a solution of the fuzzy relation equation $MM^S = XMM^S MM^S$. By Proposition 2.2, $(M^{\dagger s})^S M^{\dagger s} \in MM^S \{1, 3^S\}$. Since MM^S is secondary symmetric, $(M^{\dagger s})^S M^{\dagger s} \in MM^S \{1, 4^S\}$. By Theorem 2.4 and Corollary 2.5, $(MM^{\dagger s})^S$ exists and $(MM^S)^{\dagger s} = (M^{\dagger s})^S M^{\dagger s} MM^S (M^{\dagger s})^S M^{\dagger s} = (M^{\dagger s})^S M^{\dagger s} M (M^{\dagger s} M)^S M^{\dagger s} = (M^{\dagger s})^S M^{\dagger s} MM^{\dagger s} MM^{\dagger s} = (M^{\dagger s})^S M^{\dagger s}$. Similarly, we have $M^S M \in M_{n,n}^{\dagger s}$ and $(M^S M)^{\dagger s} = M^{\dagger s} (M^{\dagger s})^S$.
 (2) Since $MM^{\dagger s} = MM^{\dagger s} MM^{\dagger s} MM^{\dagger s}$ and $(MM^{\dagger s} MM^{\dagger s})^S = (MM^{\dagger s})^S (MM^{\dagger s})$ and $(M^{\dagger s} M)^{\dagger s} = M^{\dagger s} M$. □

Theorem 4.4. Let $M, N \in M_{m \times n}^{\dagger s}$. Then

1. if $M^S M \leq N^S N, M^{\dagger s} M \leq N^{\dagger s} N$;
2. if $MM^S \leq NN^S, MM^{dagger s} \leq NN^{\dagger s}$.

Proof. By Lemma 4.3 and Theorem 4.2 we have $(M^S M)^{\dagger s} M^S M \leq (N^S N)^{\dagger s} N^S N$. Then $M^{\dagger s} M = M^{\dagger s} MM^{\dagger s} M = M^{\dagger s} (M^{\dagger s})^S M^S M = (M^S M)^{\dagger s} M^S \leq (N^S N)^{\dagger s} N^S N = N^{\dagger s} N$.
 (2) Similar. □

5 S-ordering and other generalized inverses

Firstly, we characterize the S-ordering by the generalized $\{1, 3^S, 4^S\}$ inverse.

Theorem 5.1. Let $M, N \in M_{m,n}^{\dagger s}$. Then the following conditions are equivalent:

1. $M \leq N$;
2. $N \in M^{\dagger s} \{1, 3^S, 4^S\}$
3. $N^{\dagger s} \in M \{1, 3^S, 4^S\}$

Proof. (1) \implies (2) holds by Corollary 2.7.
 (2) \implies (1): $MM^{\dagger s} = M^{\dagger s} MN^{\dagger s} M = (M^{\dagger s} M)^S (N^{\dagger s} M)^S = (N^{\dagger s} MM^{\dagger s} M)^S = (N^{\dagger s} M)^S = N^{\dagger s} M$
 Similarly, $MM^{\dagger s} = MN^{\dagger s}$. Thus, $M \leq N$ by Theorem 3.4.
 (1) \implies (3): By Theorem 4.2, the proof is trivial. □

Theorem 5.2. Let $M \in M_{m,n}^{\dagger s}$. Then

$$H \stackrel{S}{\leq} M^{\dagger s} G, \text{ for every } G \in M\{1, 3^S, 4^S\} \text{ and every } H \in M\{2, 3^S, 4^S\} \quad (3)$$

Proof. $H \in M\{2, 3^S, 4^S\}$ implies $M \in H\{1, 3^S, 4^S\}$. Then $H^{\dagger s} \in M_{n,m}^{\dagger s}$. Hence H over set $S \leq M^{\dagger s}$ by Theorem 4.1.

By Corollary 2.5 of $G \in M\{1, 3^S, 4^S\}$ implies $M^{\dagger s} = GMG$. Then $MM^{\dagger s} = MGMG = MG$ and $MM^{\dagger s} = MGMG = GM$. Hence $M^{\dagger s} \stackrel{S}{\leq} G$ by Theorem 3.2 □

Corollary 5.3. Let $M \in M_{m,n}^{\dagger s}$. Then

$$M\{2, 3^S, 4^S\} = \{H \in M_{n,m}^{\dagger s} | H \stackrel{S}{\leq} M^{\dagger s}\} \quad (4)$$

Theorem 5.4. Let $M \in M_{m,n}^{\dagger s}$. Then, for $i = 3^S$ or 4^S ,

$$G_0 \in M\{1, i\}, G_0 \stackrel{S}{\leq} G \implies G \in M\{1\}.$$

Proof. If $i = 3$, then $M = G_0^S M^S M$, and hence $MGM = MGG_0^S M^S M = MG_0 G_0^S M^S M = MG_0 M = M$.

For $i = 4$, the result follows similarly using the equality $M = MM^S G_0^S$. □

Theorem 5.5. Let $M \in F_{m,n}$. Then

$$G_0 \in M\{1, 3^S, 4^S\}, G_0 G_0^S G \implies G \in M\{1, 3^S, 4^S\}$$

Proof. Theorem 5.4 implies that $G \in A\{1\}$, moreover, $MG = MG_0 MG = MM^S G_0^S G = MM^S G_0^S G = MG_0$, and, similarly, $GM = G_0 M$. Thus, $G \in M\{1, 3^S, 4^S\}$. □

Corollary 5.6. Let $M \in M_{m,n}^{\dagger s}$. Then

$$M\{1, 3^S, 4^S\} = \{G \in M_{n,m} | M^{\dagger s} \stackrel{S}{\leq} G\}$$

Proof. By Theorem 5.5, if $M^{\dagger s} \stackrel{S}{\leq} G$, then $G \in M\{1, 3^S, 4^S\}$. On the contrary, if $G \in M\{1, 3^S, 4^S\}$, then $M^{\dagger s} \stackrel{S}{\leq} G$ by Theorem 5.2. □

Theorem 5.7. Let $M \in M_{m,n}$ and $H, H_0 \in F_{n,m}$. Then

1. $H_0 \in M\{2\}, H \stackrel{S}{\leq} H_0 \implies M\{2\}$;
2. for $i = 3^S$ or $4^S, H \in M\{2\}, H_0 \in M\{2, i\} \implies H \in A\{2, i\}$.

Proof. (1) By Theorem 3.4(2) and 4.1(3), we have

$$HMH = HH_0^{\dagger s} H_0 M H_0 H_0^{\dagger s} H = HH_0^{\dagger s} = HH^{\dagger s} H = H$$

(2) If $i = 3$, by Theorems 3.2(2), 3.4(2) and 4.1(3), we have $MH = MH_0 H_0^{\dagger s} H = H_0^S M^S H_0^{\dagger s} H = H^S M^S H_0^S H_0 = H^S M^S H_0^{\dagger s} (H^{\dagger s})^S = H_0^S (H^{\dagger s})^S = (H^{\dagger s} H_0)^S = H_0^{\dagger s} H_0 = (H_0^{\dagger s} H_0)^S = (MH)^S$. Similarly for $i = 4$. □

Theorem 5.8. Let $M \in M_{m,n}^{\dagger s}$ and $H \in M_{n,m}^{\dagger s}$. Then $H \stackrel{S}{\leq} H_0$ implies

$$H_0 \in \{2, 3^S, 4^S\}, H \in A\{2, 3^S, 4^S\}$$

Proof. By Corollary 5.3, we have $H_0 \stackrel{S}{\leq} M^{\dagger S}$ and $H_0^{\dagger S} \in M_{m,n}^{\dagger S}$ by Theorem 3.6. Hence, $H \in M\{2, 3^S, 4^S\}$ by Corollary 5.3. \square

Theorem 5.9. Let $M, N \in M_{m,n}^{\dagger S}, M \stackrel{S}{\leq} N$. Then for every $H_M \in M\{2, 3^S, 4^S\}$, and every $G_N \in B\{1, 3^S, 4^S\}$, we have $H_M \stackrel{S}{\leq} M^{\dagger S} \stackrel{S}{\leq} N^{\dagger S} \stackrel{S}{\leq} G_N$.

Proof. Since $M \stackrel{S}{\leq} N$, implies $M^{\dagger S} \stackrel{S}{\leq} N^{\dagger S}$, the proof is trivial. \square

Theorem 5.10. Let $M, N \in M_{m,n}^{\dagger S}$. Then the following statements are equivalent:

1. $M \stackrel{S}{\leq} N$;
2. $M^{\dagger S} \stackrel{S}{\leq} G_N$ for every $G_N \in N\{1, 3^S, 4^S\}$;
3. $H_M \stackrel{S}{\leq} N$ for every $H_M \in M\{2, 3^S, 4^S\}$.

Proof. (1) \implies (2): It follows by Theorem 5.9.

(2) \implies (1): Since $N^{\dagger S} \in N \in \{1, 3^S, 4^S\}, M^{\dagger S} \stackrel{S}{\leq} N^{\dagger S}$. Then $M \stackrel{S}{\leq} N$.

(1) \implies (3): Similar. \square

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