Counterpart of Marshall-Olkin bivariate copula with negative dependence and its neutrosophic application in meteorology

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Abstract

Copulas are useful tools for modeling and describing different relationships between continuous random variables that have revived new interest through computational developments and extensive data analysis. This article contributes to the subject by generalizing the bivariate copula introduced recently in[8] and based on the concept of the counter-monotonic shock method. The proposed copula has the feature of covering the full range of negative dependence induced by two dependence parameters, which is not so common in the specialized literature. We examine the main characteristics of this copula. In particular, the absolutely continuous and singular copula components are derived. Analytical expressions of important concordance measures, such as Spearman's rho and Kendall's tau, are established, along with expressions of the product moments. A real neutrosophic data set, based on the daily quality of air in the New York Metropolitan Area, is used to illustrate the applicability of the proposed copula, with quite convincing results.

Keywords: bivariate copula; counter-monotonic; negative dependence; singularity; dependence measures; statistical modeling; neutrosophic theory

1 Introduction

Modeling the dependence structure between random variables (RVs) is of significant importance in statistical analysis. Several concepts of dependence have been proposed in the literature to capture and quantify such associations. Among them, the concept of copulas remains one of the most useful. It can be described as an innovative tool for modeling and describing different relationships among continuous RVs and, hence, providing more flexibility in building multivariate stochastic models. In the last few years, copulas have gained a lot of popularity in many applied fields, like environmental science, reliability theory, hydrology, finance, and actuarial science. More information can be found in,[21],[25],[26] and[30] for more details. From a mathematical viewpoint, a copula is a multivariate distribution function whose margins are uniformly distributed on the unit interval. In this direction, Sklar's theorem plays a crucial role in constructing copulas from a given multivariate distribution, which allows researchers to establish copula families for modeling both positive and negative dependence. Contemporary theoretical advancements on copula families can be found in,[4],[5],[8],[14],[18],[23] and.[24]

In order to understand the motivation of this paper, some fundamental copula facts need to be recalled. According to Sklar's Theorem,[27] the bivariate Marshall-Olkin (MO) copula, originally presented in,[19] is obtained from the bivariate exponential distribution in order to study complex systems where the two components are not independent and experience fatal shocks to one or both components. It is worth mentioning that the MO copula

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describes only positive dependence. In fact, most existing copulas in the literature possess some limitations in modeling negative dependence, and this is a problem for the deep analysis presenting such characteristics.

The goal of this paper is to offer a bivariate MO copula counterpart for modeling negative dependence. This can be achieved by using the counter-monotonic shock method introduced in,[11] combined with the bivariate exponential distribution with negative dependence expounded recently in.[1] More precisely, the resulting copula can be seen as a generalization of the one established in[8] to allow high flexibility in the modeling of phenomena based on real data. The proposed copula has two dependence parameters, η and γ, and correlation coefficients in the $(-1, 0)$ range, such as Kendall's tau, Spearman rho and Blomqvist's beta. It is worth noting that the results presented in[8] can be obtained in our case by setting $\eta = \gamma$. Hence, the importance and significance of the new family of copulas lie in the ability to allow more parameters of dependence in the analysis with the possibility of modeling more events in real-life situations and providing a more precise representation of the underlying dependence structure. Indeed, it is known that copulas with multiple dependence parameters can model a wide range of dependence patterns, including nonlinear, asymmetric, and tail-dependent relationships that cannot be adequately captured by a single dependence parameter copula. These copulas, with multiple dependence parameters, offer greater flexibility in capturing the complexities of real-world data, which often exhibit nontrivial dependence structures, as in financial data and environmental studies, where nonlinear and asymmetric dependencies can be encountered, for instance. To sum up, the new model fully covers the negative dependence and does not impose any constraints on the correlation structure. Motivated by these important characteristics, we provide a full theoretical and practical study of the proposed copula.

We proceed as follows: we first recall the general concept of copulas and present some well-known definitions. In Section 3, the proposed two-parameter bivariate copulas are described, followed by some useful remarks. We end this section by presenting an algorithm that provides simulated data from the proposed copula. In Section 4, we study its main properties. In particular, we demonstrate that it admits both absolutely continuous and singular components. Furthermore, we establish explicit expressions of concordance measures, namely, Spearman's rho, Kendall's tau and Blomqvist's beta. The expression of the product moments will then be stated. Estimation of the dependence parameters through the method of moments, a simulation study, and a real-life neutrosophic data analysis are presented in Section 5. Concluding comments and directions for further research are presented in Section 6, and the proofs of some propositions appear in the Appendix.

2 Preliminaries

In this section, we briefly introduce the concept of copulas for modeling the dependence structure between two continuous RVs. We examine some fundamental definitions regarding the pointwise partial ordering of copulas as well as several typical copula features.

We first start by stating Sklar's Theorem[27] which gives a link between the joint distribution, marginal distributions, and copula.

Theorem 2.1. *(Sklar's Theorem) Let X and Y be continuous RVs with cumulative distribution functions F_1 and F_2, respectively, and joint distribution H. Then, there exists a unique copula C such that*

$$H(x, y) = C(F_1(x), F_2(y)), \quad (x, y) \in \mathbb{R}^2.$$

The copula C is simply the distribution corresponding to the pair of RVs (U, V) such that $U = F_1(X)$ and $V = F_2(Y)$ are uniformly distributed over $[0, 1]$. That is, for all $(u, v) \in [0, 1]^2$,

$$C(u, v) = H(F_1^{(-1)}(u), F_2^{(-1)}(v)),$$

where $F_1^{(-1)}$ and $F_2^{(-1)}$ are pseudo-inverse functions of F_1 and F_2 respectively; when the latter are strictly increasing, $F_1^{(-1)}$ and $F_2^{(-1)}$ equal the usual inverses F_1^{-1} and F_2^{-1}, respectively, (see Nelson.[20])

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Theorem 2.2. *For any copula C, one has*

$$W(u,v) \leq C(u,v) \leq M(u,v), \quad (u,v) \in [0,1]^2,$$

where $W(u,v) = \max(u+v-1,0)$ and $M(u,v) = \min(u,v)$ are both copulas, called the lower and upper Fréchet-Hoeffding bounds, respectively.

It is worth noting that W represents the perfect negative dependence, whereas M represents the perfect positive dependence. We should also remember the independence copula, which is defined as $\Pi(u,v) = uv$.

The survival copula is also an efficient tool for modeling the dependence structure between two continuous RVs. Following Sklar's Theorem,[27] the survival copula can be defined as

$$\hat{C}(u,v) = \bar{H}(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)), \quad (u,v) \in [0,1]^2,$$

where $\bar{H}$ denotes the joint survival function, and $\bar{F}_1 = 1 - F_1$ and $\bar{F}_2 = 1 - F_2$ are the marginal survival functions.

We end these preliminaries by recalling some principal definitions about the pointwise partial ordering of copulas that we will encounter later.

Definition 2.3. If C_1 and C_2 are copulas, we say that C_1 is smaller than C_2 with respect to the concordance ordering if $C_1(u,v) \leq C_2(u,v)$, for all $(u,v) \in [0,1]^2$. In this case, it is written as $C_1 \prec C_2$.

In the sequel, we will use the following pointwise order in $\mathbb{R}^2$. For any real numbers x_1, x_2, y_1 and y_2, we say that $(x_1,y_1) \prec (x_2,y_2)$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

We also denote a family of bivariate two-parameter copulas by $\{C_{\alpha,\beta}, (\alpha,\beta) \in [0,1]^2\}$, where (α,β) represents the vector of dependency parameters associated to the copula $C_{\alpha,\beta}$.

Definition 2.4. A copula family $\{C_{\alpha,\beta}, (\alpha,\beta) \in [0,1]^2\}$ is positively ordered if $C_{\alpha_1,\beta_1} \prec C_{\alpha_2,\beta_2}$ whenever $(\alpha_1,\beta_1) \prec (\alpha_2,\beta_2)$; and negatively ordered if $C_{\alpha_1,\beta_1} \succ C_{\alpha_2,\beta_2}$ whenever $(\alpha_1,\beta_1) \prec (\alpha_2,\beta_2)$.

3 Proposed copulas

The main goal of this section is to introduce a bivariate copula, or family of bivariate copulas, in regard to the involved parameters, with two dependence parameters to allow for more flexible modeling. To this end, we adopt the concept of the counter-monotonic shock model used by Bentoumi et al.[1] This procedure is widely applied and was first described by Genest et al.[11] We follow the notations and setting of,[8] with an emphasis on the two new dependence parameters.

The notion of counter-monotonicity can be seen in relation to the lower Fréchet bound, as explained in the following definition.

Definition 3.1. Let X and Y be continuous RVs with cumulative distribution functions F_1 and F_2, respectively. The random vector (X,Y) is said to be counter-monotonic if there exists a unit uniform RV U such that $(X,Y) \stackrel{d}{=} (F_1^{-1}(U), F_2^{-1}(1-U))$. Particularly, the joint distribution function of (X,Y) is precisely the lower Fréchet-Hoeffding bound.

The proposed new family of bivariate exponential distributions with negative dependence in[1] relies on the notion of counter-monotonic shocks, and it is described by one dependence parameter, $\theta \in (0,1)$. A more general model with two dependence parameters, to allow for more flexible modeling, can be similarly constructed as in,[1] based on the definition below.

Definition 3.2. Let (X_1,X_2) and (Y_1,Y_2) be independent pairs of RVs such that X_1, X_2, Y_1 and Y_2 are exponentially distributed with parameters $\lambda_1(1-\eta)$, $\lambda_2(1-\gamma)$, $\lambda_1\eta$ and $\lambda_2\gamma$, respectively, where $\lambda_1 > 0$, $\lambda_2 > 0$ and $(\eta,\gamma) \in (0,1)^2$. The cumulative distribution functions of Y_1 and Y_2 are denoted by G_1 and G_2, respectively. Assume that

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1. Y_1 and Y_2 are counter-monotonic, that is, $Y_1 = G_1^{-1}(U)$ and $Y_2 = G_2^{-1}(1-U)$, where U is uniformly distributed over $[0,1]$.

2. X_1, X_2 and U are independent.

Then, the distribution of the pair of RVs (X, Y), defined by

$$X = \min[X_1, G_1^{-1}(U)] \quad \text{and} \quad Y = \min[X_2, G_2^{-1}(1-U)], \tag{1}$$

is called the counter-monotonic shock bivariate exponential distribution.

In what follows, we denote by $\text{BED}^-(\eta,\gamma,\Lambda)$ the set of all pairs of RVs (X, Y) defined by Equation (1), where $\Lambda = (\lambda_1, \lambda_2)$. When $\eta = \gamma$, it is reduced to the family of copulas in.[8] According to the stochastic representation in Equation (1), the $\text{BED}^-(\eta,\gamma,\Lambda)$ family only describes the negative dependence. Also, by construction of the model, the RVs X and Y are exponentially distributed with parameters λ_1 and λ_2, respectively. Therefore, the $\text{BED}^-(\eta,\gamma,\Lambda)$ family can be thought of as a family of bivariate exponential pairs of RVs with given marginal distributions.

Note that, the RVs X and Y, given in Equation (1), can be expressed as

$$X = \min\left[-\frac{\ln(U_1)}{\lambda_1(1-\eta)}, -\frac{\ln(U)}{\lambda_1\eta}\right] \quad \text{and} \quad Y = \min\left[-\frac{\ln(U_2)}{\lambda_2(1-\gamma)}, -\frac{\ln(1-U)}{\lambda_2\gamma}\right], \tag{2}$$

where U_1, U_2 and U are independent RVs uniformly distributed over $[0,1]$.

Following that, we look at some key characteristics of the $\text{BED}^-(\eta,\gamma,\Lambda)$ family.

Proposition 3.3. *For every $(\eta,\gamma) \in (0,1)^2$, the survival copula of the pair of RVs $(X,Y) \in BED^-(\eta,\gamma,\Lambda)$ is given by*

$$C_{\eta,\gamma}(u,v) = u^{1-\eta}v^{1-\gamma}W(u^\eta, v^\gamma), \quad (u,v) \in [0,1]^2, \tag{3}$$

where $W(u,v) = \max(u+v-1, 0)$ is the lower Fréchet-Hoeffding bound.

Proof. Assume $(X,Y) \in \text{BED}^-(\eta,\gamma,\Lambda)$ and denote the survival functions of X and Y as $\bar{F}_1$ and $\bar{F}_2$, respectively. It is widely known that the survival copula $C_{\eta,\gamma}$ corresponding to (X,Y) is the same distribution of the uniform pair of RVs $(V_1, V_2) = (\bar{F}_1(X), \bar{F}_2(Y))$. Making use of Equation (2), straightforward calculations lead to

$$V_1 = e^{-\lambda_1 X} = \max\left[U_1^{1/(1-\eta)}, U^{1/\eta}\right] \quad \text{and} \quad V_2 = e^{-\lambda_2 Y} = \max\left[U_2^{1/(1-\gamma)}, (1-U)^{1/\gamma}\right]. \tag{4}$$

Since U_1, U_2 and U are independent and uniformly distributed over $[0,1]$, then, for all $(u,v) \in [0,1]^2$,

$$\begin{aligned} C_{\eta,\gamma}(u,v) &= \text{P}\left(V_1 \le u, V_2 \le v\right) = \text{P}\left(U_1 \le u^{1-\eta}, U_2 \le v^{1-\gamma}, 1 - v^\gamma \le U \le u^\eta\right) \\ &= u^{1-\eta}v^{1-\gamma}W(u^\eta, v^\gamma). \end{aligned}$$

This ends the proof. □

Remark 3.4.

(a) For any cumulative distribution functions F_1 and F_2, the following function is a valid cumulative distribution function:

$$H(x,y) = C_{\eta,\gamma}\left(F_1(x), F_2(y)\right) = F_1(x)^{1-\eta}F_2(y)^{1-\gamma}\max\left(F_1(x)^\eta + F_2(y)^\gamma - 1, 0\right).$$

For motivated choices for F_1 and F_2 in a context of lifetime modeling, see.[29]

(b) The copula family $\{C_{\eta,\gamma}, (\eta,\gamma) \in (0,1)^2\}$ only describes the negative dependence.

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(c) The copula $C_{\eta,\gamma}$ is diagonally symmetric since $C_{\eta,\gamma}(u,v) = C_{\eta,\gamma}(v,u)$, for all $(u,v) \in [0,1]^2$.

(d) The independence copula Π and the Fréchet-Hoeffding lower bound copula W are limiting cases of $C_{\eta,\gamma}$ when (η,γ) goes to $(0,0)$ and $(1,1)$, respectively.

(e) The range of the dependence parameter vector (η,γ) can be extended to $[0,1]^2$, that is for all $(u,v) \in [0,1]^2$,

$$C_{\eta,\gamma}(u,v) = \begin{cases} u^{1-\eta}v^{1-\gamma}W(u^\eta, v^\gamma) & (\eta,\gamma) \in (0,1)^2 \\ \Pi(u,v) & \eta = 0 \text{ or } \gamma = 0 \\ v^{1-\gamma}W(u, v^\gamma) & (\eta,\gamma) \in \{1\} \times (0,1) \\ u^{1-\eta}W(u^\eta, v) & (\eta,\gamma) \in (0,1) \times \{1\} \\ W(u,v) & \eta = \gamma = 1. \end{cases}$$

Recall that the survival MO copula[20] is expressed in terms of the Fréchet-Hoeffding upper bound copula as follows:

$$\tilde{C}_{\eta,\gamma}(u,v) = u^{1-\eta}v^{1-\gamma}M(u^\eta, v^\gamma), \tag{5}$$

where $(\eta,\gamma) \in [0,1]^2$. This family is also known as the generalized Cuadras-Augé family.

From Proposition 3, one observes that the copula $C_{\eta,\gamma}$ has a similar expression to the MO copula given in the preceding equation, but involving the Fréchet-Hoeffding lower bound instead of the Fréchet-Hoeffding upper bound. Additionally, it can be viewed as a product of two families of copulas $C_1(u^{1-\eta}, v^{1-\gamma})$ and $C_2(u^\eta, v^\gamma)$, where, in our context, $C_1 = \Pi$ and $C_2 = W$. The construction principle for copulas having the form of a product of copulas has been introduced by Khoudraji[15] for the bivariate case and generalized by Liebscher[17] for the multivariate case. In this regard, the proposed copula $C_{\eta,\gamma}$ can be easily deduced from [6, Equation (1)], [15, Proposition 4.1] or from [17, Theorem 2.1]. Furthermore, the presented copula is easy to simulate using the stochastic representation given in Equation (4). In fact, the following algorithm provides simulated data from the copula $C_{\eta,\gamma}$:

1. Generate independent values, u_1, u_2 and u_3, from the uniform distribution over $[0,1]$;
2. Consider the maximum values: $u = \max\left[u_1^{1/(1-\eta)}, u_3^{1/\eta}\right]$ and $v = \max\left[u_2^{1/(1-\gamma)}, (1-u_3)^{1/\gamma}\right]$;
3. Take into account (u,v).

Figure 1 shows scatterplots of simulated data generated from the copula $C_{\eta,\gamma}$ for different values of (η,γ), each using 150 pairs of points. These scatterplots depict the behavior of simulated data as a function of the dependence parameter vector (η,γ). One can observe that as η and γ increase, the association between the two components becomes negatively strong.

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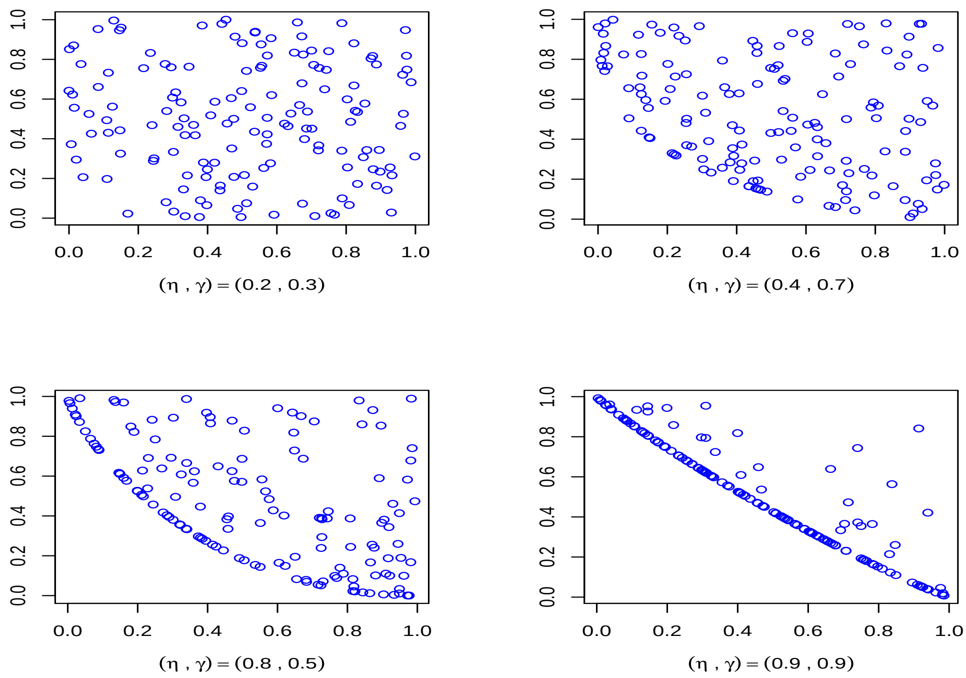

Figure 1: Examples of scatterplots of simulated data generated from the copula $C_{\eta,\gamma}$ for various values of (η, γ).

For visual representations of the proposed copulas, Figure 2 displays the shapes of $C_{\eta,\gamma}$ based on different values of (η, γ).

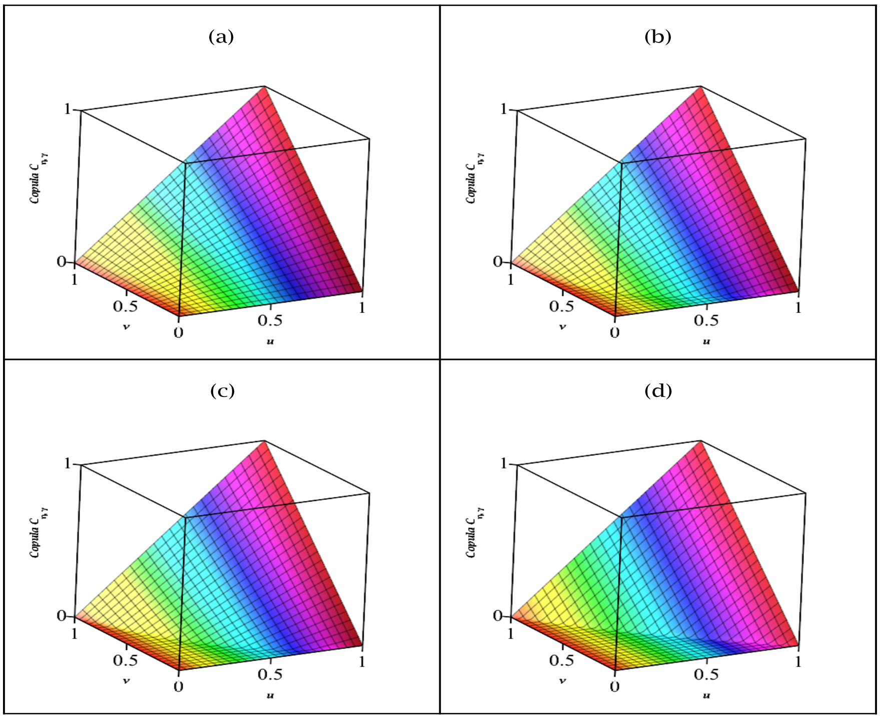

Figure 2: Copula $C_{\eta,\gamma}$ for (a) $(\eta, \gamma) = (0.2, 0.3)$, (b) $(\eta, \gamma) = (0.4, 0.7)$, (c) $(\eta, \gamma) = (0.8, 0.5)$ and (d) $(\eta, \gamma) = (0.9, 0.9)$.

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4 Key properties

The purpose of this section is to investigate the key properties of the copula $C_{\eta,\gamma}$. To begin, we derive its singular and absolutely continuous parts, then provide its corresponding density function. Next, we establish explicit expressions of widely-known concordance measures of $C_{\eta,\gamma}$, such as Spearman's rho, Kendall's tau and Blomqvist's beta. We conclude this section by looking at the mixed moments of the copula family, $\{C_{\eta,\gamma}, (\eta,\gamma) \in (0,1)^2\}$.

4.1 Singularity

The proposed copula $C_{\eta,\gamma}$, like the MO copula, is neither absolutely continuous nor singular. However, $C_{\eta,\gamma}$ has both the absolutely continuous and singular components. These two parts involve the next incomplete beta function:

$$B(x,a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt \quad a,b,x \in [0,1].$$

Note that, the incomplete beta function satisfies the following useful property:

$$B(x,a+1,b) + B(x,a,b+1) = B(x,a,b). \tag{6}$$

Furthermore, the beta function, which has the relation $B(1,a,b) = B(a,b)$, is a special case of the incomplete beta function. Furthermore, the following properties hold:

$$B(a+1,b) = B(a,b)\frac{a}{a+b} \quad \text{and} \quad B(a,b+1) = B(a,b)\frac{b}{a+b}. \tag{7}$$

The next proposition derives the singular and absolutely continuous parts of the copula $C_{\eta,\gamma}$.

Proposition 4.1. *The singular and absolutely continuous components of $C_{\eta,\gamma}$ are given, respectively, by*

$$\mathcal{S}_{\eta,\gamma}(u,v) = \begin{cases} B\left(u^\eta, \frac{1}{\eta}, \frac{1}{\gamma}\right) - B\left(1-v^\gamma, \frac{1}{\eta}, \frac{1}{\gamma}\right) & \text{if } u^\eta + v^\gamma - 1 > 0, \\ 0 & \text{if } u^\eta + v^\gamma - 1 \le 0, \end{cases}$$

and

$$\mathcal{A}_{\eta,\gamma}(u,v) = C_{\eta,\gamma}(u,v) - \mathcal{S}_{\eta,\gamma}(u,v).$$

In addition, the $C_{\eta,\gamma}$-measure of the singular component, that is $P\{U^\eta + V^\gamma - 1 = 0\}$, where (U,V) is distributed as a copula $C_{\eta,\gamma}$, is given by

$$\mathcal{S}_{\eta,\gamma}(1,1) = B\left(\frac{1}{\eta}, \frac{1}{\gamma}\right).$$

The proof of Proposition 4.1 appears in the Appendix.

4.2 Density function

In what follows, we derive the density function of the copula $C_{\eta,\gamma}$. First, we set

$$\mathcal{K}_{\eta,\gamma} = \{(u,v) \in [0,1]^2 : u^\eta + v^\gamma > 1\},$$

and

$$\mathcal{K}^*_{\eta,\gamma} = \{(u,v) \in [0,1]^2 : u^\eta + v^\gamma = 1\}.$$

Let $\mathbb{I}_A$ denote the indicator function of the set A.

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Proposition 4.2. *The density function $c_{\eta,\gamma}$ of the copula $C_{\eta,\gamma}$ can be expressed as*

$$c_{\eta,\gamma}(u,v) = c_1(u,v)\mathbb{I}_{\{(u,v)\in\mathcal{K}_{\eta,\gamma}\}} + c_0(u)\mathbb{I}_{\{(u,v)\in\mathcal{K}^*_{\eta,\gamma}\}},$$

where

$$c_1(u,v) = (1-\eta)u^{-\eta} + (1-\gamma)v^{-\gamma} - (1-\eta)(1-\gamma)u^{-\eta}v^{-\gamma},$$

and

$$c_0(u) = \eta\,(1-u^{\eta})^{1/\gamma-1}.$$

Proof. On the one hand, the function c_1 is the absolutely continuous component of the density function $c_{\eta,\gamma}$. It is then obtained from the continuous part $\mathcal{A}_{\eta,\gamma}$ of $C_{\eta,\gamma}$ described in Proposition 4.1. More specifically, for all $(u,v)\in\mathcal{K}_{\eta,\gamma}$, one has

$$c_1(u,v) = \partial^2_{u,v}\mathcal{A}_{\eta,\gamma} = (1-\eta)u^{-\eta} + (1-\gamma)v^{-\gamma} - (1-\eta)(1-\gamma)u^{-\eta}v^{-\gamma}.$$

On the other hand, the function c_0 represents the positive mass of probability distributed over the curve $\mathcal{K}^*_{\eta,\gamma}$. An explicit form of the component c_0 can be derived by adopting the approach developed in.[22] Moreover, c_0 is the singular part of the density $c_{\eta,\gamma}$ with respect to the measure ν defined by

$$\nu(B_o) = \lambda\left\{u\in[0,1] : \left(u,(1-u^{\eta})^{1/\gamma}\right)\in B_o\cap\mathcal{K}^*_{\eta,\gamma}\right\},$$

where B_o denotes a Borel set in $[0,1]^2$ and λ is the Lebesgue measure in $[0,1]$. In addition, this measure can be also seen as a product measure for all Borel sets A_1 and B_1 in $[0,1]$, defined by

$$\nu(A_1\times B_1) = \int_{A_1}\mathbb{I}_{\left\{(1-u^{\eta})^{1/\gamma}\in B_1\right\}}d\lambda(u).$$

Standard calculations show for $u^{\eta}+v^{\gamma}>1$,

$$\int_0^u\int_0^v \eta\,(1-x^{\eta})^{1/\gamma-1}\,d\nu(x,y) = B\left(u^{\eta},\frac{1}{\eta},\frac{1}{\gamma}\right) - B\left(1-v^{\gamma},\frac{1}{\eta},\frac{1}{\gamma}\right) = \mathcal{S}_{\eta,\gamma}(u,v).$$

This demonstrates that $c_0(x) = \eta\,(1-x^{\eta})^{1/\gamma-1}$ is the singular part of the density $c_{\eta,\gamma}$. It can be also shown that

$$\int_0^1\int_0^1 c_{\eta,\gamma}(u,v)du\,dv = \int_0^1\int_0^1 c_1(u,v)du\,dv + \int_0^1\int_0^1 c_0(u)d\nu(x,y) = 1,$$

The proof of the Proposition 4.2 is completed. □

Figure 3 illustrates the density function of the copula $C_{\eta,\gamma}$, for different values of (η,γ).

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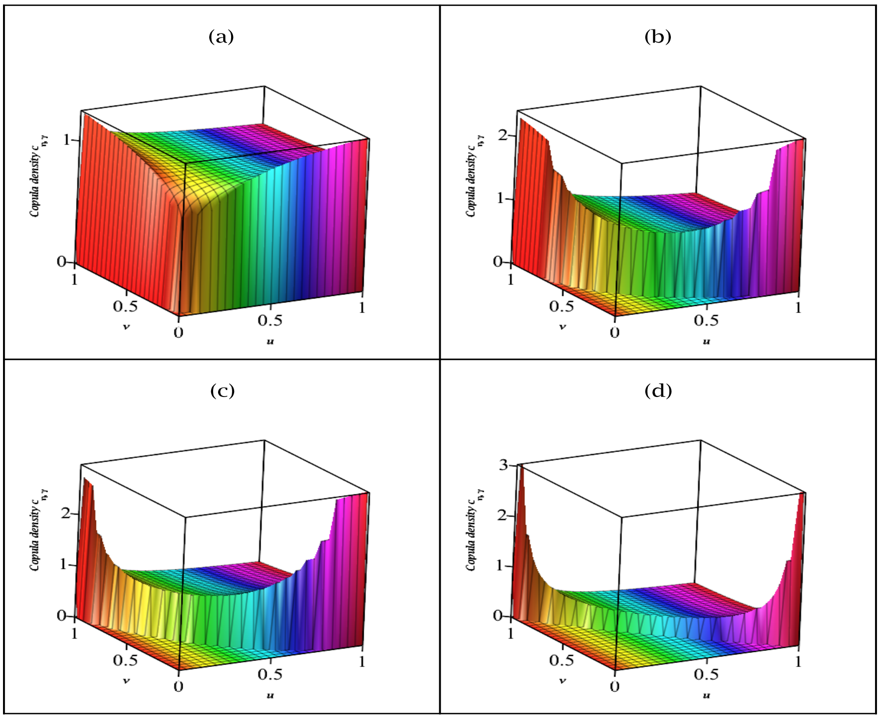

Figure 3: Copula density $c_{\eta,\gamma}$ for (a) $(\eta, \gamma) = (0.2, 0.3)$, (b) $(\eta, \gamma) = (0.4, 0.7)$, (c) $(\eta, \gamma) = (0.8, 0.5)$ and (d) $(\eta, \gamma) = (0.9, 0.9)$.

4.3 Concordance measures

First, we show that the copula family $\{C_{\eta,\gamma}, (\eta, \gamma) \in (0, 1)^2\}$ is negatively ordered with respect to concordance order throughout the next proposition.

Proposition 4.3. *The copula family $\{C_{\eta,\gamma}, (\eta, \gamma) \in (0, 1)^2\}$ is negatively ordered, i.e.,*

$$(\eta_1, \gamma_1) \prec (\eta_2, \gamma_2) \implies C_{\eta_2,\gamma_2} \prec C_{\eta_1,\gamma_1}, \quad (\eta_1, \gamma_1), (\eta_2, \gamma_2) \in (0, 1)^2.$$

Proof. Note that this is trivial in the case where $u = 0$ or $v = 0$. Otherwise, the copula $C_{\eta,\gamma}$ can be rewritten as follows:

$$C_{\eta,\gamma}(u, v) = uv \max\left[1 - (1 - u^{-\eta})(1 - v^{-\gamma}), 0\right], \quad (u, v) \in (0, 1)^2.$$

For fixed $(u, v) \in (0, 1)^2$, $\eta \to 1 - (1 - u^{-\eta})(1 - v^{-\gamma})$ and $\gamma \to 1 - (1 - u^{-\eta})(1 - v^{-\gamma})$ are decreasing functions in η and γ, respectively. Furthermore $u \to \max(u, 0)$ is an increasing function and the result can be deduced in a routine manner. □

As a consequence of the preceding proposition, the family $\{C_{\eta,\gamma}, (\eta, \gamma) \in (0, 1)^2\}$ is negatively quadrant dependent since for all $(u, v) \in [0, 1]^2$, $C_{\eta,\gamma}(u, v) \le uv$, or, equivalently, $C_{\eta,\gamma} \prec \Pi$.

Now, we are going to establish two well-known concordance measures of the copula $C_{\eta,\gamma}$, namely Spearman's rho and Kendall's tau. In this sense, we derive explicit expressions of these measures in terms of the beta function.

Proposition 4.4. *The Spearman's rho, Kendall's tau as well as the Blomqvist's beta of the copula $C_{\eta,\gamma}$ are given, respectively, by*

$$\rho_{\eta,\gamma} = -\frac{3}{(2 - \eta)(2 - \gamma)} \left[\eta\gamma - 4B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right)\right], \tag{8}$$

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$$\tau_{\eta,\gamma} = -\frac{2\eta\gamma}{(2-\eta)(2-\gamma)}\left[1-\left(\frac{2}{\eta}+\frac{2}{\gamma}-1\right)B\left(\frac{2}{\eta},\frac{2}{\gamma}\right)\right], \tag{9}$$

and

$$\beta_{\eta,\gamma} = 2^{\eta}+2^{\gamma}-2^{\eta+\gamma}-1. \tag{10}$$

The proof of Proposition 4.4 appears in the Appendix.

It is worth mentioning here that the expressions of $\rho_{\eta,\gamma}$ and $\beta_{\eta,\gamma}$ coincide exactly with the lower bounds expressed in Dolati et al. [6, Proposition 7]. Regarding Kendall's tau, a small error was found in the preceding article and was rectified.
Notice that the concordance measures Spearman's rho and Kendall's tau corresponding to the copula family $\{C_{\eta,\gamma},(\eta,\gamma)\in(0,1)^2\}$, given in the latter proposition, are linearly linked by means of the following expression:

$$\tau_{\eta,\gamma} = A_{\eta,\gamma}+B_{\eta,\gamma}\rho_{\eta,\gamma}, \tag{11}$$

where

$$A_{\eta,\gamma} = -\frac{\eta\gamma}{2} \text{ and } B_{\eta,\gamma} = \frac{2\eta+2\gamma-\eta\gamma}{6}.$$

Remark 4.5. From Propositions 4.3 and 4.4, one observes that

- $\tau_{(\cdot,\gamma)}$ and $\rho_{(\cdot,\gamma)}$ are non increasing functions with respect to the parameter η while $\tau_{(\eta,\cdot)}$ and $\rho_{(\eta,\cdot)}$ are nonincreasing functions with respect to the parameter γ;
- $\rho_{(1,1)} = \tau_{(1,1)} = -1$ since $B(2,2)=\frac{1}{6}$.

For what follows, we show that Spearman's rho and Kendall's tau associated with the copula family $\{C_{\eta,\gamma},(\eta,\gamma)\in(0,1)^2\}$ satisfy $\rho_{\eta,\gamma}\le\tau_{\eta,\gamma}\le 0$. To do this, let us first recall the definition of left-tail increasing and right-tail decreasing RVs (for more details, see,[216] and[20]).

Definition 4.6. Let X and Y be RVs.

1. X is *left-tail increasing* in Y, denoted LTI$(X|Y)$, if P$(X\le x|Y\le y)$ is nondecreasing in y for all x.
2. X is *right-tail decreasing* in Y, denoted RTD$(X|Y)$, if P$(X>x|Y>y)$ is nonincreasing in y for all x.

It was already shown by Capéraà and Genest,[2] for continuous RVs X and Y, that if LTI$(X|Y)$ and RTD$(X|Y)$ both hold, then $\rho\le\tau\le 0$. A simple proof can be found in Fredricks and Nelson.[10]

Now, let (U,V) be a pair of RVs with distribution $C_{\eta,\gamma}$. For all $(u,v)\in(0,1)^2$, we have

$$\text{P}(U\le u|V\le v) = \frac{C_{\eta,\gamma}(u,v)}{v} = u^{1-\eta}\max\left[1-(1-u^{\eta})v^{-\gamma},0\right], \tag{12}$$

and

$$\text{P}(U>u|V>v) = \frac{1-u-v+C_{\eta,\gamma}(u,v)}{1-v} = \frac{1-u-v+u^{1-\eta}v^{1-\gamma}\max(u^{\eta}+v^{\gamma}-1,0)}{1-v}. \tag{13}$$

It is clear that the function in Equation (12) is nondecreasing in v for all u, implying that LTI$(U|V)$ holds. Furthermore, simple calculations show that the function in Equation (13) is nonincreasing in v for all u, demonstrating that RTD$(U|V)$ also holds. As a result, Spearman's rho and Kendall's tau of $C_{\eta,\gamma}$ satisfy $\rho_{\eta,\gamma}\le\tau_{\eta,\gamma}\le 0$.

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4.4 Product moments

Hereafter, we derive the expression of the product moments corresponding to the copula $C_{\eta,\gamma}$. A result that will be of great use in the estimation of the dependence parameter vector (η, γ) in the next section.

Proposition 4.7. *Let (U, V) be a pair of RVs distributed as a copula $C_{\eta,\gamma}$. For any nonnegative integers i and j, the product moment of $C_{\eta,\gamma}$ is given by*

$$E\left(U^i V^j\right) = \alpha_1(i, j, \eta, \gamma) + \alpha_2(i, j, \eta, \gamma) B\left(\frac{i+1}{\eta}, \frac{j+1}{\gamma}\right), \quad (14)$$

where

$$\alpha_1(i, j, \eta, \gamma) = \frac{(i-\eta+1)(j-\gamma+1) - \eta\gamma ij}{(i+1)(j+1)(i-\eta+1)(j-\gamma+1)},$$

and

$$\alpha_2(i, j, \eta, \gamma) = \frac{ij}{(i-\eta+1)(j-\gamma+1)}.$$

The proof of Proposition 4.7 appears in the Appendix.

As noted in the Introduction, the proposed copula family $C_{\eta,\gamma}$ generalizes the results of the one presented in.[8] More precisely, one can point out that Propositions 4.1, 4.2, 4.3, 4.4 and 4.7 are reduced to [8, Propositions 2, 3, 4, 5, 6 and 9] by taking $\eta = \gamma$.

5 Simulation analysis and real data application

The purpose of this section is to use the method of moments to estimate the dependence parameter (η, γ) of the copula $C_{\eta,\gamma}$. Also, we exhibit the results of simulations assessing the performance of the estimator of (η, γ), denoted by $(\hat{\eta}, \hat{\gamma})$. In particular, we compute the bias and the mean square error corresponding to the moment estimator.

Notice that, for some copula models, it is possible to find explicit expressions that link the dependence parameter of the copula and the concordance measures, like Spearman's rho or Kendall's tau. Therefore, these measures are often used to estimate the copula parameter s. However, the measures of associations Spearman's rho and Kendall's tau of the proposed copula $C_{\eta,\gamma}$, in Proposition 4.4, are established implicitly. In addition, they are linearly linked by means of Equation (11). Thus, these results cannot be exploited to estimate the dependence parameter vector (η, γ).

The maximum likelihood estimation (MLE) method is generally used in the literature to solve the issue of estimating the dependence parameter of the copula.

Ruiz-Rivas and Cuadras[22] have established an explicit maximum likelihood estimator of the single dependence parameter of the Cuadras-Augé copula. In our context, this procedure is not applicable to estimating the dependence parameter. The reason is that the likelihood function cannot be maximized with respect to (η, γ). In fact, the likelihood function is given by

$$L(\eta, \gamma, (u_1, v_1), \ldots, (u_n, v_n)) = \prod_{(u_i, v_i) \in \mathcal{K}_{\eta,\gamma}} c_1(u_i, v_i) \prod_{(u_i, v_i) \in \mathcal{K}^*_{\eta,\gamma}} c_0(u_i),$$

where $\mathcal{K}_{\eta,\gamma} = \{(u, v) \in [0, 1]^2 : u^\eta + v^\gamma > 1\}$ and $\mathcal{K}^*_{\eta,\gamma} = \{(u, v) \in [0, 1]^2 : u^\eta + v^\gamma = 1\}$.

The main difficulty comes from the fact that the sets $\mathcal{K}_{\eta,\gamma}$ and $\mathcal{K}^*_{\eta,\gamma}$ depend on the unknown (η, γ), and we cannot identify which pairs (u_i, v_i) are in the sets $\mathcal{K}_{\eta,\gamma}$ or $\mathcal{K}^*_{\eta,\gamma}$. This is not the case for the Cuadras-Augé copula, where the singular and continuous support of the copula do not depend on the parameter of dependence.

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To avoid this problem, we have adopted the method of moments. To this end, let $(U_1, V_1), \ldots, (U_n, V_n)$ be mutually independent copies of (U, V) with copula $C_{\eta,\gamma}$. Let us denote

$$T_1 = \frac{1}{n}\sum_{i=1}^{n} U_i V_i, \quad \text{and} \quad T_2 = \frac{1}{n}\sum_{i=1}^{n} U_i^2 V_i + \frac{1}{n}\sum_{i=1}^{n} U_i V_i^2.$$

Remark that, from Proposition 4.7, one has

$$\begin{aligned}
\mathrm{E}\left(UV\right) &= \frac{(2-\eta)(2-\gamma) - 2\eta\gamma}{4(2-\eta)(2-\gamma)} + \frac{1}{(2-\eta)(2-\gamma)} B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right), \\
\mathrm{E}\left(U^2 V\right) &= \frac{(3-\eta)(2-\gamma) - 2\eta\gamma}{6(3-\eta)(2-\gamma)} + \frac{2}{(3-\eta)(2-\gamma)} B\left(\frac{3}{\eta}, \frac{2}{\gamma}\right), \\
\mathrm{E}\left(U V^2\right) &= \frac{(2-\eta)(3-\gamma) - 2\eta\gamma}{6(2-\eta)(3-\gamma)} + \frac{2}{(2-\eta)(3-\gamma)} B\left(\frac{2}{\eta}, \frac{3}{\gamma}\right).
\end{aligned}$$

By setting

$$g_1(\eta, \gamma) = \mathrm{E}\left(UV\right) \quad \text{and} \quad g_2(\eta, \gamma) = \mathrm{E}\left(U^2 V\right) + \mathrm{E}\left(U V^2\right),$$

an estimator vector of (η, γ) can be obtained by solving numerically the following optimization problem

$$\arg \min_{(\eta,\gamma)\in(0,1)^2} \sum_{i=1}^{2} (g_i(\eta, \gamma) - T_i)^2. \tag{15}$$

To check the quality of the moment-based estimator vector $(\hat{\eta}, \hat{\gamma})$ of (η, γ), consider samples generated from the copula $C_{\eta,\gamma}$, with various values of (η, γ). Different sample sizes n, are considered. After that, the moment-based estimator, bias and mean square error (MSE) of $(\hat{\eta}, \hat{\gamma})$ are calculated by means of $k = 500$ replicates. The results of simulations are summarized in Table 1. This table shows that the moment-based estimators perform well and that the bias and the MSE of these estimators decrease as the sample size n increases. We also remark that, for certain values of (η, γ), the bias of $\hat{\eta}$ and $\hat{\gamma}$ are always negative, regardless of the choice of sample size.

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Table 1: Moment estimates, Bias and MSE of $\hat{\eta}$ and $\hat{\gamma}$.

		n	$\hat{\eta}$	Bias($\hat{\eta}$)	MSE($\hat{\eta}$)	$\hat{\gamma}$	Bias($\hat{\gamma}$)	MSE($\hat{\gamma}$)
$\eta = 0.1$	$\gamma = 0.3$	100	0.1391	0.0391	0.0298	0.3561	0.0561	0.1329
		200	0.1286	0.0286	0.0227	0.3502	0.0502	0.1186
		300	0.1188	0.0188	0.0184	0.3431	0.0431	0.1140
		400	0.1034	0.0034	0.0152	0.3298	0.0298	0.1114
		500	0.1007	0.0007	0.0133	0.3161	0.0161	0.0971
$\eta = 0.1$	$\gamma = 0.4$	100	0.1439	0.0439	0.0318	0.3465	−0.0535	0.1332
		200	0.1277	0.0277	0.0212	0.3518	−0.0482	0.1175
		300	0.1199	0.0199	0.0181	0.3569	−0.0431	0.1125
		400	0.1143	0.0143	0.0150	0.3586	−0.0415	0.1124
		500	0.1091	0.0091	0.0138	0.3638	−0.0362	0.1095
$\eta = 0.3$	$\gamma = 0.6$	100	0.2668	−0.0332	0.0422	0.5309	−0.0691	0.1270
		200	0.2733	−0.0267	0.0226	0.6444	0.0444	0.0820
		300	0.2743	−0.0257	0.0187	0.5753	−0.0247	0.0744
		400	0.2805	−0.0195	0.0157	0.6142	0.0142	0.0654
		500	0.2902	−0.0098	0.0007	0.6095	0.0095	0.0542
$\eta = 0.4$	$\gamma = 0.7$	100	0.3786	−0.0214	0.0163	0.7271	0.0271	0.0386
		200	0.3793	−0.0207	0.0152	0.7224	0.0224	0.0385
		300	0.3872	−0.0127	0.0129	0.7181	0.0181	0.0274
		400	0.3906	−0.0094	−0.0005	0.7087	0.0087	0.0050
		500	0.3964	−0.0036	0.0004	0.7081	0.0081	0.0007
$\eta = 0.5$	$\gamma = 0.8$	100	0.5270	0.0270	0.0296	0.7508	−0.0492	0.0510
		200	0.5189	0.0189	0.0175	0.7789	−0.0211	0.0313
		300	0.5173	0.0173	0.0141	0.7861	−0.0139	0.0200
		400	0.5074	0.0074	0.0116	0.7901	−0.0099	0.0164
		500	0.5054	0.0054	0.0111	0.7990	−0.0010	0.0134
$\eta = 0.7$	$\gamma = 0.3$	100	0.5757	−0.1243	0.1009	0.3293	0.0293	0.0624
		200	0.6335	−0.0664	0.0612	0.3234	0.0234	0.0407
		300	0.6554	−0.0445	0.0519	0.3114	0.0114	0.0262
		400	0.6684	−0.0316	0.0415	0.3109	0.0109	0.0184
		500	0.6871	−0.0128	0.0352	0.3089	0.0089	0.0150
$\eta = 0.8$	$\gamma = 0.3$	100	0.6415	−0.1585	0.1071	0.3521	0.0521	0.0506
		200	0.6764	−0.1236	0.0577	0.3505	0.0505	0.0250
		300	0.6980	−0.1019	0.0452	0.3501	0.0501	0.0212
		400	0.7065	−0.0935	0.0374	0.3485	0.0485	0.0185
		500	0.7282	−0.0718	0.0304	0.3463	0.0463	0.0162
$\eta = 0.9$	$\gamma = 0.6$	100	0.8518	−0.0482	0.0160	0.6481	0.0481	0.0325
		200	0.8557	−0.0443	0.0098	0.6437	0.0437	0.0178
		300	0.8577	−0.0423	0.0087	0.6337	0.0337	0.0141
		400	0.8601	−0.0399	0.0087	0.6336	0.0336	0.0092
		500	0.8641	−0.0359	0.0077	0.6282	0.0282	0.0094
$\eta = 0.9$	$\gamma = 0.7$	100	0.8837	−0.0163	0.0075	0.7244	0.0244	0.0285
		200	0.8878	−0.0122	0.0048	0.7163	0.0163	0.0143
		300	0.8891	−0.0109	0.0043	0.7161	0.0161	0.0114
		400	0.8901	−0.0010	0.0040	0.7096	0.0096	0.0075
		500	0.8906	−0.0094	0.0041	0.7065	0.0065	0.0065
$\eta = 0.9$	$\gamma = 0.9$	100	0.9198	0.0198	0.0030	0.8833	−0.0166	0.0137
		200	0.9151	0.0151	0.0022	0.8898	−0.0102	0.0104
		300	0.9115	0.0115	0.0020	0.8900	−0.0010	0.0089
		400	0.9084	0.0084	0.0018	0.8928	−0.0072	0.0079
		500	0.9078	0.0078	0.0018	0.8939	−0.0061	0.0069

The following will be devoted to comparing the fitting of our copula to the ones modeling the negative dependence and previously introduced in[8] and[12] by means of a neutrosophic data set, "airquality", based on the daily quality of air in the New York Metropolitan Area. The sample size of the data set is $n = 153$. The average

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wind speed (in miles per hour) and the mean ozone level (in parts per billion) are the variables to be explored here. For a thorough description of the data, one can refer to Chambers et al.,[3] Appendix, Data set 2. This data set is also available in the R package "datasets".

Figures 4 and 5 indicate a negative dependence between the average wind speed and the mean ozone level, which are supported by negative values of empirical Spearman's rho and Kendall's tau coefficients, -0.59 and -0.43, respectively.

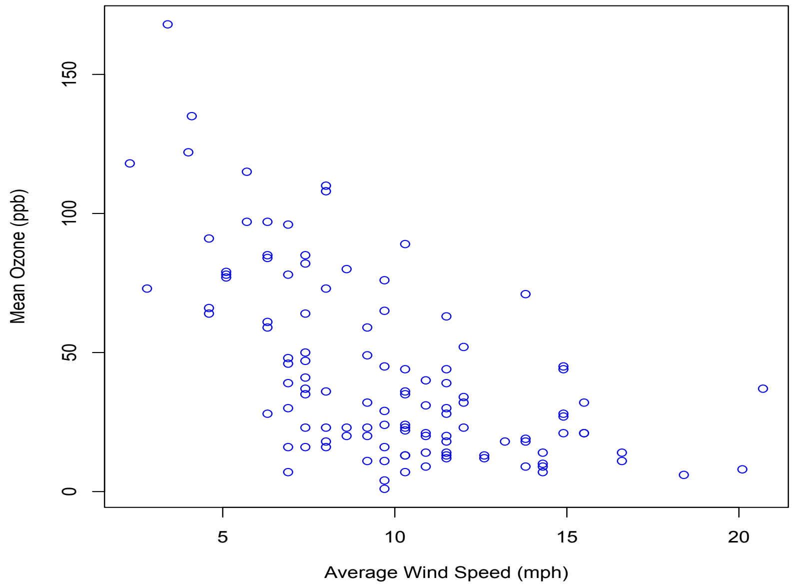

Figure 4: Scatter plot of average wind speed versus mean ozone

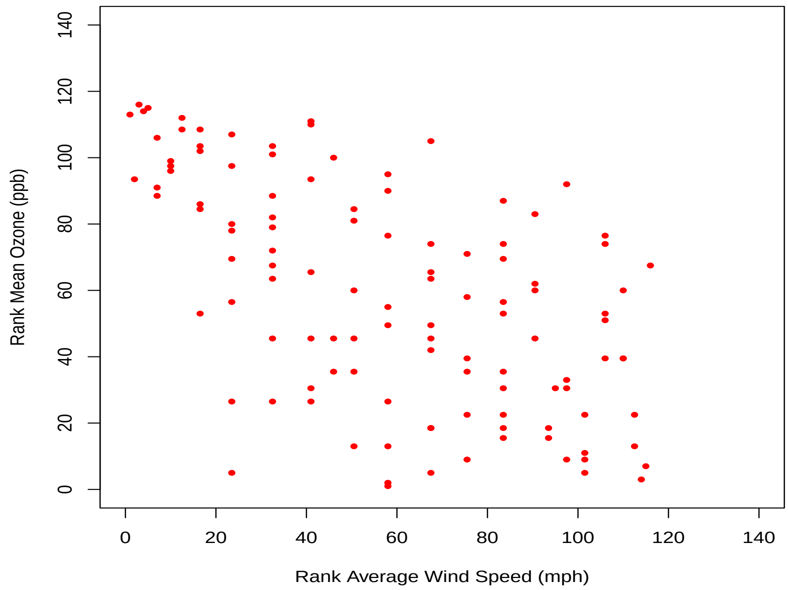

Figure 5: Rank-Rank plot of average wind speed versus mean ozone

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To estimate the dependence parameter s of the copula, we make use of Equation (15) and found that $(\hat{\eta}, \hat{\gamma}) = (0.96, 0.56)$. By means of Equations (8) and (9), the concordance measures Spearman's rho and Kendall's tau corresponding to the proposed copula family are, respectively, $\rho_{\hat{\eta},\hat{\gamma}} = -0.63$ and $\tau_{\hat{\eta},\hat{\gamma}} = -0.53$. Now, we evaluate the good-of-fit tests of the proposed copula $C_{\eta,\gamma}$, based on Kolmogorov–Smirnov (KS) and Cramér–von Mises (CVM) statistics using the bootstrap algorithm proposed by Genest et al.[11] Many classical families of copulas, commonly used for modeling negative dependence, were tested to fit our data set. No significant result has been found, though. We were limited in our ability to compare our results to those previously described in[8] and,[12] respectively, based on the following negatively dependent copulas:

$$C_\theta(u, v) = u^{1-\theta} v^{1-\theta} W(u^\theta, v^\theta)$$

and

$$C_\alpha(u,v) = \begin{cases} v - (1-u) + \dfrac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} (1-u)^{1+\alpha} v^{-\alpha}, 0 < v \le \dfrac{\alpha}{1+\alpha}, 1 - \dfrac{(1+\alpha)v}{\alpha} < u < 1 \\ u - (1-v)\left[1 - (1-u)^{1+\alpha}\right], 0 < u < 1, \dfrac{\alpha}{1+\alpha} < v < 1, \end{cases}$$

where $(\theta, \alpha) \in (0, 1)^2$.

Table 2 shows that our model fits very well the data set and displays the highest p-values. It also demonstrates its superiority when compared to the one parameter dependence models presented in[8] and.[12]

Table 2: Goodness-of-fit test for the considered copulas

	KS		CVM	
	Test statistic	p-value	Test statistic	p-value
$C_{\eta,\gamma}$	$T_n = 0.082$	0.636	$S_n = 0.150$	0.344
C_θ	$T_n = 0.078$	0.585	$S_n = 0.120$	0.219
C_α	$T_n = 0.099$	0.241	$S_n = 0.244$	0.055

6 Conclusion

The main purpose of this paper was to introduce a new bivariate copula and its related family with two-dependence parameters to allow for more flexible modeling and more precise representation of the underlying dependence structure. The construction technique for this family was based on the counter-monotonic shock method. Various of its properties were studied, and moment-based estimators of model parameters were established. The performance of the proposed estimators was illustrated by means of simulation studies and a real-life neutrosophic case study dealing with the daily air quality measurements for the New York Metropolitan Area. We argued that the proposed model overpowered the one described in.[8] Lastly, it should be emphasized that the proposed copula covers the full range of negative dependence. In this regard, it would be interesting to develop a general model that describes both positive and negative dependence. An extension of the current model to higher dimensions could also be explored. These questions are currently under investigation.

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References

[1] Bentoumi, R., El Ktaibi, F. and Mesfioui, M. A new family of bivariate exponential distributions with negative dependence based on counter-monotonic shock method. Entropy, 23, 548 (2021).

[2] Capéraà, P. and Genest, C. Spearman's is larger than Kendall's for positively dependent random variables. J. Nonparametr. Statist. 2, 183-194 (1993).

[3] Chambers, John M., William S. Cleveland, Beat Kleiner, and Paul A. Tukey. 1983. Graphical Methods for Data Analysis. London: Taylor & Francis/CRC.

[4] Chesneau, C. Theoretical contributions to three generalized versions of the Celebioglu-Cuadras copula. Analytics, 2023, 2, 31-54 (2023).

[5] Diaz, W. and Cuadras, C.M. An extension of the Gumbel-Barnett family of copulas. Metrika, 85, 913-926 (2022).

[6] Dolati, Ali, Soheyla Mohseni, and Manuel Úbeda-Flores. 2014. Some results on a transformation of copulas and quasi-copulas. Information Sciences 257: 176–82.

[7] Durante, F. Construction of non-exchangeable bivariate distribution functions. Statist. Papers, 50(2), 383-391 (2009).

[8] El Ktaibi, F., Bentoumi, R., Sottocornola, N., and Mesfioui, M. (2022). Bivariate copulas based on counter-monotonic shock method. Risks, 10(11), 202.

[9] Ferguson, T.S. (1996). A Course in Large Sample Theory. Chapman & Hall, London.

[10] Fredricks,G. A. and Nelsen, R. B. J. On the relationship between Spearman's rho and Kendall's tau for pairs of continuous random variables. Stat. Plann. Inference 137, 2143-2150 (2007).

[11] Genest, C., Mesfioui, M. and Schulz, J. A new bivariate Poisson common shock model covering all possible degrees of dependence. Stat. Probab. Lett . 140, 202–209 (2018).

[12] Ghosh, S., Bhuyan, P. and Finkelstein, M. On a bivariate copula for modeling negative dependence: application to New York air quality data. J. Statistical Methods & Applications., 31:1329–1353 (2022).

[13] Hering, C. and Mai, J. F. Moment-based estimation of extendible Marshall-Olkin copulas, Metrika, 75(5), 601-620 (2012).

[14] Ju Wu, Lianming Mou, Fang Liu , Haobin Liu, and Yi Liu (2020). Archimedean Copula-Based Hesitant Fuzzy Information Aggregation Operators for Multiple Attribute Decision Making. Mathematical Problems in Engineering, 2020. DOI: https://doi.org/10.1155/2020/6284245

[15] Khoudraji, A. Contributions à l'étude des copules et à la modélisation des valeurs extrêmes bivariées. PhD thesis, Université de Laval, Québec (Canada), 1995.

[16] Lehmann, E. L. Some concepts of dependence. Ann. Math. Statist. 37, 1137-1153, (1966).

[17] Liebscher, E. Construction of asymmetric multivariate copulas. J. Multivariate Anal., 99(10), 2234-2250 (2008).

[18] Manstavičius, M. and Bagdonas, G. A class of bivariate independence copula transformations. Fuzzy Sets and Systems, 428, 58-79 (2022).

[19] Marshall, A.W. and Olkin, I. A multivariate exponential distribution. J. Am. Stat. Assoc. 62, 30–44 (1967).

[20] Nelsen, R. B. An Introduction to Copulas, 2nd edition (Springer-Verlang, New York, 2006).

[21] Roberts, D. J. and Zewotir, T. Copula geoadditive modelling of anaemia and malaria in young children in Kenya, Malawi, Tanzania and Uganda. Journal of Health, Population and Nutrition, 39, 8 (2020).

[22] Ruiz-Rivas, C. and Cuadras, C. M. Inference properties of a one-parameter curved exponential family of distributions with given marginals. J. Multivariate Anal., 27, 447-456 (1988).

https://doi.org/10.54216/IJNS.250124
Received: December 19, 2023 Revised: February 15, 2024 Accepted: July 03, 2024

[23] Saali, T., Mesfioui, M. and Shabri, A. (2023). Multivariate extension of Raftery copula. Mathematics 2023, 11, 414.

[24] Saali, T., Mesfioui, M. and Shabri, A. (2024). A novel multivariate copula of Raftery type with multiple dependence parameters and its neutrosophic application in finance. Journal of International Journal of Neutrosophic Science, 23 (2), 296-307. DOI: https://doi.org/10.54216/IJNS.230224

[25] Safari-Katesari, H., Samadi, S. Y. and Zaroudi, S. Modelling count data via copulas. Statistics, 54, 6, 1329-1355 (2020).

[26] Shiau, J.-T. and Lien, Y. -C. Copula-based infilling methods for daily suspended sediment loads. Water, 13, 1701 (2021).

[27] Sklar, A. Fonction de répartion à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris.* 8 229-231, (1959).

[28] Smarandache, F. Introduction To Neutrosophic Measure, Neutrosophic Integral, And Neutrosophic Probability; Sitech: Craiova, Romania, 2011

[29] Taketomi, N., Yamamoto, K., Chesneau, C. and Emura, T. Parametric distributions for survival and reliability analyses, a review and historical sketch. Mathematics 2022, 10, 3907 (2022).

[30] Tavakol, A., Rahmani, V. and Harrington, J. Jr. Probability of compound climate extremes in a changing climate: A copula-based study of hot, dry, and windy events in the central United States. Environmental Research Letters, 15, 10, 104058 (2020).

7 Appendix

Proof of Proposition 4.1

Proof. The singular component can be written as

$$\mathcal{S}_{\eta,\gamma}(u,v) = C_{\eta,\gamma}(u,v) - \mathcal{A}_{\eta,\gamma}(u,v),$$

where

$$\mathcal{A}_{\eta,\gamma}(u,v) = \int_0^u \int_0^v \partial^2_{x,y} C_{\eta,\gamma}(x,y)dx\,dy,$$

where $\partial^2_{x,y} = \partial^2/(\partial x \partial y)$.

First, let us show that $\mathcal{A}_{\eta,\gamma}(u,v) = 0$ if $u^\eta + v^\gamma - 1 \le 0$. By choosing $(x,y) \in [0,u] \times [0,v]$ and noting that $x^\eta + y^\gamma - 1 \le u^\eta + v^\gamma - 1 \le 0$, it follows that $C_{\eta,\gamma}(x,y) = 0$ for all $(x,y) \in [0,u] \times [0,v]$ and $\mathcal{A}_{\eta,\gamma}(u,v) = 0$ as a result.
Next, suppose that $u^\eta + v^\gamma - 1 > 0$ and set $\mathcal{K}_{u,v} = \{(x,y) \in [0,u] \times [0,v] : x^\eta + y^\gamma > 1\}$. The following decomposition holds:

$$\begin{aligned}
\mathcal{A}_{\eta,\gamma}(u,v) &= \int_{\mathcal{K}_{u,v}} \partial^2_{x,y} C_{\eta,\gamma}(x,y)dx\,dy \\
&= \int_{(1-v^\gamma)^{1/\eta}}^u \left\{ \int_{(1-x^\eta)^{1/\gamma}}^v \left[(1-\eta)x^{-\eta} + (1-\gamma)y^{-\gamma} - (1-\eta)(1-\gamma)x^{-\eta}y^{-\gamma}\right] dy \right\} dx \\
&= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,
\end{aligned}$$

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where

$$\mathcal{J}_1 = (1-\eta)\int_{(1-v^\gamma)^{1/\eta}}^{u}\int_{(1-x^\eta)^{1/\gamma}}^{v} x^{-\eta} dy\, dx,$$
$$\mathcal{J}_2 = (1-\gamma)\int_{(1-v^\gamma)^{1/\eta}}^{u}\int_{(1-x^\eta)^{1/\gamma}}^{v} y^{-\gamma} dy\, dx,$$
$$\mathcal{J}_3 = -(1-\eta)(1-\gamma)\int_{(1-v^\gamma)^{1/\eta}}^{u}\int_{(1-x^\eta)^{1/\gamma}}^{v} x^{-\eta}y^{-\gamma} dy\, dx.$$

Simple calculations lead to

$$\begin{aligned}\mathcal{J}_1 &= (1-\eta)\int_{(1-v^\gamma)^{1/\eta}}^{u}\left[v-(1-x^\eta)^{1/\gamma}\right]x^{-\eta}dx\\ &= v\left[u^{1-\eta}-(1-v^\gamma)^{1/\eta-1}\right]-(1-\eta)\int_{(1-v^\gamma)^{1/\eta}}^{u} x^{-\eta}(1-x^\eta)^{1/\gamma}dx.\end{aligned}$$

By the change of variabes $z = x^\eta$, we obtain

$$\begin{aligned}\mathcal{J}_1 &= v\left[u^{1-\eta}-(1-v^\gamma)^{1/\eta-1}\right]-\frac{1-\eta}{\eta}\int_{1-v^\gamma}^{u^\eta} z^{1/\eta-2}(1-z)^{1/\gamma}dz\\ &= \frac{1-\eta}{\eta}\left[B\left(1-v^\gamma,\frac{1}{\eta}-1,\frac{1}{\gamma}+1\right)-B\left(u^\eta,\frac{1}{\eta}-1,\frac{1}{\gamma}+1\right)\right]\\ &\quad + v\left[u^{1-\eta}-(1-v^\gamma)^{1/\eta-1}\right].\end{aligned}$$

Analogous arguments lead to

$$\begin{aligned}\mathcal{J}_2 &= \frac{1}{\eta}\left[B\left(1-v^\gamma,\frac{1}{\eta},\frac{1}{\gamma}\right)-B\left(u^\eta,\frac{1}{\eta},\frac{1}{\gamma}\right)\right]\\ &\quad + v^{1-\gamma}\left[u-(1-v^\gamma)^{1/\eta}\right],\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_3 &= -\frac{1-\eta}{\eta}\left[B\left(1-v^\gamma,\frac{1}{\eta}-1,\frac{1}{\gamma}\right)-B\left(u^\eta,\frac{1}{\eta}-1,\frac{1}{\gamma}\right)\right]\\ &\quad - v^{1-\gamma}\left[u^{1-\eta}-(1-v^\gamma)^{1/\eta-1}\right].\end{aligned}$$

By virtue of Equations (6) and (7), the absolutely continuous component of the copula $C_{\eta,\gamma}$ can be expressed as

$$\mathcal{A}_{\eta,\gamma}(u,v) = \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3 = C_{\eta,\gamma}(u,v) - B\left(u^\eta,\frac{1}{\eta},\frac{1}{\gamma}\right)+B\left(1-v^\gamma,\frac{1}{\eta},\frac{1}{\gamma}\right).$$

This ends the proof of Proposition 4.1. □

Proof of Proposition 4.4

Proof. Let U and V be two independent RVs and uniformly distributed over $[0,1]$. We can define and calculate the Spearman's rho as follows:

$$\begin{aligned}\rho_{\eta,\gamma} &= 12\mathrm{E}[C_{\eta,\gamma}(U,V)]-3\\ &= 12\int_{\mathcal{K}_{\eta,\gamma}}\left(vu^{1-\eta}+uv^{1-\gamma}-u^{1-\eta}v^{1-\gamma}\right)du\,dv-3\\ &= 12\int_0^1\left[\int_{(1-u^\eta)^{1/\gamma}}^1\left(vu^{1-\eta}+uv^{1-\gamma}-u^{1-\eta}v^{1-\gamma}\right)dv\right]du-3\\ &= -\frac{3\eta\gamma}{(2-\eta)(2-\gamma)}-12\left(\frac{1}{2}\mathcal{I}_1+\frac{1}{2-\gamma}\mathcal{I}_2-\frac{1}{2-\gamma}\mathcal{I}_3\right),\end{aligned}$$

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where

$$\mathcal{I}_1 = \int_0^1 u^{1-\eta}(1-u^\eta)^{2/\gamma} du,\ \mathcal{I}_2 = \int_0^1 u(1-u^\eta)^{2/\gamma-1} du,\ \mathcal{I}_3 = \int_0^1 u^{1-\eta}(1-u^\eta)^{2/\gamma-1} du.$$

By applying the change of variables $x = u^\eta$ and considering the following formula:

$$\int_0^1 u^a(1-u^\eta)^b du = \frac{1}{\eta}\int_0^1 x^{(a+1)/\eta-1}(1-x)^b dx = \frac{1}{\eta} B\left(\frac{a+1}{\eta}, b+1\right),$$

we can express $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ in terms of beta function. All combined, we get

$$\rho_{\eta,\gamma} = -\frac{3\eta\gamma}{(2-\eta)(2-\gamma)} - \frac{12}{2\eta} B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}+1\right) - \frac{12}{\eta(2-\gamma)} B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right) + \frac{12}{\eta(2-\gamma)} B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}\right). \tag{16}$$

By virtue of Equation (7), we have

$$B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}+1\right) = \frac{2\eta}{(2-\eta)\gamma} B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right), \tag{17}$$

and

$$B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}\right) = \frac{2\eta+2\gamma-\eta\gamma}{(2-\eta)\gamma} B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right). \tag{18}$$

Substituting Equations (17) and (18) in Equation (16), we obtain the desired expression of $\rho_{\eta,\gamma}$, i.e.,

$$\rho_{\eta,\gamma} = -\frac{3}{(2-\eta)(2-\gamma)}\left[\eta\gamma - 4B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right)\right].$$

Next, we address Kendall's tau measure corresponding to the copula $C_{\eta,\gamma}$. It is known that

$$\tau_{\eta,\gamma} = 1 - 4\int_0^1\int_0^1 \partial_u C_{\eta,\gamma}(u,v)\partial_v C_{\eta,\gamma}(u,v) du\, dv,$$

where $\partial_u = \partial/(\partial u)$ and $\partial_v = \partial/(\partial_v)$, and

$$\partial_u C_{\eta,\gamma}(u,v) = v^{1-\gamma} + (1-\eta)u^{-\eta}v - (1-\eta)u^{-\eta}v^{1-\gamma},$$
$$\partial_v C_{\eta,\gamma}(u,v) = u^{1-\eta} + (1-\gamma)uv^{-\gamma} - (1-\gamma)u^{1-\eta}v^{-\gamma}.$$

Consequently, we can decompose τ as

$$\tau_{\eta,\gamma} = 1 - 4\sum_{j=1}^{9} \mathcal{I}_j, \tag{19}$$

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where

$$\mathcal{I}_1 = \int_{\mathcal{K}_{\eta,\gamma}} u^{1-\eta}v^{1-\gamma}du\,dv = \frac{1}{(2-\eta)(2-\gamma)} - \frac{1}{\eta(2-\gamma)}B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}\right),$$
$$\mathcal{I}_2 = (1-\gamma)\int_{\mathcal{K}_{\eta,\gamma}} uv^{1-2\gamma}du\,dv = \frac{1}{4} - \frac{1}{2\eta}B\left(\frac{2}{\eta}, \frac{2}{\gamma}-1\right),$$
$$\mathcal{I}_3 = -(1-\gamma)\int_{\mathcal{K}_{\eta,\gamma}} u^{1-\eta}v^{1-2\gamma}du\,dv = -\frac{1}{2(2-\eta)} + \frac{1}{2\eta}B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}-1\right),$$
$$\mathcal{I}_4 = (1-\eta)\int_{\mathcal{K}_{\eta,\gamma}} vu^{1-2\eta}du\,dv = \frac{1}{4} - \frac{1-\eta}{2\eta}B\left(\frac{2}{\eta}-2, \frac{2}{\gamma}+1\right),$$
$$\mathcal{I}_5 = (1-\eta)(1-\gamma)\int_{\mathcal{K}_{\eta,\gamma}} u^{1-\eta}v^{1-\gamma}du\,dv = \frac{(1-\eta)(1-\gamma)}{(2-\eta)(2-\gamma)} - \frac{(1-\eta)(1-\gamma)}{\eta(2-\gamma)}B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}\right),$$
$$\mathcal{I}_6 = -(1-\eta)(1-\gamma)\int_{\mathcal{K}_{\eta,\gamma}} u^{1-2\eta}v^{1-\gamma}du\,dv = -\frac{1-\gamma}{2(2-\gamma)} + \frac{(1-\eta)(1-\gamma)}{\eta(2-\gamma)}B\left(\frac{2}{\eta}-2, \frac{2}{\gamma}\right),$$
$$\mathcal{I}_7 = -(1-\eta)\int_{\mathcal{K}_{\eta,\gamma}} u^{1-2\eta}v^{1-\gamma}du\,dv = -\frac{1}{2(2-\gamma)} + \frac{1-\eta}{\eta(2-\gamma)}B\left(\frac{2}{\eta}-2, \frac{2}{\gamma}\right),$$
$$\mathcal{I}_8 = -(1-\eta)(1-\gamma)\int_{\mathcal{K}_{\eta,\gamma}} u^{1-\eta}v^{1-2\gamma}du\,dv = -\frac{1-\eta}{2(2-\eta)} + \frac{1-\eta}{2\eta}B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}-1\right),$$
$$\mathcal{I}_9 = (1-\eta)(1-\gamma)\int_{\mathcal{K}_{\eta,\gamma}} u^{1-2\eta}v^{1-2\gamma}du\,dv = \frac{1}{4} - \frac{1-\eta}{2\eta}B\left(\frac{2}{\eta}-2, \frac{2}{\gamma}-1\right).$$

The previous integrals can be expressed with a common beta function as follows:

$$B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}\right) = \frac{2\eta+2\gamma-\eta\gamma}{\gamma(2-\eta)}B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right),$$
$$B\left(\frac{2}{\eta}, \frac{2}{\gamma}-1\right) = \frac{2\eta+2\gamma-\eta\gamma}{\eta(2-\gamma)}B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right),$$
$$B\left(\frac{2}{\eta}-1, \frac{2}{\gamma}-1\right) = \frac{2(\eta+\gamma-\eta\gamma)(2\eta+2\gamma-\eta\gamma)}{\eta\gamma(2-\eta)(2-\gamma)}B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right),$$
$$B\left(\frac{2}{\eta}-2, \frac{2}{\gamma}\right) = \frac{(\eta+\gamma-\eta\gamma)(2\eta+2\gamma-\eta\gamma)}{\gamma^2(1-\eta)(2-\eta)}B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right),$$
$$B\left(\frac{2}{\eta}-2, \frac{2}{\gamma}+1\right) = \frac{\eta(2\eta+2\gamma-\eta\gamma)}{\gamma^2(1-\eta)(2-\eta)}B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right),$$
$$B\left(\frac{2}{\eta}-2, \frac{2}{\gamma}-1\right) = \frac{(\eta+\gamma-\eta\gamma)(2\eta+2\gamma-\eta\gamma)(2\eta+2\gamma-3\eta\gamma)}{\eta\gamma^2(1-\eta)(2-\eta)(2-\gamma)}B\left(\frac{2}{\eta}, \frac{2}{\gamma}\right).$$

The desired result follows upon substitution in Equation (19).
By recalling the definition of Blomqvist's beta, $\beta(C_{\eta,\gamma}) = 4C_{\eta,\gamma}(1/2, 1/2) - 1$, Equation (10) is in force. □

Proof of Proposition 4.7

Proof. For $i = 0$ or $j = 0$, the proof is straightforward and will be omitted. Suppose now $i \geq 1$ and $j \geq 1$. Let (U, V) be a pair of RVs with cumulative distribution function $C_{\eta,\gamma}$. For all $(u, v) \in (0, 1)^2$, one has

$$\begin{aligned}\mathrm{E}\left(U^iV^j\right) &= \int_0^1\int_0^1 iju^{i-1}v^{j-1}\mathrm{P}(U > u, V > v)dudv \\ &= \int_0^1\int_0^1 iju^{i-1}v^{j-1}\left[1-u-v+C_{\eta,\gamma}(u,v)\right]du\,dv = \mathcal{L}_1 + \mathcal{L}_2,\end{aligned}$$

where

$$\mathcal{L}_1 = \int_0^1\int_0^1 iju^{i-1}v^{j-1}(1-u-v)du\,dv = \frac{1-ij}{(i+1)(j+1)},$$

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and

$$\mathcal{L}_2 = \int_0^1 \int_0^1 iju^{i-1}v^{j-1}C_{\eta,\gamma}(u,v)du\,dv = \int_0^1 \int_0^1 iju^{i-\eta}v^{j-\gamma}\max(u^\eta + v^\gamma - 1, 0)du\,dv$$
$$= m_1 + m_2 - m_3,$$

where, after standard calculations as previously,

$$m_1 = ij\int_0^1 u^i \left[\int_{(1-u^\eta)^{1/\gamma}}^1 v^{j-\gamma}dv\right]du = \frac{ij}{j-\gamma+1}\left\{\frac{1}{i+1} - \frac{1}{\eta}B\left(\frac{i+1}{\eta}, \frac{j+1}{\gamma}\right)\right\},$$
$$m_2 = ij\int_0^1 v^j \left[\int_{(1-v^\gamma)^{1/\eta}}^1 u^{i-\eta}du\right]dv = \frac{ij}{i-\eta+1}\left\{\frac{1}{j+1} - \frac{1}{\gamma}B\left(\frac{i+1}{\eta}, \frac{j+1}{\gamma}\right)\right\},$$
$$m_3 = ij\int_0^1 v^{j-\gamma} \left[\int_{(1-v^\gamma)^{1/\eta}}^1 u^{i-\eta}du\right]dv$$
$$= \frac{ij}{(i-\eta+1)(j-\gamma+1)}\left\{1 - \left(\frac{i+1}{\eta} + \frac{j+1}{\gamma} - 1\right)B\left(\frac{i+1}{\eta}, \frac{j+1}{\gamma}\right)\right\}.$$

All of the above quantities add up to the expression given in Equation (14). □