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On The Computational Properties of 3-Cyclic and 4-Cyclic Refined Matrices and the Diagonalization Algorithm

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Abstract:

This paper is concerned with studying the matrix computations of 3-cyclic refined neutrosophic matrices and 4-cyclic refined neutrosophic matrices with 3cyclic/4-cyclic real entries, where we introduce a novel method to compute eigenvalues and vectors of these matrix classes. Also, we provide a novel algorithm for diagonalization these matrices and to determine whether an n-cyclic refined matrix is diagonalizable or not for n=3, 4.

Keywords: 3-cyclic refined matrix; 4-cyclic refined matrix; Diagonalization problem; Matrix computation; Eigenvalue

1. Introduction

The concept of n-cyclic refined structures was proposed for the first time in [11], as a direct application of n-cyclic neutrosophic sets into algebraic structures. In [12-17], authors have studied many related algebraic structures such as groups, equations, and vector spaces. They have suggested many open research problems concerning these structures and their properties. In [6-8], the relations between these structures were studied by using algebraic mappings, where we find many interesting results, especially those that are related to the computation of units, and the group of unit's problem [1-5]. The 2-cyclic refined numbers have been used in cryptography, where we find generalized versions of RSA and El-Gamal algorithms [9-10].
This has motivated us to studying the matrix computations of 3-cyclic refined neutrosophic matrices and 4-cyclic refined neutrosophic matrices with 3cyclic/4-cyclic real entries, where we introduce a novel method to compute eigenvalues and vectors of these matrix classes. Also, we provide a novel algorithm for diagonalization these matrices and to determine whether an n-cyclic refined matrix is diagonalizable or not for n=3, 4.

2. Main Discussion

Definition:
The 3-cyclic refined real square matrix is defined as:

$$M = M_0 + \sum_{i=1}^{3} M_i I_i \, , M_i \in R_{n\times n}$$

The set of all 3-cyclic refined real square matrices is denoted by $M_{R_3(I)\times R_3(I)}$.

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Addition on $M_{R_3(I)\times R_3(I)}$ is defined as follows:

$(M_0 + \sum_{i=1}^{3} M_i\, I_i\,) + (N_0 + \sum_{i=1}^{3} N_i\, I_i\,) = (M_0 + N_0) + \sum_{i=1}^{3}(M_i + N_i)\, I_i$.

Multiplication is defined as follows:

$(M_0 + \sum_{i=1}^{3} M_i\, I_i\,) \times (N_0 + \sum_{i=1}^{3} N_i\, I_i\,) = M_0N_0 + (\sum_{i+j\equiv 1(mod\ 3)} M_iN_j)I_1 + (\sum_{i+j\equiv 2(mod\ 3)} M_iN_j)I_2 + (\sum_{i+j\equiv 3(mod\ 3)} M_iN_j)I_3$.

$(M_{R_3(I)\times R_3(I)}, +, \times)$ is a non-commutative ring.

Remark: [1]

$R_3(I) \cong \mathbb{R}\times\mathbb{R}\times\mathbb{C}$, with $g: R_3(I) \to R\times R\times C$ as:

$g: (a_0 + \sum_{i=1}^{3} a_i\, I_i\,) = (a_0, \sum_{i=0}^{4} a_i\,, a_0 + a_3 + a_1 e^{\frac{2\pi}{3}i} + a_2 e^{\frac{4\pi}{3}i})$.

Result:

By a similar discussion of the previous remark, there exists a ring isomorphism

$f: M_{R_3(I)\times R_3(I)} \to \mathbb{R}_{n\times n}\times\mathbb{R}_{n\times n}\times\mathbb{C}_{n\times n}$.

Where $f(M_0 + \sum_{i=1}^{3} M_i\, I_i\,) = (M_0, \sum_{i=0}^{3} M_i\,, M_0 + M_3 + M_1 e^{\frac{2\pi}{3}i} + M_2 e^{\frac{4\pi}{3}i})$.

For $M_0 + M_3 + M_1 e^{\frac{2\pi}{3}i} + M_2 e^{\frac{4\pi}{3}i} = M_1' + M_2' i\ \ ; M_1', M_2' \in R_{n\times n}$.

The inverse isomorphism is:

$f^{-1}: (\mathbb{R}_{n\times n}\times R_{n\times n}\times C_{n\times n}) \to M_{R_3(I)\times R_3(I)}$ such that:

$f^{-1}(\,M_0, M_1, M_2 + M_3 i) = M_0 + I_1\left[\frac{1}{3}(M_1 - M_2) + \frac{1}{\sqrt{3}}M_3\right] + I_2\left[\frac{1}{3}(M_1 - M_2) - \frac{1}{\sqrt{3}}M_3\right] + I_3\left[\frac{1}{3}M_1 - M_0 + \frac{2}{3}M_2\right]$.

Theorem:

Let $N = N_0 + \sum_{i=1}^{3} N_i\, I_i\ \in M_{R_3(I)\times R_3(I)}$, then:

1] N is invertible if and only if: $N_0, \sum_{i=0}^{3} N_i\,, N_1' + N_2' i$ are invertible matrices, where $N_1' + N_2' i = N_0 + N_3 + N_1 e^{\frac{2\pi}{3}i} + N_2 e^{\frac{4\pi}{3}i}$.

2] $N^{-1} = f^{-1}({N_0}^{-1}, (\sum_{i=0}^{3} N_i)^{-1}, (N_1' + N_2' i)^{-1})$.

3] $N^{s} = f^{-1}({N_0}^{s}, (\sum_{i=0}^{3} N_i)^{s}, (N_1' + N_2' i)^{s})\ \ ; s \in \mathbb{N}$.

Proof:

1] N is invertible in $M_{R_3(I)\times R_3(I)}$ if and only if $f(N)$ is invertible in $R_{n\times n}\times R_{n\times n}\times C_{n\times n}$, so that it is equivalent to: $N_0, \sum_{i=0}^{3} N_i\,, N_1' + N_2' i$ are invertible.

2] It holds directly from the isomorphism properties.

3] It holds directly from the isomorphism properties.

Example:

Consider the 3-cyclic refined 2 × 2-real matrix:

$$N = \begin{pmatrix} 1 + \left(-\frac{2}{3} + \frac{1}{\sqrt{3}}\right)I_1 + \left(-\frac{2}{3} - \frac{1}{\sqrt{3}}\right)I_2 + \frac{7}{3}I_3 & \frac{2}{3}I_1 - \frac{2}{3}I_2 - \frac{1}{3}I_3 \\ -2I_1 - 2I_2 + 5I_3 & 1 + \left(1 + \frac{1}{\sqrt{3}}\right)I_1 + \left(1 - \frac{1}{\sqrt{3}}\right)I_2 - 2I_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} +$$

$$I_1\begin{pmatrix} -\frac{2}{3} + \frac{1}{\sqrt{3}} & \frac{2}{3} \\ -2 & 1 + \frac{1}{\sqrt{3}} \end{pmatrix} + I_2\begin{pmatrix} -\frac{2}{3} - \frac{1}{\sqrt{3}} & \frac{2}{3} \\ -2 & 1 - \frac{1}{\sqrt{3}} \end{pmatrix} + \begin{pmatrix} \frac{7}{3} & -\frac{1}{3} \\ 5 & -2 \end{pmatrix} I_3 = N_0 +\ N_1I_1 + N_2I_2 + N_3I_3.$$

$$\sum_{i=0}^{3} N_i = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, N_1 e^{\frac{2\pi}{3}i} = \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)\begin{pmatrix} -\frac{2}{3} + \frac{1}{\sqrt{3}} & \frac{2}{3} \\ -2 & 1 + \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} - \frac{1}{2\sqrt{3}} + i\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right) & \frac{-1}{3} + \frac{1}{\sqrt{3}}i \\ 1 - \sqrt{3}i & \frac{-1}{2} - \frac{1}{2\sqrt{3}} + i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) \end{pmatrix},$$

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$$N_2 e^{\frac{4\pi}{3}i} = \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)\begin{pmatrix} -\frac{2}{3} - \frac{1}{\sqrt{3}} & \frac{2}{3} \\ -2 & 1 - \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{3} + \frac{1}{2\sqrt{3}}\right) + i\left(\frac{1}{\sqrt{3}} + \frac{1}{2}\right) & \frac{-1}{3} - \frac{1}{\sqrt{3}}i \\ 1 + \sqrt{3}i & \left(\frac{-1}{2} + \frac{1}{2\sqrt{3}}\right) + i\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}\right) \end{pmatrix},$$

$N_0 + N_3 = \begin{pmatrix} \frac{10}{3} & -\frac{1}{3} \\ 5 & -1 \end{pmatrix}, N_0 + N_3 + N_1 e^{\frac{2\pi}{3}i} + N_2 e^{\frac{4\pi}{3}i} = \begin{pmatrix} \frac{10}{3} & -\frac{1}{3} \\ 5 & -1 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} + i & -\frac{2}{3} \\ 2 & -1 + i \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 7 & -2 \end{pmatrix} + i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$ $N_1' + N_2' i.$

$$M_0 = N_0^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_1 = \left(\sum_{i=0}^{3} N_i\right)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, M_2 + M_3 i = (N_1' + N_2' i)^{-1} = \begin{pmatrix} \frac{3}{4} + \frac{1}{4}i & \frac{-1}{4} - \frac{1}{4}i \\ \frac{7}{4} + \frac{7}{4}i & \frac{-3}{4} - \frac{5}{4}i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{-1}{4} \\ \frac{7}{4} & \frac{-3}{4} \end{pmatrix} + i\begin{pmatrix} \frac{1}{4} & \frac{-1}{4} \\ \frac{7}{4} & -\frac{5}{4} \end{pmatrix}$$

So that, $N^{-1} = f^{-1}(M_0, M_1, M_2 + M_3 i) = M_0 + I_1\left[\frac{1}{3}(M_1 - M_2) + \frac{1}{\sqrt{3}}M_3\right] + I_2\left[\frac{1}{3}(M_1 - M_2) - \frac{1}{\sqrt{3}}M_3\right] + I_3\left[\frac{1}{3}M_1 - M_0 + \frac{2}{3}M_2\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + I_1\left[\frac{1}{3}\begin{pmatrix} \frac{1}{4} & \frac{-3}{4} \\ \frac{-11}{4} & \frac{11}{4} \end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix} \frac{1}{4} & \frac{-1}{4} \\ \frac{7}{4} & \frac{-5}{4} \end{pmatrix}\right] + I_2\left[\frac{1}{3}\begin{pmatrix} \frac{1}{4} & \frac{-3}{4} \\ \frac{-11}{4} & \frac{11}{4} \end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix} \frac{1}{4} & \frac{-1}{4} \\ \frac{7}{4} & \frac{-5}{4} \end{pmatrix}\right] + I_3\left[\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} \frac{3}{4} & \frac{-1}{4} \\ \frac{7}{4} & \frac{-3}{4} \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + + I_1\begin{pmatrix} \frac{1}{12} + \frac{1}{4\sqrt{3}} & \frac{-1}{4} - \frac{1}{4\sqrt{3}} \\ \frac{-11}{12} + \frac{7}{4\sqrt{3}} & \frac{11}{12} - \frac{5}{4\sqrt{3}} \end{pmatrix} + I_2\begin{pmatrix} \frac{1}{12} - \frac{1}{4\sqrt{3}} & \frac{-1}{4} + \frac{1}{4\sqrt{3}} \\ \frac{-11}{12} - \frac{7}{4\sqrt{3}} & \frac{11}{12} + \frac{5}{4\sqrt{3}} \end{pmatrix} + I_3\begin{pmatrix} -\frac{1}{6} & -\frac{1}{2} \\ \frac{5}{6} & -\frac{5}{6} \end{pmatrix}.$

Definition:

Let $N = N_0 + \sum_{i=1}^{3} N_i I_i$ be a 3-cyclic refined real square matrix, then N is called diagonalizable if there exists a 3-cyclic refined diagonal matrix L and a 3-cyclic refined invertible matrix A such that $N = A\,L\,A^{-1}$.

Algorithm for the diagonalization:

Let $N = N_0 + \sum_{i=1}^{3} N_i I_i \in M_{R_3(I) \times R_3(I)}$, then:

$f(N) = (N_0, \sum_{i=0}^{3} N_i, N_1' + iN_2')$. The matrix N is diagonalizable if and only if: $N_0, \sum_{i=0}^{3} N_i, N_1' + iN_2'$ are diagonalizable.

Assume that $N_0, \sum_{i=0}^{3} N_i, N_1' + iN_2'$ are diagonalizable, then, $N_0 = A_0\,L_0\,A_0^{-1}, \sum_{i=0}^{3} N_i = A_1\,L_1\,A_1^{-1}, (N_1' + iN_2') = (A_2 + A_3 i)(L_2 + L_3 i)(A_2 + A_3 i)^{-1}$, where:

$$\begin{cases} A_0, A_1, A_2 + A_3 i & \text{are invertible matrices} \\ L_0, L_1, L_2 + L_3 i & \text{are diagonal matrices} \end{cases}$$

So that:

$f(N) = \left(A_0\,L_0\,A_0^{-1}, A_1\,L_1\,A_1^{-1}, (A_2 + A_3 i)(L_2 + L_3 i)(A_2 + A_3 i)^{-1}\right) =$
$(A_0, A_1, A_2 + A_3 i) \times (L_0, L_1, L_2 + L_3 i) \times A_0^{-1}, A_1^{-1}, (A_2 + A_3 i)^{-1}).$

Thus

$$N = f^{-1}(A_0, A_1, A_2 + A_3 i) f^{-1}(L_0, L_1, L_2 + L_3 i) f^{-1}(A_0^{-1}, A_1^{-1}, (A_2 + A_3 i)^{-1}) = A\,L\,A^{-1}.$$

Example:

Take the 3-cyclic 2 × 2-real refined matrix:

$$N = \begin{pmatrix} 3 + \frac{2}{3}I_1 + \frac{2}{3}I_2 - \frac{10}{3}I_3 & \frac{2}{3}I_1 + \frac{2}{3}I_2 - \frac{1}{3}I_3 \\ 2 - 2I_3 & 1 + \frac{1}{3}I_1 + \frac{1}{3}I_2 + \frac{1}{3}I_3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} + + I_1\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} + I_2\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} + I_3\begin{pmatrix} -\frac{10}{3} & -\frac{1}{3} \\ -2 & \frac{1}{3} \end{pmatrix} =$$

$N_0 + N_1 I_1 + N_2 I_2 + N_3 I_3$.

We have:

$$N_0 = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} = A_0\, L_0\, {A_0}^{-1},$$

$$A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, L_0 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, {A_0}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$\sum_{i=0}^{3} N_i = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = A_1\, L_1\, {A_1}^{-1}, A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

$${A_1}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$N_0 + N_3 + N_1 e^{\frac{2\pi}{3} i} + N_2 e^{\frac{4\pi}{3} i} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = (A_2 + A_3 i)(L_2 + L_3 i)(A_2 + A_3 i)^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}^{-1};$$

$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, L_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, {A_2}^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}, A_3 = L_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

now let's compute.

$$f^{-1}(A_0, A_1, A_2 + A_3 i) = \begin{pmatrix} 1 & +I_3 \\ 1 - I_3 & 1 + I_1 + I_2 - \frac{4}{3} I_3 \end{pmatrix} = A,$$

$$f^{-1}(L_0, L_1, L_2 + L_3 i) = \begin{pmatrix} 3 + \frac{2}{3} I_1 + \frac{2}{3} I_2 - \frac{10}{3} I_3 & 0 \\ 0 & 1 + \frac{1}{3} I_1 + \frac{1}{3} I_2 + \frac{1}{3} I_3 \end{pmatrix} = L,$$

Hence $N = A\, L\, A^{-1}$.

Definition:

Let $N = N_0 + \sum_{i=1}^{3} N_i\, I_i$ be a 3-cyclic refined square real matrix, then $m = m_0 + \sum_{i=1}^{3} m_i\, I_i \in R_3(I)$ is called 3-cyclic refined Eigen value if and only if:

$$\begin{cases} N \cdot X = m \cdot X \\ X = x_0 + \displaystyle\sum_{i=1}^{3} x_i\, I_i \ \in V_3(I) \text{ is called } 3-\text{cyclic refined Eigen vector.} \end{cases}$$

Remark that $\{x_i\,;\ 0 \le i \le 3\}$ are vector in $\mathbb{R}^n$.

Theorem:

Let $N = N_0 + \sum_{i=1}^{3} N_i\, I_i \in M_{R_3(I) \times R_3(I)}$, and $: R_3^n(I) \to R^n \times R^n \times C^n$ with

$h: (x_0 + \sum_{i=1}^{n} x_i\, I_i) = (x_0, \sum_{i=0}^{3} x_i\,, x_0 + x_3 + x_1 e^{\frac{2\pi}{3} i} + x_2 e^{\frac{4\pi}{3} i})$ and

$g: R_3(I) \to R \times R \times C$ with $g(m_0 + \sum_{i=1}^{3} m_i\, I_i) = \left(m_0, \sum_{i=0}^{3} m_i\,, m_0 + m_3 + m_1 e^{\frac{2\pi}{3} i} + m_2 e^{\frac{4\pi}{3} i}\right)$

are the semi-module isomorphism and the ring isomorphism defined in [ ], we have:

$m = m_0 + \sum_{i=1}^{3} m_i\, I_i \in R_3(I)$ is an Eigen value of N with $X = x_0 + \sum_{i=1}^{3} x_i\, I_i$ as an Eigen vector if and only if $g(m)$ is an Eigen value of $f(N)$ with $h(X)$ as an Eigen vector.

Proof:

Assume that $N \cdot X = m \cdot X$, this is equivalent to:

$h(NX) = h(m \cdot X) \Longrightarrow f(N) h(X) = g(m) \cdot h(X)$, there for:

$$\begin{cases} N_0 x_0 = m_0 x_0, \left(\displaystyle\sum_{i=0}^{3} N_i\right)\left(\displaystyle\sum_{i=0}^{3} x_i\right) = (\displaystyle\sum_{i=0}^{3} m_i)(\displaystyle\sum_{i=0}^{3} x_i) \\ (N_0 + N_3 + N_1 e^{\frac{2\pi}{3} i} + N_2 e^{\frac{4\pi}{3} i})(x_0 + x_3 + x_1 e^{\frac{2\pi}{3} i} + x_2 e^{\frac{4\pi}{3} i}) \\ = (m_0 + m_3 + m_1 e^{\frac{2\pi}{3} i} + m_2 e^{\frac{4\pi}{3} i})(x_0 + x_3 + x_1 e^{\frac{2\pi}{3} i} + x_2 e^{\frac{4\pi}{3} i}) \end{cases}$$

This implies the proof.

Example:

Consider the 3-cyclic refined 2×2-real matrix:

$$N = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} + +I_1 \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} + I_2 \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} + I_3 \begin{pmatrix} -\frac{10}{3} & -\frac{1}{3} \\ -2 & \frac{1}{3} \end{pmatrix}.$$

The Eigen values of $N_0 = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$ are $\{3,1\}$,

with $\{(1,1),(0,1)\}$ are the Eigen vectors.

The Eigen values of $\sum_{i=0}^{3} N_i = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ are $\{1,2\}$,

with $\{(1,0),(1,1)\}$ as the Eigen vectors.

The Eigen values of $N_0 + N_3 + N_1 e^{\frac{2\pi}{3}i} + N_2 e^{\frac{4\pi}{3}i} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ are $\{-1,1\}$,

with $\{(1,0),(1,-2)\}$ as the Eigen vectors.

The Eigen values of $f(N)$ are:

$E_1 = (3,1,-1), E_2 = (3,1,1)$,

$E_3 = (3,2,-1), E_4 = (3,2,1)$,

$E_5 = (1,1,-1), E_6 = (1,1,1)$,

$E_7 = (1,2,-1), E_8 = (1,2,1)$.

The corresponding Eigen vectors are:

$V_1 = \big((1,1)(1,0)(1,0)\big), V_2 = \big((1,1)(1,0)(1,-2)\big)$,

$V_3 = \big((1,1)(1,1)(1,0)\big), V_4 = \big((1,1)(1,1)(1,-2)\big)$,

$V_5 = \big((0,1)(1,0)(1,0)\big), V_6 = \big((0,1)(1,0)(1,-2)\big)$,

$V_7 = \big((0,1)(1,1)(1,0)\big), V_8 = \big((0,1)(1,1)(1,-2)\big)$,

This implies that the Eigen values of N are:

$g^{-1}(E_1), g^{-1}(E_2), g^{-1}(E_3), g^{-1}(E_4), g^{-1}(E_5), g^{-1}(E_6), g^{-1}(E_7), g^{-1}(E_8)$.

The Eigen vectors of N are:

$h^{-1}(V_1), h^{-1}(V_2), h^{-1}(V_3), h^{-1}(V_4), h^{-1}(V_5), h^{-1}(V_6), h^{-1}(V_7), h^{-1}(V_8)$.

4-cyclic refined matrices:

Definition:

The 4-cyclic refined real square matrix is defined as follows:

$$M = M_0 + \sum_{i=1}^{4} M_i I_i, M_i \in R_{n\times n}$$

The set of all 4-cyclic real square matrices is denoted by: $M_{R_4(I)\times R_4(I)}$.

Addition on $M_{R_4(I)\times R_4(I)}$ is defined as:

$(M_0 + \sum_{i=1}^{4} M_i I_i) + (N_0 + \sum_{i=1}^{4} N_i I_i) = (M_0 + N_0) + \sum_{i=1}^{4} (M_i + N_i) I_i$,

Multiplication is defined as:

$(M_0 + \sum_{i=1}^{4} M_i I_i) \times (N_0 + \sum_{i=1}^{4} N_i I_i) = M_0 N_0 + (\sum_{i+j\equiv 1 (mod\ 4)} M_i N_j) I_1 + (\sum_{i+j\equiv 2 (mod\ 4)} M_i N_j) I_2 + (\sum_{i+j\equiv 3 (mod\ 4)} M_i N_j) I_3 + (\sum_{i+j\equiv 4 (mod\ 4)} M_i N_j) I_4$.

It is clear that $(M_{R_4(I)\times R_4(I)}, +, \times)$ is a non-commutative ring.

Remark: [ ]

$R_4(I) \cong R \times R \times R \times \mathbb{C}$, with $g: R_4(I) \to R \times R \times R \times C$ as:

$g: (m_0 + \sum_{i=1}^{4} m_i I_i) = (m_0, \sum_{i=0}^{4} m_i, m_0 + m_2 + m_4 - m_1 - m_3, m_0 - m_2 + m_4 + i(m_1 - m_3))$.

Result:

from the previous remark, we can see that: $f: M_{R_4(I)\times R_4(I)} \to R_{n\times n} \times R_{n\times n} \times R_{n\times n} \times C_{n\times n}$ such that:

$f(M_0 + \sum_{i=1}^{4} M_i I_i) = (M_0, \sum_{i=0}^{4} M_i, M_0 + M_2 + M_4 - M_1 - M_3, M_0 - M_2 + M_4 + i(M_1 - M_3))$is a ring isomorphism.

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The inverse isomorphism is:

$f^{-1}: (R_{n\times n} \times R_{n\times n} \times R_{n\times n} \times C_{n\times n}) \to M_{R_4(I)\times R_4(I)}$ such that:

$f^{-1}(M_0, M_1, M_2 + M_3 i) = M_0 + I_1\left[\frac{1}{4}(M_1 - M_2) + \frac{1}{2}M_4\right] + I_2\left[\frac{1}{4}(M_1 + M_2) - \frac{1}{2}M_3\right] + I_3\left[\frac{1}{4}(M_1 - M_2) - \frac{1}{2}M_3\right] + I_4\left[\frac{1}{4}(M_1 + M_2) + \frac{1}{2}M_3 - M_0\right].$

Theorem:

Let $N = N_0 + \sum_{i=1}^{4} N_i I_i \in M_{R_4(I)\times R_4(I)}$, then:

1] N is invertible if and only if:

$N_0, \sum_{i=0}^{4} N_i, N_0 + N_2 + N_4 - N_1 - N_3, N_0 - N_2 + N_4 + i(N_1 - N_3)$ are invertible matrices.

2] $N^{-1} = f^{-1}\left(N_0{}^{-1}, (\sum_{i=0}^{4} N_i)^{-1}, (N_0 + N_2 + N_4 - N_1 - N_3)^{-1}, \left(N_0 - N_2 + N_4 + i(N_1 - N_3)\right)^{-1}\right).$

3] $N^s = f^{-1}\left(N_0{}^s, (\sum_{i=0}^{4} N_i)^s, (N_0 + N_2 + N_4 - N_1 - N_3)^s, \left(N_0 - N_2 + N_4 + i(N_1 - N_3)\right)^s\right).$

The proof similar to the 3-cyclic refined case.

Example:

Consider the 4-cyclic refined 2 × 2-real matrix:

$$N = \begin{pmatrix} 1 - I_1 + I_2 + I_3 + 2I_4 & 1 + I_2 \\ 2 - I_2 - 3I_4 & 1 - I_1 + I_3 + I_4 \end{pmatrix} = N_0 + N_1 I_1 + N_2 I_2 + N_3 I_3 + N_4 I_4 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + I_1\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + I_2\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + I_3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + I_4\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}.$$

$$N_0 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, N_0{}^{-1} = -1\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix},$$

$$\sum_{i=0}^{4} N_i = \begin{pmatrix} 4 & 2 \\ -2 & 2 \end{pmatrix}, (\sum_{i=0}^{4} N_i)^{-1} = \frac{1}{12}\begin{pmatrix} 2 & -2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{-1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix},$$

$$N_0 + N_2 + N_4 - N_1 - N_3 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix},$$

$$(N_0 + N_2 + N_4 - N_1 - N_3)^{-1} = \frac{1}{12}\begin{pmatrix} 2 & -2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{-1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix},$$

$$N_0 - N_2 + N_4 + i(N_1 - N_3) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} + i\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + i\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 - 2i & 0 \\ 0 & 2 - 2i \end{pmatrix},$$

$$\left(N_0 - N_2 + N_4 + i(N_1 - N_3)\right)^{-1} = \frac{1}{-8i}\begin{pmatrix} 2 - 2i & 0 \\ 0 & 2 - 2i \end{pmatrix} = \frac{1}{8}i\begin{pmatrix} 2 - 2i & 0 \\ 0 & 2 - 2i \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{1}{4}i & 0 \\ 0 & \frac{1}{4} + \frac{1}{4}i \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} + i\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix},$$

So that, $N^{-1} = f^{-1}(f(N)^{-1})$.

Definition:

Let $N = N_0 + \sum_{i=1}^{4} N_i I_i \in M_{R_4(I)\times R_4(I)}$, then N is called diagonalizable if there exists a 4-cyclic refined diagonal matrix L and a 4-cyclic refined invertible matrix A such that $N = A\,L\,A^{-1}$.

Algorithm for the diagonalization:

Let $N = N_0 + \sum_{i=1}^{4} N_i I_i \in M_{R_4(I)\times R_4(I)}$, then:

$f(N) = (N'_0, N'_1, N'_2, N'_3 + iN'_4)$;

$$N'_0 = N_0, N'_1 = \sum_{i=1}^{4} N_i,$$

$N'_2 = N_0 + N_2 + N_4 - N_1 - N_3, N'_3 = N_0 - N_2 + N_4, N'_4 = N_1 - N_3.$

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Assume that $N'_0, N'_1, N'_2, (N'_3 + iN'_4)$ are diagonalizable matrices (which is the necessary and sufficient condition for the diagonalization of N).

We have:

$$N'_0 = A_0\, L_0\, {A_0}^{-1}, N'_1 = A_1\, L_1\, {A_1}^{-1}, N'_2 = A_2\, L_2\, {A_2}^{-1}, (N'_3 + iN'_4) = (A_3 + A_4 i)(L_3 + L_4 i)(A_3 + A_4 i)^{-1}, \text{ where:}$$

$$\begin{cases} A_0, A_1, A_2, A_3 + A_4 i \text{ are invertible matrices} \\ L_0, L_1, L_2, L_3 + L_4 i \text{ are diagonal matrices} \end{cases}$$

There for:

$f(N) = \left(A_0\, L_0\, {A_0}^{-1}, A_1\, L_1\, {A_1}^{-1}, A_2\, L_2\, {A_2}^{-1}, (A_3 + A_4 i)(L_3 + L_4 i)(A_3 + A_4 i)^{-1}\right)$, thus:

$N = f^{-1}(A_0, A_1, A_2, A_3 + A_4 i) f^{-1}(L_0, L_1, L_2, L_3 + L_4 i) f^{-1}\left({A_0}^{-1}, {A_1}^{-1}, {A_2}^{-1}, (A_3 + A_4 i)^{-1}\right) = A\, L\, A^{-1}.$

Example:

Consider the following 4-cyclic 2 × 2-real refined matrix:

$$N = \begin{pmatrix} 1 + \frac{9}{4}I_1 - \frac{1}{4}I_2 + \frac{1}{4}I_3 - \frac{1}{4}I_4 & \frac{-1}{4}I_1 + \frac{1}{4}I_2 - \frac{1}{4}I_3 + \frac{1}{4}I_4 \\ 1 + \frac{1}{2}I_1 - \frac{1}{2}I_3 - I_4 & 2 + \frac{1}{4}I_1 + \frac{5}{4}I_2 - \frac{3}{4}I_3 - \frac{7}{4}I_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + +I_1 \begin{pmatrix} \frac{9}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} + I_2 \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{5}{4} \end{pmatrix} + I_3 \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{-3}{4} \end{pmatrix} + I_4 \begin{pmatrix} \frac{-1}{4} & \frac{1}{4} \\ -1 & \frac{-7}{4} \end{pmatrix} = N_0 + N_1 I_1 + N_2 I_2 + N_3 I_3 + N_4 I_4.$$

We have:

$$N'_0 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, N'_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, N'_2 = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}, N'_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, N'_4 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, N'_3 + iN'_4 = \begin{pmatrix} 1 + 2i & 0 \\ 1 & -1 + i \end{pmatrix},$$

The Eigen values of N'_0 are $\{e_0 = 1, e'_0 = 2\}$,

The Eigen values of N'_0 are $\{V_0 = (1, -1), V'_0 = (0,1)\}$,

The Eigen values of N'_1 are $\{e_1 = 3, e'_1 = 1\}$,

The Eigen values of N'_1 are $\{V_1 = (1,0), V'_1 = (0,1)\}$,

The Eigen values of N'_2 are $\{e_2 = -2, e'_2 = 2\}$,

The Eigen values of N'_2 are $\{V_2 = (1,0), V'_2 = (1,4)\}$,

The Eigen values of $N'_3 + iN'_4$ are $\{e_3 + ie_4 = 1 + 2i, e'_3 + ie'_4 = -1 + i\}$,

The Eigen values of $N'_3 + iN'_4$ are $\{V_3 + iV_4 = (2,1) + i(1,0), V'_3 + iV'_4 = (0,1)\}$,

This implies that the Eigen values of *N* are:

$E_1 = g^{-1}(1,3,-2,1 + 2i), E_2 = g^{-1}(1,3,-2,-1 + i)$,

$E_3 = g^{-1}(1,1,-2,1 + 2i), E_4 = g^{-1}(1,1,-2,-1 + i)$,

$E_5 = g^{-1}(1,3,2,1 + 2i), E_6 = g^{-1}(1,3,2,-1 + i)$,

$E_7 = g^{-1}(1,1,2,1 + 2i), E_8 = g^{-1}(1,1,2,-1 + i)$,

$E_9 = g^{-1}(2,3,-2,1 + 2i), E_{10} = g^{-1}(2,3,-2,-1 + i)$,

$E_{11} = g^{-1}(2,3,2,1 + 2i), E_{12} = g^{-1}(2,3,2,-1 + i)$,

$E_{13} = g^{-1}(2,1,-2,1 + 2i), E_{14} = g^{-1}(2,1,-2,-1 + i)$,

$E_{15} = g^{-1}(2,1,2,1 + 2i), E_{16} = g^{-1}(2,1,2,-1 + i)$.

The Eigen vectors of *N* are:

$w_1 = h^{-1}\left((1,-1),(1,0),(1,0),(2,1) + i(1,0)\right), w_2 = h^{-1}\left((1,-1),(1,0),(1,0),(0,1) + i(0,0)\right)$,

$w_3 = h^{-1}\left((1,-1),(1,0),(1,4),(2,1) + i(1,0)\right), w_4 = h^{-1}\left((1,-1),(1,0),(1,4),(0,1) + i(0,0)\right)$,

$w_5 = h^{-1}\left((1,-1),(0,1),(1,0),(2,1) + i(1,0)\right), w_6 = h^{-1}\left((1,-1),(0,1),(1,0),(0,1) + i(0,0)\right)$,

$w_7 = h^{-1}\left((1,-1),(0,1),(1,4),(2,1) + i(1,0)\right), w_8 = h^{-1}\left((1,-1),(0,1),(1,4),(0,1) + i(0,0)\right)$,

$w_9 = h^{-1}\left((0,1),(1,0),(1,0),(2,1) + i(1,0)\right), w_{10} = h^{-1}\left((0,1),(1,0),(1,0),(0,1) + i(0,0)\right)$,

$w_{11} = h^{-1}\left((0,1),(1,0),(1,4),(2,1) + i(1,0)\right), w_{12} = h^{-1}\left((0,1),(1,0),(1,4),(0,1) + i(0,0)\right)$,

$w_{13} = h^{-1}\left((0,1),(0,1),(1,0),(2,1) + i(1,0)\right), w_{14} = h^{-1}\left((0,1),(0,1),(1,0),(0,1) + i(0,0)\right)$,

$w_{15} = h^{-1}\left((0,1),(0,1),(1,4),(2,1) + i(1,0)\right), w_{16} = h^{-1}\left((0,1),(0,1),(1,4),(0,1) + i(0,0)\right)$,

On the other hand, we have:

$N_0' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = A_0\, L_0\, {A_0}^{-1},$

$N_1' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A_1\, L_1\, {A_1}^{-1},$

$N_2' = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & \frac{-1}{4} \\ 0 & \frac{1}{4} \end{pmatrix} = A_2\, L_2\, {A_2}^{-1},$

$N_3' + iN_4' = \begin{pmatrix} 2+i & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1+2i & 0 \\ 1 & -1+i \end{pmatrix}\begin{pmatrix} \frac{2}{5}-\frac{1}{5}i & 0 \\ \frac{-2}{5}+\frac{1}{5}i & 1 \end{pmatrix} = (A_3 + A_4 i)(L_3 + L_4 i)(A_3 + A_4 i)^{-1},$

Thus $N = A\, L\, A^{-1}$; where:
$A = f^{-1}(A_0, A_1, A_2, A_3 + A_4 i), L = f^{-1}(L_0, L_1, L_2, L_3 + L_4 i), A^{-1} = f^{-1}\left({A_0}^{-1}, {A_1}^{-1}, {A_2}^{-1}, (A_3 + A_4 i)^{-1}\right).$

3. Conclusion

In this paper, we studied the matrix computations of 3-cyclic refined neutrosophic matrices and 4-cyclic refined neutrosophic matrices with 3cyclic/4-cyclic real entries, where we introduced a novel method to compute eigenvalues and vectors of these matrix classes. Also, we provided a novel algorithm for diagonalization these matrices and to determine whether an n-cyclic refined matrix is diagonalizable or not for n=3, 4.

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