Some results on approximation in neutrosophic normed space

Alaa Adnan Auad[1, *], Mohammed A. Hilal[2]

[1]Department of Mathematics, College of Education for pure science, University of Al-Anbar, Iraq

[2]Middle Technical University, Technical Institute of Baquba, Diyala, Iraq

Emails: alaa.adnan.auad@uoanbar.edu.iq; mohammed_azeez_hilal@mtu.edu.iq

Abstract

Neutrosophic normed linear spaces are the main significant notion in the study of classical functional analysis under a neutrosophic environment to handle indeterminate and inconsistent information. Where the neutrosophic norm function assigns to each vector in the linear space a neutrosophic number, which is a number with a truth, indeterminacy, and falsity component. The main aim of this work is to study and discuss the important properties of proximinality of specific sets and new results for a large class in neutrosophic normed space. Moreover, we show some results closely related proximainality of classes to the normed construction in the space. Also, we prove achieved for generalized sets in neutrosophic normed space, most marks on convexity and Cheby-shevity classes are considered.

Keywords: Neutrosophic set; proximinality; Convexity class: Cheby-shevity class; Neutrosophic normed space

1. Introduction

Algebraic branches, both geometric and analytical, are of great importance in mathematics by creating algebraic structures that can be used to solve everyday problems. Neutrosophy or neutrosophic put forward by Smarandache [1] as a new branch of philosophy and as an extension of some fuzziness models [2]. Neutrosophic has influenced deeply all the different scientific fields such as in topology field [3, 4], complex field [5, 6], algebraic field [7, 8], and a lot of fields [9, 10]. The study of neutrosophic normed linear spaces is an active area of research in the field of neutrosophic mathematics, providing a flexible and comprehensive framework for dealing with uncertainty and indeterminacy in linear spaces. Researchers have explored topics like linear independence, completeness, Cauchy sequences, and convergence in the context of neutrosophic normed linear spaces.

Using the concepts of metric space and neutrosophic set Kristi [11] proposed neutrosophic metric space. Bera and Mahapatra [12] defined neutrosophic norm and neutrosophic norm soft linear space. Abobala [13] defined for the first time both concepts of inner product in a neutrosophic vector space and neutrosophic vector space. Şahin et.al [14] have shown to be true that neutrosophic vector space is isomorphic. N-refined neutrosophic vector space proved by Smarandache et.al [15]. Al-Sharqi et al [16, 17] used neutrosophic to prodact some algebraic with uncertainty vector space. Al-Quran et al. [18, 19] proposed some application on neutrosophic vector space in real field.

On the other hand, in the field of functional analysis the study of proximinality of subclasses in spaces of functional analysis has been introduced by many authors [20, 21]. We presented the main results of Roshdi [22] and other works there where the problems are processed fully detailed in new space its normed space. Mazheri [23] discussed the concept of proximinality briefly in metric space. Some results recently have been obtained which are relevant of proximinality problem. Alaa et al. [24, 25] proposed some algebraic approximation of unbounded functions to introduce and investigate with best simultaneous approximation of unbounded functions. Indumathi and N.

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Prakash in 2017 [26] they proved that E-proximinality is unchanging by $c_\circ$-direct sum of complete normed space and given some example of E-proximinality, Jayanarayanan in 2019 [27] introduced properties finite co-dimensional strappingly ball proximinal subset of L^1-predual space and prove that they exactly the finite co-dimensional strappingly proximal subsets, Auad and Al-sharqi in [28] presented some results of co-proximal and co-Chebyshev in weighted space and obtained of new characterizations of best co-proximal of unbounded mappings in the same space.

In this work, some new results which are closely related by proximinality of classes to the nutrosophic normed spaces. Amongst other concepts that Mazheri result for a larger set of normed space. In addition, convexity nutrosophic normed spaces and Chebyshevity nutrosophic normed spaces are considered. Co-proximinal nutrosophic normed subspace of the space is compact set. In the second section, a number of new definitions and results about nutrosophic normed spaces were presented and highlighted, and in the third section, a number of theories and results related to Best approximation in nutrosophic normed space were presented.

2. Nutrosophic normed spaces

Definition 2.1: For $n_1, n_2, n_3, n_4 \in [0,1]$, then if the following operation $\diamond: [0,1] \times [0,1] \to [0,1]$ has been satisfied the following point:

i. $n_1 \diamond 1 = n_1$.
ii. If $n_1 \le n_3$ and $n_2 \le n_4$ then $n_1 \diamond n_2 \le n_3 \diamond n_4$.
iii. $\diamond$ is continuous.
iv. $\diamond$ is continuous and associative.

Then, we called the operation $\diamond$ is continuous TN.

Definition 2.2: For $n_1, n_2, n_3, n_4 \in [0,1]$, then if the following operation $\star: [0,1] \times [0,1] \to [0,1]$ has been satisfied the following point:

i. $n_1 \star 0 = n_1$.
ii. If $n_1 \le n_3$ and $n_2 \le n_4$ then $n_1 \star n_2 \le n_3 \star n_4$.
iii. $\star$ is continuous.
iv. $\star$ is continuous and associative.

Then, we called the operation $\diamond$ is continuous TCN.

Definition 2.3: Let $\diamond$ and $\star$ refer to continuous TN and continuous TCN and $\mathbb{F}$ refer to vector space (VS). The structure $\mathcal{N} = \{< w, T(w), I(w), F(w) >: w \in \mathbb{F}\}$ be neutrosophic normed space such that $\mathcal{N}: \mathbb{F} \times R^+ \to [0,1]$.Then the four terms $K = (\mathbb{F}, \mathcal{N}, \diamond, \star)$ called neutrosophic normed space if satisfied the following points:

Definition 2.4: Let $(\mathbb{F}, \mathcal{N}, \|.\|, ,\diamond,\star)$ be a neutrosophic normed space, for $n \in \mathcal{N}$; $\epsilon > 0$, we denoted of open ball by $\mathcal{O}_\epsilon(n)$ and given $\mathcal{O}_\epsilon(n) = \{m \in \mathcal{N};\ \|m - n\| < \epsilon\}$, closed ball by $\mathcal{C}_\epsilon(n)$ and given $\mathcal{C}_\epsilon(n) = \{m \in \mathcal{N};\ \|m - n\| \le \epsilon\}$ and sphere $\mathcal{S}_\epsilon(n)$ define by $\mathcal{S}_\epsilon(n) = \{m \in \mathcal{N};\ \|m - n\| = \epsilon\}$. We call $(\mathcal{N}, \|.\|)$ convex normed space if $\mathcal{C}_{\epsilon_1}(n_1) \cap \mathcal{C}_{\epsilon_2}(n_2) \neq \emptyset$ whenever $\|n_1 - n_2\| \le \epsilon_1 + \epsilon_2$.

We denoted the set of points of the line segment connecting n_1 and n_2 by $\mathcal{K}(n_1, n_2)$.

Definition 2.5: A normed space $(\mathcal{N}, \|.\|)$ is said to be $\mathcal{N}$-normed space if every two vectors n_1 and n_2 in $\mathcal{N}$ with $\|n_1 - n_2\| = \alpha$ and for every $\epsilon \in [0, \alpha]$, there is alone $m_\epsilon \in \mathcal{N}$ such that $\mathcal{O}_\epsilon(n_1) \cap \mathcal{C}_{\alpha-\epsilon}(n_2) = \{m_\epsilon\}$.

If $= \frac{1}{2}\alpha$, the vector m_ϵ is called midway between n_1 and n_2, and denoted by $m_\epsilon = h(n_1, n_2)$.

For n_1 and n_2 are vectors in $\mathcal{N}$, the curve connected n_1 to n_2 in $\mathcal{N}$ is the image under a injective continuous function $\mathcal{F}$ of $[u, v]$ into $\mathcal{N}$ such that $\mathcal{F}(n_1) = u$ and $\mathcal{F}(n_2) = v$. If $\mathcal{F}$ is a line connected n_1 to n_2 in $\mathcal{N}$, then the distance of $\mathcal{F}$ is given $\mathcal{L}(\mathcal{F}) = \lim_{\Delta_m \to 0} \sum_{k=1}^{n} \|\mathcal{F}(u_{k-1}) - \mathcal{F}(u_k)\|$,

where $\Delta_m = \underbrace{sup}_{1 \le k \le n} |u_k - u_{k-1}|, \{u = u_0, u_1, \dots, u_n = v\}$ is a segmentation of $[u, v] = [\mathcal{F}^{-1}(n_1), \mathcal{F}^{-1}(n_2)]$.

If $\|n_1 - n_2\| = \alpha$ and $\{m\} = \mathcal{O}_{(1-q)\alpha}(n_1) \cap \mathcal{C}_{q\alpha}(n_2)$, $0 \le q \le 1]$, then $m = q n_1 + (1 - q) n_2$.

Now, we give sufficient description of $\mathcal{N}$-normed space through the following two theorems.

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Theorem 2.6: Let $(\mathcal{N}, \|.\|)$ be an nutrosophic normed spaces and $n_1, n_2 \in \mathcal{N}$. Then $\mathcal{K}(n_1, n_2)$ smallest length of n_1 to n_2.

Theorem 2.7: Let $(\mathcal{N}, \|.\|)$ be a nutrosophic normed spaces. Then $(\mathcal{N}, \|.\|)$ is a nutrosophic normed spaces if and only if any two vectors $n_1, n_2 \in \mathcal{N}$ can connect them with a unique straight line.

The proofs of theorem 2.6 and theorem 2.7 are easy and the reader can see to the [15].

3. Best approximation in $\mathcal{N}$-normed space

Definition 3.1: Let $(\mathcal{N}, \|.\|)$ be a normed space and $\mathcal{S}$ a closed subset of $\mathcal{N}$, we can define the following set $\mathcal{P}(\mathcal{S}, n) = inf\{m \in \mathcal{S}; \|m - n\|\}$. Then $\mathcal{S}$ is called proximinal set in $\mathcal{N}$ if $\mathcal{P}(n, \mathcal{S})$ is non-empty and its Cheby-shev if $\mathcal{P}(n, \mathcal{S})$ includes just one point. Many research has been complete determine the proximinality of closed subset of metric space. It doesn't our interest to search on this problematic. The characterization of the norm function in conditionally of proximinality and Cheby-shevity of some subspace in $\mathcal{N}$ have been considered.

Theorem 3.2: If $(\mathcal{N}, \|.\|)$ nutrosophic normed spaces, then the next relationships are equivalents

i. $(\mathcal{N}, \|.\|)$ is $\mathcal{N}$- normed space.
ii. $\mathcal{C}_\epsilon(m)$ is Cheby-shev for all $m \in \mathcal{N}$, $\epsilon > 0$.
iii. $\mathcal{P}(\mathcal{C}, n_1) \cap \mathcal{P}(\mathcal{C}, n_2) = \emptyset$ for $n_1 \neq n_2$ and $\mathcal{C}$ subset in $\mathcal{N}$.

Proof: Now to prove $(i) \to (ii)$.

Let $\mathcal{C}_\epsilon(m)$ be any closed ball and $n_1 \in \mathcal{N} - \mathcal{C}_\epsilon(m)$ and $\|n_1 - m\| = \epsilon + \alpha$. Then $inf\|\mathcal{C}_\epsilon(m) - n_1\| = \alpha$ and from (i) we have $\mathcal{C}_\epsilon(m) \cap \mathcal{C}_\alpha(n_1) = \{n_2\}$ for some $n_2 \in \mathcal{N}$. Clearly $inf\|\mathcal{C}_\epsilon(m) - n_1\| = \{\alpha\}$ and $\mathcal{C}_\epsilon(m)$ is Chebyshev set.

$(i) \to (iii)$: If $\mathcal{P}(\mathcal{S}, n_1) \cap \mathcal{P}(\mathcal{S}, n_2) \neq \emptyset$ for some closed ball $\mathcal{S} = \mathcal{C}_\epsilon(m)$ and for some $n_1, n_2 \in \mathcal{S}$ then, there is $n_3 \in \mathcal{P}(\mathcal{S}, n_1) \cap \mathcal{P}(\mathcal{S}, n_2)$ such that

$inf\|\mathcal{C}_\epsilon(m) - n_1\| = \|n_3 - n_1\| = \|n_3 - n_2\|$. This is contradiction with (ii).

$(i) \to (iii)$: Assume that $(\mathcal{N}, \|.\|)$ not nutrosophic normed spaces with $n_1, n_2 \in \mathcal{N}$

and $\|n_2 - n_1\| = \alpha$. From convexity of $(\mathcal{N}, \|.\|)$ there is r such that

$\mathcal{C}_{(1-r)\alpha}(n_1) \cap \mathcal{C}_{r\alpha}(n_2) \neq \emptyset$. If there are $m_1, m_2 \in \mathcal{C}_{(1-r)\alpha}(n_1) \cap \mathcal{C}_{r\alpha}(n_2)$

with $m_1 \neq m_2$, then

$n_1 \in \mathcal{P}(\mathcal{C}_{r\alpha}(n_2), m_1) \cap \mathcal{P}(\mathcal{C}_{r\alpha}(n_2), m_2) \neq \emptyset$.

This contradiction with (iii).

Hence $(\mathcal{N}, \|.\|)$ nutrosophic normed spaces

Definition 3.3: A set $\mathcal{S}$ subset of $\mathcal{N}$ is called convex nutrosophic normed spaces if for each $n_1, n_2 \in \mathcal{S}$, then $\mathcal{C}_{(1-r)\alpha}(n_1) \cap \mathcal{C}_{r\alpha}(n_2) \subseteq \mathcal{S}$ for all $0 \leq r \leq 1$, where $\|n_1 - n_2\| = \alpha$.

A convex normed space $(\mathcal{N}, \|.\|)$ is called finely convex nutrosophic normed spaces if, for $n_1, n_2 \in \mathcal{C}_\epsilon(m)$ with $\alpha = \|n_1 - n_2\|$, then $\mathcal{C}_{(1-r)\alpha}(n_1) \cap \mathcal{C}_{r\alpha}(n_2) \subseteq \mathcal{C}_\epsilon(m)$ for all $0 \leq r \leq 1$.

Theorem 3.4: Let $(\mathcal{N}, \|.\|)$ be an nutrosophic normed spaces. The following are equivalent

i. Shut- in balls in $(\mathcal{N}, \|.\|)$ are convex nutrosophic normed spaces.
ii. If $\mathcal{S}$ is shut- in convex set in $(\mathcal{N}, \|.\|)$ and $n_1 \notin \mathcal{S}$, then $\mathcal{P}(\mathcal{S}, n)$ is convex.

Proof: (i)→(ii). Let $\mathcal{S}$ be a shut- in convex set in $(\mathcal{N}, \|.\|)$ and $n_1 \notin \mathcal{S}$. If $\mathcal{P}(\mathcal{S}, n_1)$ is empty set, then clearly its convex.

If $\mathcal{P}(\mathcal{S}, n_1)$ is a non-empty set, implies there are $m_1, m_2 \in \mathcal{P}(\mathcal{S}, n_1)$,

$\|m_1 - m_2\| = \alpha$ and $\mathcal{P}(\mathcal{S}, n_1) = \mathcal{C}_\epsilon(n_1)$. Since $\mathcal{S}$ is convex set, then all points of the line connecting m_1 and m_2 lie within $\mathcal{P}(\mathcal{S}, n_1)$.

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So, $\mathcal{P}(\mathcal{S}, n_1)$ convex.

(ii)→(i). Let $\mathcal{C}_\epsilon(m_1)$ be shut- in ball in $(\mathcal{N}, \|.\|)$ and $n_1, n_2 \in \mathcal{C}_\epsilon(m_1)$.

If $\{\mathcal{C}_{(1-r)\alpha}(n_1) \cap \mathcal{C}_{r\alpha}(n_2)\} \nsubseteq \mathcal{C}_\epsilon(m)$ where $\alpha = \|n_1 - n_2\|$ for some

$0 \leq r \leq 1$.

From theorem 3.4 vectors n_1 and n_2 can be connected only one curve of α. If $\mathcal{F}$ is again curve which connected between n_1 and n_2 also from theorem $\mathcal{F}$ is a convex in $\mathcal{N}$. From connectedness of $\mathcal{F}$ there are two points m_1 and m_2 such that $\{m_1, m_2\} \subseteq \mathcal{F} \cap \mathcal{S}_\epsilon(m)$. Then m_1 and m_2 belongs in $\mathcal{P}(\mathcal{F}, m)$, we obtain $\mathcal{P}(\mathcal{F}, m)$ not convex, this contradiction with (ii).

So, $\mathcal{C}_\epsilon(m_1)$ convex.

Theorem 3.5: Let $(\mathcal{N}, \|.\|)$ be an nutrosophic normed spaces and for each proximinal convex set is Cheby-shev set. Then $\mathcal{C}_\epsilon(m)$ is convex, for $m \in \mathcal{N}$ and $\epsilon > 0$.

Poof: Let n_1 , $n_2 \in \mathcal{C}_\epsilon(m)$, $\alpha = \|n_1 - n_2\|$ and for some $0 \leq r \leq 1$

$\{\mathcal{C}_{(1-r)\alpha}(n_1) \cap \mathcal{C}_{r\alpha}(n_2)\} \nsubseteq \mathcal{C}_\epsilon(m)$.

From theorem 2.3, there are two different vectors $m_1, m_2 \in \mathcal{S}_\epsilon(m)$ such that $\mathcal{F}$ is not consisted in $\mathcal{C}_\epsilon(m)$, we see that $\mathcal{F}$ is proximinal but $m_1, m_2 \in \mathcal{P}(\mathcal{F}, m)$ this contradiction with Chebyshev set of $\mathcal{F}$.

Since $\mathcal{F}$ is convex implies $\mathcal{C}_\epsilon(m)$ is convex.

Theorem 3.6: Let $(\mathcal{N}, \|.\|)$ be a finely convex normed space. Then $(\mathcal{N}, \|.\|)$ is nutrosophic normed spaces.

Proof: Let n_1 , $n_2 \in \mathcal{N}$, $\alpha = \|n_1 - n_2\|$ and from convexity of $(\mathcal{N}, \|.\|)$ we have for all $0 < r < 1$ then $\mathcal{C}_{(1-r)\alpha}(n_1) \cap \mathcal{C}_{r\alpha}(n_1) \neq \emptyset$.

If we can take two vectors m_1, m_2 belong to the set $\mathcal{C}_{(1-r)\alpha}(n_1) \cap \mathcal{C}_{r\alpha}(n_1)$. From the definition of finely convex of $(\mathcal{N}, \|.\|)$, we obtain

$\mathcal{C}_{(1-u)\beta}(m_1) \cap \mathcal{C}_{u\beta}(m_2) \subseteq \mathcal{C}_{(1-u)\beta}(n_1) \cap \mathcal{C}_{r\alpha}(n_1)$, for all $0 < u < 1$, where $\beta = \|m_1 - m_2\|$. This is available just in case $m_1 = m_2$.

Implies , $(\mathcal{N}, \|.\|)$ is a nutrosophic normed spaces.

Theorem 3.7: Let $(\mathcal{N}, \|.\|)$ be a finely convex nutrosophic normed spaces. Then proximinal convex set in $(\mathcal{N}, \|.\|)$ is Cheby-shev.

Proof: Let $\mathcal{S} \subseteq \mathcal{N}$ be a convex proximinal set. If we can take two vector $m \in \mathcal{N} \setminus \mathcal{S}$ such that $\mathcal{P}(\mathcal{S}, m)$ includes more than one item.

Let $= \|\mathcal{S} - m\|$,$\{m_1, m_2\} \subseteq \mathcal{P}(\mathcal{S}, m)$ and $\alpha = \|m_1 - m_2\|$.

Since, ,$\{m_1, m_2\} \subseteq \mathcal{C}_\epsilon(m)$ and from the finely convexity of $(\mathcal{N}, \|.\|)$ that

$\mathcal{C}_{(1-r)\alpha}(m_1) \cap \mathcal{C}_{r\alpha}(m_2) \subset \mathcal{O}_\epsilon(m)$.

The convexity of $\mathcal{S}$ implies $\mathcal{C}_{(1-r)\alpha}(m_1) \cap \mathcal{C}_{r\alpha}(m_2) \subset \mathcal{C}_\epsilon(m)$.

This is contradiction with $m_1 \neq m_2$.

So, $m_1 = m_2$ and $\mathcal{P}(\mathcal{S}, m)$ contains only one vector.

Implies $\mathcal{S}$ is Chebyshev.

Theorem 3.8: Let $(\mathcal{N}, \|.\|)$ be a nutrosophic normed spaces. Then the next relationships are equivalents:

i. $(\mathcal{N}, \|.\|)$ is finely convex nutrosophic normed spaces.
ii. Every proximinal convex set is Cheby-shev nutrosophic normed spaces.

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Proof: It be adequate to prove that if every proximinal convex subset of $\mathcal{N}$ is Cheby-shev then every shut- in ball is finely convex.

Let $\mathcal{C}_{\epsilon}(m)$ be any ball and it's not finely convex. Then there are x, y, z that are vectors of $\mathcal{S}_{\epsilon}(m)$ such that x belong to the set of points of the line segment connecting y and z and dented its (y , z). From the proof of theorem 3.4, then $\mathcal{K}(y , z)$ is proximinal implies its Cheby-shev set and thus by close interconnectedness of $\mathcal{K}(y , z)$ and norm function is continuous, there are vectors $n_1 , n_2 \in \mathcal{K}(y , z)$ such that x belong to the set of points of the line segment connecting n_1 and n_2 and dented its $\mathcal{K}(n_1 , n_2)$ with

$\|m - n_2\| = \|m - n_1\| = \eta < \|m - x\|$.

So, $\notin \mathcal{K}(m , \eta)$ implies $\mathcal{K}(m , \eta)$ is not convex , this is contradiction with theorem 3.4 . we obtain

In $(\mathcal{N}, \|.\|)$ be an nutrosophic normed space, every proximinal convex set is Cgeby-shev set if and only if $(\mathcal{N}, \|.\|)$ is finely convex.

Theorem 3.9:

If $\mathcal{S}$ is sub space of the nutrosophic normed space$(\mathcal{N}, \|.\|)$ and $0 \neq n \in \mathcal{N}$, then

(i) $\mathcal{S}$ is a co-proximinal subspace of $(\mathcal{N}, \|.\|) \Leftrightarrow (\mathcal{N}, \|.\|) = \mathcal{S} + \mathcal{H}_{\mathcal{S}}^{-1}(0)$
(ii) $\mathcal{S}$ is a co-chebyshev subspace of $(\mathcal{N}, \|.\|) \Leftrightarrow (\mathcal{N}, \|.\|) = \mathcal{S} \oplus \mathcal{H}_{\mathcal{S}}^{-1}(0)$
Proof: (i) Assume That $\mathcal{S}$ is co-proximinal , $n \in (\mathcal{N}, \|.\|)$ and $m \in \mathcal{H}_{\mathcal{S}}(n, m)$, then, $n - m \in \mathcal{H}_{\mathcal{S}}^{-1}(0)$.

So, $n = m + (n - m) \in \mathcal{S} + \mathcal{H}_{\mathcal{S}}^{-1}(0)$

Since n is random mapping in $(\mathcal{N}, \|.\|)$, Hence $(\mathcal{N}, \|.\|) = \mathcal{S} + \mathcal{H}_{\mathcal{S}}^{-1}(0)$.

Conversely . Now assume $n \in \mathcal{N}, \|.\|) = \mathcal{S} + \mathcal{H}_{\mathcal{S}}^{-1}(0)$, and to prove $\mathcal{S}$ is aproximinal so $n= m + \sigma, m \in \mathcal{S}$, $\sigma \in \mathcal{H}_{\mathcal{S}}^{-1}(0)$, and $0 \in \mathcal{H}_{\mathcal{S}}^{-1}(\sigma, m) = \mathcal{H}_{\mathcal{S}}^{-1}(n - m, p)$

Since: $\|(p - (n - m) - \sigma)\|$ is continuous mapping.

Thus , $\|(m - n) - \sigma\| \geq \|((m + p) - n) - \sigma\|$

Where $p + n \in \mathcal{S}$; hence $n \in \mathcal{H}_{\mathcal{S}}^{-1}(n, m)$, Therefor $\mathcal{S}$ is co-proximinal.

(ii)Suppose that $\mathcal{S}$ is co-chebyshev nutrosophic normed subspace, $n \in (\mathcal{N}, \|.\|)$,and

$n = p + q = r + s$

Where $p, r \in \mathcal{S}$ and $q, s \in \mathcal{H}_{\mathcal{S}}^{-1}(0)$

We show that $p = r$ and $q = s$

Since,
$n = p + q = r + s$.

Then
$n - p = q,\ n - r = s$.This implies that
$p, r \in \mathcal{H}_{\mathcal{S}}^{-1}(n, m)$, therefor $p = s$ Thus $(\mathcal{N}, \|.\|), = \mathcal{S} \oplus \mathcal{H}_{\mathcal{S}}^{-1}(0)$ conversly . Suppose that

$(\mathcal{N}, \|.\|), = \mathcal{S} \oplus \mathcal{H}_{\mathcal{S}}^{-1}(0)$ and $n \in (\mathcal{N}, \|.\|)$. There exist , $r \in \mathcal{H}_{\mathcal{S}}^{-1}(n, m)$. Then $n - p , n - r \in \mathcal{H}_{\mathcal{S}}^{-1}(0)$

And therefor, $n = p + q = r + s$

Where $= n - p$, $s = n - q$, it follows p, r and $q = s$.

Implies, $\mathcal{S}$ is co-chebsher nutrosophic normed subspace of $(\mathcal{N}, \|.\|)$.

Theorem 3.10:

If $\mathcal{S} \neq \emptyset)$ is subset of the nutrosophic normed space $(\mathcal{N}, \|.\|)$ and $0 \neq m \in (\mathcal{N}, \|.\|)$, then

(i) $\mathcal{H}_{\mathcal{S}+\mathcal{Z}}(n + p, m) = \mathcal{H}_{\mathcal{S}}(n, m) + p$ for every $n , p \in (\mathcal{N}, \|.\|)$,
(ii) $\mathcal{H}_{b\mathcal{S}}(bn, m) = b\mathcal{H}_{\mathcal{S}}(n, m)$ for every $n \in (\mathcal{N}, \|.\|)$ *and* $b \in \mathcal{R}/\{0\}$
(iii) $\mathcal{S}$ is co-proximinal(respectively co-chebysher) $\Leftrightarrow \mathcal{S} + p$ is co-proximinal(respectively co-chebsyher), for any $p \in (\mathcal{N}, \|.\|)$.

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(iv) $\mathcal{S}$ is co-proximinal (respectively co-chebysher) iff $\mathscr{b}\mathcal{S}$ is $|\mathscr{b}|$ co-proximinal (respectively $|\mathscr{b}|$ co-chebyshev) , $\forall\ \beta \in \mathscr{b} \in \mathcal{R}/\{0\}$.

Proof:- (i) For $m, p \in (\mathcal{N}, \|.\|)$, and $0 \neq p$,

$r \in \mathcal{H}_{\mathcal{S}+Z}(u+v, w)$ iff $\|((r - s + q) - m)\| \geq \|(n + p - (s + p)) - m\|$for all $(s + \mathcal{S}) \in \mathcal{S} + p \Leftrightarrow \|(r - p) - s) + m)\| \geq \| (n - s) - m) \|_{p,w}$ $\forall\, s \in \mathcal{S}$, $\Leftrightarrow r - p \in \mathcal{H}_{\mathcal{S}}(fn, m)$,

such that, $r \in \mathcal{H}_{\mathcal{S}}(n, m) + p$.

(ii) For each $n \in (\mathcal{N}, \|.\|)$, $\mathscr{b} \in \mathcal{R}/\{0\}$ and $m \in (\mathcal{N}, \|.\|)$, $\mathcal{H}_{\mathscr{b}\mathcal{S}}(\mathscr{b}n, m) = \mathscr{b}\mathcal{H}_{\mathcal{S}}(n, m) \Leftrightarrow$

$\|(r - \mathscr{b}s) - m)\| \geq \|(\mathscr{b}n - \mathscr{b}s) - km)\|$ for all $\ell \in \mathcal{A}\ iff$

$\left\|(\frac{1}{\mathscr{b}}r - s) - m\right\| \geq \|(n - s) - m\|$ for all $s \in \mathcal{S} \Leftrightarrow \frac{1}{\mathscr{b}}r \in \mathcal{H}_{\mathcal{S}}(n, m) \Leftrightarrow$

$r \in \mathcal{H}_{\mathcal{S}}(n, m)$, there fore ,$\mathcal{H}_{\mathscr{b}\mathcal{S}}(\mathscr{b}n, m) = \mathscr{b}\mathcal{H}_{\mathcal{S}}(n, m)$.

(iii) The of this part immediately consequence from (i) .

(iv The of this part immediately consequence from (ii) .

Theorem 3.11:

Let $\mathcal{S}$ be co-proximinal nutrosophic normed subspace of the space $(\mathcal{N}, \|.\|)$, and $0 \neq m \in (\mathcal{N}, \|.\|)$

Then

(i) If $\mathcal{H}^{-1}{}_{m}(0)$ is compact set , then $\mathcal{A}$ is quasi co-chebyshev .
(ii) If $\mathcal{H}^{-1}{}_{m}(0)$ is a closed , then $\mathcal{H}^{-1}{}_{m}(n, m)$ is closed for every $n \in (\mathcal{N}, \|.\|)$.

Proof :-

(i) suppose $n \in (\mathcal{N}, \|.\|)$ and $\{m_n\}$ is a sequence in$\mathcal{H}^{-1}{}_{m}(n, m)$, since $n - m_n \in \mathcal{H}^{-1}{}_{m}(0)$ and $\mathcal{H}^{-1}{}_{m}(0)$ is a compact set , there exists a sub sequence $\{n - m_{nk}\}$ that convergence to $n - m_o \in \mathcal{H}^{-1}{}_{m}(0)$. consequently , $\{m_n\}$ has a sub sequence $m_{nk} \to g_o \in \mathcal{H}^{-1}{}_{m}(n, m)$ and hence $\mathcal{S}$ is quasi co- chebyshev .

(ii) suppose $n \in (\mathcal{N}, \|.\|)$ and $\{m_n\}$ is a sequence in $\mathcal{H}^{-1}{}_{m}(n, m)$, since $n - m_n \in \mathcal{H}^{-1}{}_{m}(0)$ and $\mathcal{H}_{\mathcal{S}}^{-1}(0)$ is a closed set , there exists a sub sequence $\{n - m_{nk}\}$ that convergence to $n - m_o \in \mathcal{H}^{-1}{}_{m}(0)$.

Consequently , $\{m_n\}$ has a sub sequence $m_{nk} \to m_o \in \mathcal{H}^{-1}{}_{\mathcal{S}}(n, m)$ because $\mathcal{H}^{-1}{}_{\mathcal{S}}(0)$ is closed and hence $\mathcal{H}^{-1}{}_{\mathcal{S}}(n, m)$ is closed .

Theorem 3.12 :

Let $\mathcal{S}$ be a nutrosophic normed subspace of the space $(\mathcal{N}, \|.\|)$, if $\mathcal{S} \perp n - m_o$, then

$m_o \in \mathcal{H}^{-1}{}_{\mathcal{S}}(n, m)$, and $0 \neq m \in (\mathcal{N}, \|.\|)$.

Proof:- Assume $n \in (\mathcal{N}, \|.\|)$ & $\mathcal{S} \perp n - m_o$. then

$\|(m + \alpha(n - m_o) - p)\| \leq \|m - p\|\ \forall\, m \in \mathcal{S}$ & $\alpha \in \mathcal{R}, \alpha \neq 0$.

So, $\| (n - m_o + \alpha^{-1}m) - p) \|_{p,w} \leq \| (\alpha^{-1}m - p) \|_{p,w}$

Hence $\|n - m_1 - p\| \leq \|m_o - m_1 - p\|$.

Where $m_1 = m_o - \alpha^{-1}m$, Now

$\|(m_3 - m_o) - p)\| \geq \|(n - m_3) - p)\|\ \forall l\ m_3 \in \mathcal{S}$, & $m_o \in \mathcal{H}^{-1}{}_{\mathcal{S}}(n, m)$ &and $0 \neq m \in (\mathcal{N}, \|.\|)$.

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4. Conclusion

The proximinality, Cheby-shev and convexity have been studied in neutrosophic normed space. Through these paper, we conclude some new results which are closely related by proximinality sets, Cheby-shev nutrosophic normed sets and convexity concept in neutrosophic normed space also every proximainal set is Chebyshev set. Furthermore, finely convex normed space implies neutrosophic normed space and has been conclude there is equivalent relation between finely convex nutrosophic normed spaces with every proximinal convex set is Cheby-shev. Finally, in future studies, we will work to develop this mathematical structure by integrating and crushing it with other flexible algebraic concepts [29, 30] of a complex nature.

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