An Outer Generalized Prime System and Some Discrete Examples

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Abstract

Beurling (or generalized) prime system has been defined by Arne Beurling in 1937, and several couthers have been working on this during the last century. This work focuses on addressing some concrete examples of an outer generalized prime system involving Beurling zeta function. The core of this work is to create a discrete generalized prime system under a fixed condition to give a new upper bound for Beurling zeta function.

Keyword: Generalized Prime Systems; Beurling Zeta Function.

1. Introduction

Arne Beurling (see [1]) introduced the concept generalized primes and integers in 1937. This concept has been studied by many couthers since that time tell now (see for example [5-11]). Which means that for any positive real sequence $\dot{P} = \{b_1, b_2, b_3 \dots\}$ satisfying $1< b_1 \le b_2 \le \cdots \le b_i \le \cdots$ and $b_i \to \infty$ called generalized primes (as $i \to \infty$). Applying the rule of the fundamental theorem of arithmetic, one can get the sequence of generalized integers.

An outer generalized prime system is defined as a pair of right continuous, and increasing functions $(\dot{\Pi}_p, \dot{N}_p)$, where $\dot{\Pi}_p \in B_0^+$ and $\dot{N}_p \in B_1^+$. In fact, $\dot{N}_p = \exp * \dot{\Pi}_p$. This definition is somewhat more inclusive than the usual "generalized primes", since it does not include the equivalent of the prime counting function $\dot{\pi}_p$. In his usage of the outer generalized prime system, Hilberdink [4] uses the name generalized prime system to one for which there exists an increasing function $\dot{\pi}_{\dot{P}} \in B_0$ that is associated with the function $\dot{\Pi}_p \in B_0^+$. The aim of this work is to present concrete examples of an outer generalized prime system involving the Beurling zeta function, and to create a discrete generalized prime system under a specified condition in order to provide a new upper bound for the Beurling zeta function.

After presenting the basic concepts required, our work starts with the definition of continuous functions, which correspond to the classical weighted generalized prime counting functions and presenting some basic properties of these functions. After that, we find the functions associated with these functions, and we show some of their properties. Based on these concepts and by Hilberdinck's usage of the outer generalized prime system, we introduce examples of outer generalized prime systems and show the connection of these systems to the Beurling zeta function. Finally, we create a discrete generalized prime system under a fixed condition to give a new upper bound for Beurling zeta function. Also, we clarify the relationship between σ and the indeterminate part of the neutrosophic measure. For further reading on the concept of neutrosophic and the neutrosophic measure, refer to (for example, [14-20]).

2. Some Basic Concepts

In this section, we go over some basic definitions that are used in the work's next sections.

2.1.1. Mellin-Stieltjes Convolution

Let B be the space of all local bounded variation, right continuous, zero on $(-\infty, 1)$ functions $h:\mathbb{R} \to \mathbb{C}$. Let B^+ be the subspace of B consisting of all increasing functions.

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Furthermore, let $B_a = \{h \in B; h(1) = a\}$ for a in R, and let $B_a^+ = B^+ \cap B_a$.
The definition of the Mellin-Stieltjes convolution for h, k ∈B is $(h*k)(t) = \int_{1-}^{t} h\left(\frac{t}{x}\right) dk(x)$.
It is seen that B is closed under ∗, and that ∗ is commutative and associated. When x≥ 1, the identity (with respect to ∗) is $i(x) = 1$, and otherwise it is zero.

- If h or k is continuous (on R), then (h∗k) is as well.
- If h is in B_1, then there is k in B_0 such that h= exp ∗ k, that is

$$h = \sum_{n=0}^{\infty} \frac{k^{*n}}{n!} \quad (\text{where}, k^{*n} = k * k^{*(n-1)}, k^{*0} = i),$$

additionally, h= exp ∗ k ↔ $h * k_L = h_L$, here h_L in B is the function given by $h_L = \int_1^t \log x \, dh(x)$ for t ≥1.

- The definition of the Mellin transform for h ∈ B is $\hat{h}(s) = \int_{1-}^{\infty} \frac{1}{x^s} dh(x)$.

This is true if, for some D ≥ 0, $h(x) = O(x^D)$. Note that $\widehat{h * k} = \hat{h}\,\hat{k}$ and $\widehat{\exp * h} = \exp \hat{h}$.
(see e.g. [2]).
2.1.2. Theorem[13] For any generalized prime system of polynomial growth, the relation $d\dot{N}_p = \exp * d\dot{\Pi}_p$ holds.
2.1.3 Theorem [12] Let H be a right continuous function that is non-decreasing and tends to ∞. Its H(1) value is 0 and it satisfies the Chebyshev bound $H(t) \ll \frac{t}{\log t}$. Then there exists a sequence of real numbers that are positive $\dot{P} = \{b_j\}_j^{\infty}$ satisfying $1 < b_1 \le b_2 \le \cdots$ such that $\left|\dot{\pi}_p(t) - H(t)\right| \le 2$ and that $\left| \sum_{b_j \le t} b_j^{-ix} - \int_1^t u^{-ix} \, dH(u) \right| \ll \sqrt{t} + \sqrt{\frac{t \log(|x|+1)}{\log(t+1)}}$ for any t ≥ 1 and for any real x.
2.1.4. Theorem [14] Suppose that $\dot{\psi}_p(t) = t + O(t^c)$, for some $0 < c < 1$. Then, for some positive constants $\rho > 0$,

$$\dot{N}_p(t) = \rho t + O(t\, e^{-\frac{\sqrt{1-c}}{4} (\log t \log\log t)^{\frac{1}{2}}}).$$

2.2. Generalized Prime System (ġ-prime system)
This system's structure is a sequence of real numbers that are positive $\dot{P} = \{b_1, b_2, b_3 \dots\}$ satisfying

$$1 < b_1 \le b_2 \le \cdots \le b_i \le \cdots$$

and $b_i \to \infty$ called generalized primes (for short, ġ-primes) (as $i \to \infty$). From these can be formed a new increasing real number sequence

$$1 < m_1 \le m_2 \le \cdots \le m_i \le \cdots$$

that represents each possible products

$$b_1^{a_1} b_2^{a_2} \dots b_k^{a_k},$$

where $k \in \mathbb{N}$ and $a_1, a_2, \dots, a_k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
These fresh values are represented by $\dot{N}$ (i.e. $\dot{N} = \{m_i\}_{i=1}^{\infty}$) and are referred to as generalized integers (for short, ġ-integers) linked to $\dot{P}$.
Attached to these systems is the counting function of ġ-primes and ġ-integers

$$\dot{\pi}_p(t) = \sum_{\substack{b \le t \\ b \in \dot{P}}} 1,$$

$$\dot{N}_p(t) = \sum_{\substack{m \le t \\ m \in \dot{N}}} 1.$$

With step functions $\dot{\pi}_p$ and $\dot{N}_p$ and integer jumps, these systems are discrete systems. The generalized Chebyshev function $\dot{\psi}_p$ over $\dot{N}$ with the sum extending over all ġ-prime numbers $p \in \dot{P}$ that are less than or equal to t is defined as follows:

$$\dot{\psi}_p(t) = \sum_{\substack{b^k \le t \\ b \in \dot{P} \\ t \in \mathbb{N}}} \log b = \sum_{\substack{m \le t \\ m \in \dot{N}}} \dot{\Lambda}_p(m),$$

where $\dot{\Lambda}_p$ is generalized Von Mangoldt function:

$$\dot{\Lambda}_p(m) = \begin{cases} \log b & \text{if } m = b^k \text{ for some } b \in \dot{P} \text{ and integer } k \ge 1 \\ 0 & \text{otherwise} \end{cases}.$$

The weighted generalized prime counting function $\dot{\Pi}_p$ is introduced as in classical prime number theory

$$\dot{\Pi}_p(t) = \sum_{n=1}^{\infty} \frac{1}{n} \dot{\pi}_p\left(t^{\frac{1}{n}}\right).$$

It is related to $\dot{\psi}_p$ via

$$\dot{\psi}_p(t) = \int_1^t \log u \left(\dot{\Pi}_p(u)\right)' du.$$

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These systems linked with the generalized (or Beurling) zeta function

$$\dot{Z}_p(s) = \prod_{i=1}^{\infty} \frac{1}{1-b_i^{-s}} = \prod_{b\in\dot{P}} \frac{1}{1-b^{-s}} = \sum_{i=0}^{\infty} \frac{1}{m_i^{s}} = \sum_{m\in\dot{N}} m^{-s}.$$

have been investigated by many researchers.
It is related to $\dot{\Pi}_p$, $\dot{N}_p$ and $\dot{\psi}_p$ via

$$\dot{\zeta}_p(s) = \sum_{i=0}^{\infty} \frac{1}{m_i^{s}} = \sum_{m\in\dot{N}} m^{-s} = \exp\{\int_1^{\infty} t^{-s}\, d\dot{\Pi}_p(t)\} \tag{2.1}$$

$$\dot{\zeta}_p(s) = \sum_{i=0}^{\infty} \frac{1}{m_i^{s}} = \sum_{m\in\dot{N}} m^{-s} = \int_{1-}^{\infty} \frac{1}{t^s} d\dot{N}_p(t) \tag{2.2}$$

$$-\frac{\dot{\zeta}_p(s)'}{\dot{\zeta}_p(s)} = s\int_1^{\infty} t^{-(s+1)}\, \dot{\psi}_p(t)\, dt \tag{2.3}$$

For every complex number s that the $\sum_{m\in\dot{N}} m^{-s}$ converges to.
A generalized prime system $\dot{P}$ is said to have an abscissa of convergence σ_c if and only if $\sum_{m\in\dot{N}} m^{-s}$ converges for $R_e(s) > \sigma_c$ and diverges for $R_e(s) < \sigma_c$. The set of points s such that $R_e(s) > \sigma_c$ is called a half-plane. Within the half-plane of convergence, the series represents an analytical function for the complex variable s. The product of the generalized zeta function is known as the Euler product. The generalized Dirichlet series is represented by the sum $\sum_{m\in\dot{N}} m^{-s}$. The abscissa of convergence of $\sum_{m\in\dot{N}} m^{-s}$ and $\sum_{b\in\dot{P}} b^{-s}$ is the same.
If $\dot{P} = \{3, 5, 7, 11, 13 \dots\} = \mathbb{P}\backslash\{2\}$, i.e. $b_n = n^{th}$ odd prime, we have

- $\dot{N} = \{1, 3, 5, 7, 9, 11, 13, \dots\}$ which is the set of odd numbers.
- $\dot{\pi}_{\dot{P}}(t) = \sum_{\substack{b\le t\\ b\in\dot{P}}} 1 = \left(\sum_{\substack{b\le t\\ b\in\mathbb{P}}} 1\right) - 1$ for $t \ge 2$.

The behavior of this the counting function $\dot{\pi}_{\dot{P}}$ for large t is

$$\frac{t}{\log t}(1+ O(1)) \text{ by the PNT.}$$

- $\dot{N}_{\dot{P}}(t) = \sum_{\substack{m\le t\\ m\in\dot{N}}} 1 = \sum_{\substack{(2k-1)\le t\\ k\in\mathbb{N}}} 1 = \sum_{\substack{k\le\frac{t+1}{2}\\ k\in\mathbb{N}}} 1 = \left[\frac{t+1}{2}\right]$.

The behavior of this the counting function $\dot{N}_{\dot{P}}$ for large t is

$$\frac{t}{2} + O(1).$$

- The associated generalized zeta function $\dot{\zeta}_p(s)$ is given by the Euler product

$$\dot{\zeta}_p(s) = \prod_{\substack{b>2\\ b\in\mathbb{P}}} \frac{1}{1-b^{-s}} = \prod_{\substack{b>2\\ b\in\mathbb{P}}} (1 - \frac{1}{b^s})^{-1}.$$

$\sum_{m\in\dot{N}} m^{-s} = \sum_{\substack{m=2k-1\\ k\in\mathbb{N}}} m^{-s}$ converges for $R_e(s) > 1$ and diverges for $R_e(s) \le 1$. That means that generalized prime system $\dot{P}$ has an abscissa of convergence 1.

2.3. Outer Generalized Prime System

The previously defined notion of ġ-prime numbers can be generalized such that, rather than always being step functions $\dot{\pi}_p$ and $\dot{N}_p$ are thought of as general increasing functions. This leads to the following:

2.3.1. Definition A generalized prime system is an outer generalized prime system for which there exists right continuous, local bounded variation and increasing function $\dot{\pi}_p$ with $\dot{\pi}_p(1) = 0$ such that

$$\dot{\Pi}_p(t) = \sum_{n=1}^{\infty} \frac{1}{n}\, \dot{\pi}_p\left(t^{\frac{1}{n}}\right) \tag{2.4}$$

If there exists such a function $\dot{\pi}_p \in B_0^+$, then $\dot{N}_p$ is said to determine a generalized prime system. As such $\dot{\pi}_p(t)$ is shown by

$$\dot{\pi}_p(t) = \sum_{w=1}^{\infty} \frac{\mu(w)}{w}\, \dot{\Pi}_p\left(t^{\frac{1}{w}}\right) \tag{2.5}$$

Since $\dot{\Pi}_p\left(t^{\frac{1}{w}}\right)$ decreases with w, $\sum_{w=1}^{\infty} \frac{\mu(w)}{w}$ converges, hence in actuality, this sum always converges for $\dot{\Pi}_p \in B^+$. But of course, the function $\dot{\pi}_p$ does not need to be increasing.
(see [4]).

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2.3.2. Remark

a) If $\dot{\pi}_p$ is a step function with integer jumps, then a generalized prime system is a discrete system. Here, the multiplicity is the step and the discontinuities $\dot{\pi}_p$ are the primes.

b) In $(1,\infty)$, if $\dot{N}_p$ (and hence, $\dot{\Pi}_p$) is continuous, then so is the outer generalized prime $(\dot{\Pi}_p, \dot{N}_p)$.

c) Let $\dot{\psi}_p(t) = \dot{\Pi}_{p_L}(t) = \int_1^t \log u \left(\dot{\Pi}_p(u)\right)'$ be the Chebyshev function for an outer generalized prime system $(\dot{\Pi}_p, \dot{N}_p)$. Note that $\dot{\psi}_p * \dot{N}_p = \dot{N}_{p_L}$ is equivalent to $\dot{N}_p = \exp * \dot{\Pi}_p$.

(More details can be seen in, [1] [2], [3], [4] and [13]).

3.1. $\dot{\Pi}_p$ and the associated function $\dot{\pi}_p$.

In this section, we introduce the continuous functions $\dot{\Pi}_{pk}$, $\dot{\Pi}_{\overline{pa}}$, and $\dot{\Pi}_{\overline{p\sigma}}$, which correspond to the classical weighted generalized prime counting function, and we show some basic properties of these functions. We also find the associated functions $\dot{\pi}_{pk}$, $\dot{\pi}_{\overline{pa}}$, and $\dot{\pi}_{\overline{p\sigma}}$ and a few of their properties.

3.1.1. Function $\dot{\Pi}_{\overline{p\sigma}}$ For $t \geq 1$ and $-1 \leq \sigma \leq 1$, the counting function $\dot{\Pi}_{\overline{p\sigma}}$ is defined as follows:

$$\dot{\Pi}_{\overline{p\sigma}}(t) := \int_1^t \frac{u^{\sigma} - (1-\sigma)u^{-1} - \sigma}{\log u} du.$$

3.1.2. Function $\dot{\Pi}_{\overline{pa}}$ For $t \geq 1$ and $0 < a < 2$, the counting function $\dot{\Pi}_{\overline{pa}}$ is defined as follows:

$$\dot{\Pi}_{\overline{pa}}(t) := \int_1^t \frac{1 - u^{-a}}{\log u} du.$$

3.1.3. Function $\dot{\Pi}_{pk}$ For $t \geq 1$ and $0 < k \leq 2$, the counting functions $\dot{\Pi}_{pk}$ is defined as follows:

$$\dot{\Pi}_{pk}(t) := \int_1^t \frac{u^{k-1} - u^{-1}}{\log u} du.$$

Our selection of values for σ, a, and k within the intervals [-1,1], (0,2) and (0,2] does not imply that these are the only possible values. These intervals were chosen because they provide a suitable and reasonable curve for studying systems associated with them. Therefore, it's important to understand that these specified intervals serve as a framework for study and are not strict boundaries that completely exclude values outside of them.

3.1.4. Proposition For $t \geq 1$, we have

a. $\dot{\pi}_{\overline{p\sigma}}(t) := \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \dot{\Pi}_{\overline{p\sigma}}\left(t^{\frac{1}{w}}\right) = \sum_{n=1}^{\infty} \frac{((1+\sigma)^n - \sigma)(\log t)^n}{n!\, n\, \dot{\zeta}(n+1)}$.

b. $\dot{\pi}_{\overline{pa}}(t) := \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \dot{\Pi}_{\overline{pa}}\left(t^{\frac{1}{w}}\right) = \sum_{n=1}^{\infty} \frac{(1-(1-a)^n)(\log t)^n}{n!\, n\, \dot{\zeta}(n+1)}$.

c. $\dot{\pi}_{pk}(t) := \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \dot{\Pi}_{pk}\left(t^{\frac{1}{w}}\right) = \sum_{n=1}^{\infty} \frac{(k)^n (\log t)^n}{n!\, n\, \dot{\zeta}(n+1)}$, where $\dot{\zeta}$ is the Riemann zeta function and μ is the Mobius function.

Proof. a

$$\begin{aligned}
\dot{\pi}_{\overline{p\sigma}}(t) &= \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \dot{\Pi}_{\overline{p\sigma}}\left(t^{\frac{1}{w}}\right) \\
&= \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \int_1^{t^{\frac{1}{w}}} \frac{u^{\sigma} - (1-\sigma)u^{-1} - \sigma}{\log u} du \\
&= \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \int_1^{t^{\frac{1}{w}}} \frac{u^{\sigma} - u^{-1} - \frac{1}{2}\sigma + \frac{1}{2}\sigma u^{-1} - \frac{1}{2}\sigma + \frac{1}{2}\sigma u^{-1}}{\log u} du \\
&= \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \int_1^{t^{\frac{1}{w}}} \frac{u^{\sigma} - u^{-1} - \frac{1}{2}\sigma(1-u^{-1}) - \frac{1}{2}\sigma(1-u^{-1})}{\log u} du \\
&= \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \int_1^{t^{\frac{1}{w}}} \frac{u^{\sigma} - u^{-1}}{\log u} du - \sigma \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \int_1^{t^{\frac{1}{w}}} \frac{(1-u^{-1})}{\log u} du \\
&= \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \int_1^{t} \frac{x^{\frac{\sigma}{w}} - x^{\frac{-(1)}{w}}}{\log x} x^{\frac{1}{w}-1} dx - \sigma \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \int_1^{t} \frac{1 - x^{\frac{-(1)}{w}}}{\log x} x^{\frac{1}{w}-1} dx.
\end{aligned}$$

This implies that

$$\dot{\pi}_{\overline{p\sigma}}(t) = \int_1^t \frac{1}{x \log x} \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \left(x^{\frac{1+\sigma}{w}} - 1\right) dx - \sigma \int_1^t \frac{1}{x \log x} \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \left(x^{\frac{1}{w}} - 1\right) dx \qquad (3.1)$$

But $x^{\frac{1+\sigma}{w}} - 1 = \sum_{n=1}^{\infty} \frac{(1+\sigma)^n \left(\frac{\log x}{w}\right)^n}{n!}$ and $x^{\frac{1}{w}} - 1 = \sum_{n=1}^{\infty} \frac{\left(\frac{\log x}{w}\right)^n}{n!}$.

Therefore, by (3.1), we get

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$$\dot{\pi}_{\overline{p\sigma}}(t) = \int_1^t \frac{1}{x\log x}\sum_{w=1}^{\infty}\frac{\mu(w)}{w}\left(\sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)\left(\frac{\log x}{w}\right)^n}{n!}\right)dx$$
$$= \int_1^t \frac{1}{x}\sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)(\log x)^{n-1}}{n!}\left(\sum_{w=1}^{\infty}\frac{\mu(w)}{w^{n+1}}\right)dx$$
$$= \int_1^t \frac{1}{x}\sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)(\log x)^{n-1}}{n!\,\dot{\zeta}(n+1)}dx$$
$$= \sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)(\log t)^n}{n!\,n\,\dot{\zeta}(n+1)}.$$

Proof. b.

Let $\alpha = -a$ and $\dot{\pi}_{\overline{\overline{pa}}}(t) = \sum_{w=1}^{\infty}\frac{\mu(w)}{w}\,\dot{\Pi}_{\overline{\overline{pa}}}\left(t^{\frac{1}{w}}\right)$.

This implies that

$$-\dot{\pi}_{\overline{\overline{pa}}}(t) = \sum_{w=1}^{\infty}\frac{\mu(w)}{w}\int_1^{t^{\frac{1}{w}}}\frac{u^{\alpha}-1}{\log u}du$$
$$= \sum_{w=1}^{\infty}\frac{\mu(w)}{w}\int_1^{t}\frac{x^{\frac{\alpha}{w}}-1}{\log x}\,x^{\frac{1}{w}-1}dx$$
$$= \int_1^t \frac{1}{x\log x}\sum_{w=1}^{\infty}\frac{\mu(w)}{w}\left(x^{\frac{1+\alpha}{w}}-x^{\frac{1}{w}}\right)dx \qquad (3.2).$$

But $\left(x^{\frac{1+\alpha}{w}}-x^{\frac{1}{w}}\right) = \left(\sum_{n=1}^{\infty}\frac{\left(\frac{(1+\alpha)\log x}{w}\right)^n}{n!}-\sum_{n=1}^{\infty}\frac{\left(\frac{\log x}{w}\right)^n}{n!}\right)$.

Therefore, by (3.2), we get

$$-\dot{\pi}_{\overline{\overline{pa}}}(t) = \int_1^t \frac{1}{x\log x}\sum_{w=1}^{\infty}\frac{\mu(w)}{w}\left(\sum_{n=1}^{\infty}\frac{\left(\frac{(1+\alpha)\log x}{w}\right)^n}{n!}-\sum_{n=1}^{\infty}\frac{\left(\frac{\log x}{w}\right)^n}{n!}\right)dx$$
$$= \int_1^t \frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{(1+\alpha)^n(\log x)^{n-1}}{n!}-\sum_{n=1}^{\infty}\frac{(\log x)^{n-1}}{n!}\right)\left(\sum_{w=1}^{\infty}\frac{\mu(w)}{w^{n+1}}\right)dx$$
$$= \int_1^t \frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{(1+\alpha)^n(\log x)^{n-1}}{n!\,\zeta(n+1)}-\sum_{n=1}^{\infty}\frac{(\log x)^{n-1}}{n!\,\zeta(n+1)}\right)dx$$
$$= \sum_{n=1}^{\infty}\left(\frac{(1+\alpha)^n(\log t)^n}{n!\,n\,\dot{\zeta}(n+1)}-\frac{(\log t)^n}{n!\,n\,\dot{\zeta}(n+1)}\right)$$
$$= \sum_{n=1}^{\infty}\frac{((1+\alpha)^n-1)(\log t)^n}{n!\,n\,\zeta(n+1)})$$

This means that

$$\dot{\pi}_{\overline{\overline{pa}}}(t) = \sum_{n=1}^{\infty}\frac{(1-(1-a)^n)(\log t)^n}{n!\,n\,\zeta(n+1)}).$$

Proof. c In the same way as the [a and b]. ©

3.1.5. Proposition If $-1 \le \sigma \le 1$, we have $\dot{\pi}_{\overline{p\sigma}} \in B_0^+$.

Proof. It is clear that $\dot{\pi}_{\overline{p\sigma}}$ is continuous and that $\dot{\pi}_{\overline{p\sigma}}(1) = 0$. If we can demonstrate that $\left(\dot{\pi}_{\overline{p\sigma}}\right)'$ is non-negative, we may demonstrate that the function $\dot{\pi}_{\overline{p\sigma}}$ is increasing because it is absolutely continuous.

Writing $x^{\frac{1}{w}}-1 = \sum_{n=1}^{\infty}\frac{\left(\frac{\log x}{w}\right)^n}{n!}$, $x^{\frac{(1+\sigma)}{w}}-1 = \sum_{n=1}^{\infty}\frac{\left(\frac{(1+\sigma)\log x}{w}\right)^n}{n!}$ and by (3.1), we find

$$\dot{\pi}_{\overline{p\sigma}}(t) = \int_1^t \frac{1}{x}\sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)(\log x)^{n-1}}{n!}\left(\sum_{w=1}^{\infty}\frac{\mu(w)}{w^{n+1}}\right)dx$$
$$= \int_1^t \frac{1}{x}\sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)(\log x)^{n-1}}{n!\,\zeta(n+1)}dx$$
$$\Rightarrow \left(\dot{\pi}_{\overline{p\sigma}}(t)\right)' = \frac{1}{(\log t)\,t}\sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)(\log t)^n}{n!\,\zeta(n+1)}$$
$$\ge \frac{1}{t(\log t)}\left(\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)(\log t)^n}{n!}\right)$$
$$> \frac{1}{\zeta(2)t} > 0, \text{ for all } t > 1.$$

Thus, $\dot{\pi}_{\overline{p\sigma}}(t)$ is increasing. ©

3.1.6. Proposition If $0 < k \le 2$, we have $\dot{\pi}_{pk} \in B_0^+$.

Proof. In the same way as the 3.1.5. Prop.

3.1.7. Proposition If $0 < a < 2$, we have $\dot{\pi}_{\overline{\overline{pa}}} \in B_0^+$.

Proof. In the same way as the 3.1.5. Prop.

3.1.8. Proposition For $t \ge 1$, we have

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d. $\dot{\Pi}_{\overline{p\sigma}}(t) = \sum_{n=1}^{\infty} \frac{((1+\sigma)^n - \sigma)\,(\log t)^n}{n!\,n}$.

e. $\dot{\Pi}_{\overline{\overline{pa}}}(t) = \sum_{n=1}^{\infty} \frac{(1-(1-a)^n)\,(\log t)^n}{n!\,n}$.

f. $\dot{\Pi}_{pk}(t) = \sum_{n=1}^{\infty} \frac{(k)^n\,(\log t)^n}{n!\,n}$,

Proof. It can be seen from the 3.1.4 Prop.

3.1.9. Remark The functions $\dot{\Pi}_{\overline{\overline{pa}}}$, $\dot{\pi}_{\overline{\overline{pa}}}$, $\dot{\Pi}_{pk}$, and $\dot{\pi}_{pk}$ are not increasing if $a, k < 0$.

3.1.10. Remark The functions $\dot{\pi}_{\overline{\overline{pa}}}$ is not monotonic increasing if $a > 2$.

3.1.11. Proposition For $R_e(s) > (1+\sigma)$, $-1 < \sigma \leq 1$, we have

$$\int_1^{\infty} t^{-s}\, d\dot{\Pi}_{\overline{p\sigma}}(t) = \log \frac{(s-1)^{\sigma} s^{(1-\sigma)}}{s-(1+\sigma)}.$$

Proof. By 3.1.1. Function $\dot{\Pi}_{\overline{p\sigma}}$, we find

$$\int_1^{\infty} t^{-s}\, d\dot{\Pi}_{\overline{p\sigma}}(t) = \int_1^{\infty} t^{-s}\left(\frac{t^{\sigma} - (1-\sigma)t^{-1} - \sigma}{\log t}\right) dt\,.$$

Thus,

$$\begin{aligned}
\frac{d\int_1^{\infty} t^{-s}\left(\frac{t^{\sigma}-(1-\sigma)t^{-1}-\sigma}{\log t}\right)dt}{ds} &= -\int_1^{\infty} t^{-s}(t^{\sigma} - (1-\sigma)t^{-1} - \sigma)dt \\
&= \left(-\frac{t^{-s+\sigma+1}}{-s+\sigma+1} - \frac{\sigma\, t^{1-s}}{s-1} + \frac{(\sigma-1)t^{-s}}{s}\right)_1^{\infty}. \\
&= \frac{\sigma}{s-1} - \frac{1}{s-(1+\sigma)} + \frac{1-\sigma}{s}, \text{ for } R_e(s) > (1+\sigma). \\
&= \frac{d\left(\log \frac{(s-1)^{\sigma} s^{(1-\sigma)}}{s-(1+\sigma)}\right)}{ds}, \text{ for } R_e(s) > (1+\sigma).
\end{aligned}$$

$$\Longrightarrow \int_1^{\infty} t^{-s}\left(\frac{t^{\sigma}-(1-\sigma)t^{-1}-\sigma}{\log t}\right)dt = \log \frac{(s-1)^{\sigma} s^{(1-\sigma)}}{s-(1+\sigma)} + C_2, \text{ for some } C_2.$$

The constant C_2 is 0

since $\lim_{s\to+\infty} \int_1^{\infty} t^{-s}\left(\frac{t^{\sigma}-(1-\sigma)t^{-1}-\sigma}{\log t}\right)dt = 0 = \lim_{s\to+\infty} \log \frac{(s-1)^{\sigma} s^{(1-\sigma)}}{s-(1+\sigma)}$.

Hence,

$$\int_1^{\infty} t^{-s}\, d\dot{\Pi}_{\overline{p\sigma}}(t) = \int_1^{\infty} t^{-s}\left(\frac{t^{\sigma}-(1-\sigma)t^{-1}-\sigma}{\log t}\right)dt = \log \frac{(s-1)^{\sigma} s^{(1-\sigma)}}{s-(1+\sigma)}. \qquad ©$$

3.1.12. Proposition For $R_e(s) > k$, $0 < k \leq 2$, we have $\int_1^{\infty} t^{-s}\, d\dot{\Pi}_{pk}(t) = \log \frac{s}{s-k}$.

Proof. In the same way as the 3.1.11 Prop**.**

3.1.13. Proposition For $R_e(s) > 1$, $0 < a < 2$, we have $\int_1^{\infty} t^{-s}\, d\dot{\Pi}_{\overline{\overline{pa}}}(t) = \log \frac{s-(1-a)}{s-1}$.

Proof. In the same way as the 3.1.11 Prop**.**

3.1.14. Remake When $\sigma = 0$ or -1, the function $\dot{\zeta}_{\overline{p\sigma}}(s)$ is near the classical "Riemman zeta function" on the half-line $\{s: s > 1\}$. But in the complex half-plan $\{s: R_e(s) > 1\}$, the two functions are not near. For $\dot{\zeta}_{\overline{\overline{pa}}.}$ and $\dot{\zeta}_{pk.}$, the remake is similarly holds for k = 1 and a = 1.

3.2. Outer Generalized Prime System Examples

Based on the previous concepts, we introduce examples of outer generalized prime systems and show the connection of these systems to the Beurling zeta function.

3.2.1. Remake For an outer generalized prime system $(\dot{\Pi}_p, \dot{N}_p)$, we have

$$\dot{N}_p = \exp * \dot{\Pi}_p \Leftrightarrow \left(s\int_1^{\infty} t^{-(s+1)} \dot{N}_p(t)\, dt = \exp\{\int_1^{\infty} t^{-s}\, d\dot{\Pi}(t)\}\right).$$

3.2.2.Example Let $\dot{\Pi}_{\overline{p\sigma}}(t) := \int_1^{t} \frac{u-1}{\log u} du$ (i.e. $\sigma = 1$). This means that caerly $\dot{\Pi}_{\overline{p\sigma}} \in B_0^{+}$, and

$$\dot{\psi}_{\overline{p\sigma}}(t) = \int_1^{t} \log u \left(\dot{\Pi}_{\overline{p\sigma}}(u)\right)' du = \frac{t^2}{2} - t + \frac{1}{2}, t \geq 1, \dot{\psi}_{\overline{p\sigma}} \in B_0^{+}.$$

By [3.1.4. Proposition (a)], we fined

$$\dot{\pi}_{\overline{p\sigma}}(t) = \sum_{n=1}^{\infty} \frac{((2)^n - 1)\,(\log t)^n}{n!\,n\,\dot{\zeta}(n+1)}.$$

This shows that $\dot{\pi}_{\overline{p\sigma}}(t) \in B_0^{+}$ [by (3.1.5. Proposition)] and hence we have a generalized prime system.

For $R_e(s) > 2$, the "Beurling zeta function" associated with $\dot{\Pi}_{\overline{p\sigma}}$ by [3.1.11. Proposition] is

$$\dot{\zeta}_{\overline{p\sigma}}(s) = \frac{s-1}{s-2}.$$

To determine the associated generalized integer counting function $\dot{N}_{\overline{p\sigma}}$, we express $\dot{\zeta}_{\overline{p\sigma}}(s)$ as Mellin transform

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$$s\int_1^{\infty} t^{-(s+1)}(\tfrac{1}{2}t^2+\tfrac{1}{2})dt=\frac{s-1}{s-2},\ R_e(s)>2.$$

Invoking the relationship

$$s\int_1^{\infty} t^{-(s+1)}\dot{N}_p(t)dt=\dot{\zeta}_p(s) \qquad (3.3)$$

the choice

$$\dot{N}_{\overline{p\sigma}}(t):=\frac{1}{2}(t^2+1),\ \text{for } t\geq 1,$$

is the associated generalized integer counting function with $\dot{\Pi}_{\overline{p\sigma}}$ that satisfies (3.3). It follows that the system $(\dot{\Pi}_{\overline{p\sigma}},\dot{N}_{\overline{p\sigma}})$, is an outer generalized prime system. It is also a continuous generalized prime system.

3.2.3 Example Let $\dot{\Pi}_{\overline{\overline{pa}}}(t):=\int_1^t\frac{1-u^{-\frac{1}{4}}}{\log u}du$ $\left(\text{i.e.}\, a=\frac{1}{4}\right)$ and $t\geq 1$.

This means that caerly $\dot{\Pi}_{\overline{\overline{pa}}}\in B_0^+$, and

$$\dot{\psi}_{\overline{\overline{pa}}}(t)=\int_1^t\log u\left(\dot{\Pi}_{\overline{\overline{pa}}}(u)\right)'du$$
$$=t-\frac{4}{3}t^{\frac{3}{4}}+\frac{1}{3},\dot{\psi}_{\overline{\overline{pa}}}\in B_0^+.$$

By [3.1.4. Proposition (b)], we find

$$\dot{\pi}_{\overline{\overline{pa}}}(t)=\sum_{n=1}^{\infty}\frac{(1-(\frac{3}{4})^n)\,(\log t)^n}{n!\,n\,\zeta(n+1)}).$$

This shows that $\dot{\pi}_{\overline{\overline{pa}}}(t)\in B_0^+$ [by (3.1.7. Proposition)] and hence we have a generalized prime system. For $R_e(s)>1$, the "Beurling zeta function" associated with $\dot{\Pi}_{\overline{\overline{pa}}}$ by [3.1.13. Proposition] is

$$\dot{\zeta}_{\overline{\overline{pa}}}(s)=\frac{s-\frac{3}{4}}{s-1}.$$

To determine the associated generalized integer counting function $\dot{N}_{\overline{\overline{pa}}}$, we express $\dot{\zeta}_{\overline{\overline{pa}}}(s)$ as Mellin transform

$$s\int_1^{\infty} t^{-(s+1)}\frac{1}{4}(t+3)dt=\frac{s-\frac{3}{4}}{s-1},\ \text{for } R_e(s)>1.$$

Invoking the relationship

$$s\int_1^{\infty} t^{-(s+1)}\dot{N}_p(t)dt=\dot{\zeta}_p(s),$$

the choice

$$\dot{N}_{\overline{\overline{pa}}}(t):=\frac{1}{4}(t+3)\ \text{for } t\geq 1,$$

is the associated generalized integer counting function with $\dot{\Pi}_{\overline{\overline{pa}}}$ that satisfies (3.3). It follows that System $(\dot{\Pi}_{\overline{\overline{pa}}},\dot{N}_{\overline{\overline{pa}}})$ is an outer generalized prime system. It is also a continuous generalized prime system.

3.2.4.Example Let ε be a positive value close to zero. Let $\dot{\Pi}_{pk}(t):=\int_1^t\frac{u^{\varepsilon}-u^{-1}}{\log u}du$ $(\text{i.e.}\, k=(1+\varepsilon))$ and $t\geq 1$.

This means that $\dot{\Pi}_{\overline{pc}}\in B_0^+$, and

$$\dot{\psi}_{pk}(t)=\int_1^t\log u\left(\dot{\Pi}_{pk}(u)\right)'du$$
$$=\frac{t^{(1+\varepsilon)}-1}{k}-\log t,\ \dot{\psi}_{pk}\in B_0^+.$$

By [3.1.4. Proposition (c)], we fined

$$\dot{\pi}_{pk}(t)=\sum_{n=1}^{\infty}\frac{((1+\varepsilon))^n(\log t)^n}{n!\,n\,\zeta(n+1)}).$$

This shows that $\dot{\pi}_{pk}(t)\in B_0^+$ [by (3.1.6. Proposition)]and hence we have a generalized prime system.

The associated generalized integer counting function in this case is

$$\dot{N}_{pk}(t)=t^{(1+\varepsilon)}\ \text{for } t\geq 1$$

since by (2.3.2. Remark (c)), we have

$$\int_1^t\log u\left(\dot{N}_{pk}(u)\right)'du=\int_1^t\dot{N}_{pk}(\tfrac{t}{u})\left(\dot{\psi}_{pk}(u)\right)'du$$
$$\Rightarrow \log t\,\dot{N}_{pk}(t)-\int_1^t\frac{\dot{N}_{pk}(u)}{u}du=\int_1^t\dot{N}_{pk}(\tfrac{t}{u})\,(u^{\varepsilon}-u^{-1})\,du.$$

If we put $x=\frac{t}{u}$, then

$$\log t\,\dot{N}_{pk}(t)-\int_1^t\frac{\dot{N}_{pk}(u)}{u}du=\int_1^t\dot{N}_{pk}(x)\left((\tfrac{t}{x})^{\varepsilon}-\tfrac{x}{t}\right)\frac{t}{x^2}dx$$
$$\Rightarrow \log t\,\dot{N}_{pk}(t)=t^{(1+\varepsilon)}\int_1^t\frac{\dot{N}_{pk}(x)}{x^{(\varepsilon+2)}}dx$$

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$$\Longrightarrow \left(\frac{\log t\,\dot{N}_{pk}(t)}{t^{(1+\varepsilon)}}\right)' = \frac{\dot{N}_{pk}(t)}{t^{(\varepsilon+2)}}$$

$$\Longrightarrow\ t\ \log t\ \left(\dot{N}_{pk}(t)\right)' + \dot{N}_{pk}(t) - (1+\varepsilon)\log t\ \dot{N}_{pk}(t) = \dot{N}_{pk}(t)$$

$$\Longrightarrow t\log t\ \left(\dot{N}_{pk}(t)\right)' = (1+\varepsilon)\log t\ \dot{N}_{pk}(t)$$

$$\Longrightarrow \frac{\left(\dot{N}_{pk}(t)\right)'}{(1+\varepsilon)\ \dot{N}_{pk}(t)} = \frac{1}{t}.$$

By the solution of differential equation, we get

$$\int \frac{\left(\dot{N}_{pk}(t)\right)'}{\dot{N}_{pk}(t)}\,dt = \int\ \frac{(1+\varepsilon)}{t}\ dt.$$

This gives that $\dot{N}_{pk}(t) = c\,t^{(1+\varepsilon)}$, but $\dot{N}_{pk}(1) = 1$, which means c=1.
In this case, we have

$$\dot{N}_{pk}(t) = t^{(1+\varepsilon)}.$$

For $R_e(s) > (1+\varepsilon)$, the "Beurling zeta function" associated with this system is

$$\begin{aligned}\dot{\zeta}_{pk}(s) &= \int_{1-}^{\infty} \frac{1}{t^s}\,d\dot{N}_{pk}(t)\\ &= s\int_1^{\infty}\ t^{-(s+1)}t^{(1+\varepsilon)}\ dt\\ &= s\int_1^{\infty}\ t^{-s+(1+\varepsilon)-1}\ dt\\ &= \frac{s}{s-(1+\varepsilon)}\ .\end{aligned}$$

It follows that the system $(\dot{\Pi}_{pk}, \dot{N}_{pk})$ is an outer generalized prime system. It is also a continuous generalized prime system.

In the following theorem, we use the function $\dot{\pi}_{\overline{p\sigma}}$ and set σ ∈ [-1,0]. The reason for this is that when σ ∈ [-1,0], the [2.1.3 Theorem], can be applied. On the other hand, we note that the function $\dot{\pi}_{\overline{p\sigma}}$ when (σ = 0 or -1) is equivalent to the function $\dot{\pi}_{\overline{pa}}$ and the function $\dot{\pi}_{\overline{pk}}$ as long as k = a = 1.

3.2.5. Lemma For $t \geq 1$ and σ ∈ [-1,0], we have

$$\dot{\pi}_{\overline{p\sigma}}(t) = \sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)\,(\log t)^n}{n!\,n\,\dot{\zeta}(n+1)} \ll \frac{t}{\log t}\,.$$

Proof.

$$\begin{aligned}\dot{\pi}_{\overline{p\sigma}}(t) &= \sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)\,(\log t)^n}{n!\,n\,\dot{\zeta}(n+1)} \leq\ \sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)\,(\log t)^n}{n!\,n}\\ &= \dot{\Pi}_{\overline{p\sigma}}(t)\\ &\ll \frac{t}{\log t}.\end{aligned}$$

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3.2.6. Theorem There exists a discrete generalized prime system $\dot{P}$ such that

- for some σ ∈ [-1,0],

$$\dot{\Pi}_P(t) = \dot{\Pi}_{\overline{p\sigma}}(t) + O(\log\log t).$$

- for some positive constants $\rho > 0$ and $\beta \in \left(0,\frac{1}{2}\right)$,

$$\dot{N}_P(t) = \rho t + O\left(t\,e^{-\frac{\sqrt{1-\beta}}{4}(\log t\log\log t)^{\frac{1}{2}}}\right).$$

Which implies that $\dot{\zeta}_p(\theta + ix) \ll (x^q)$ as $x \to \infty$ in this region where

$$1 - \frac{1}{4}\sqrt{\frac{(1-\beta)\log x}{x}} \ll \theta,\ \text{for some } q > 0.$$

Proof. According to [3.2.5. Lemma and 3.1.4. Proposition], the function $\dot{\pi}_{\overline{p\sigma}}(t) = \sum_{n=1}^{\infty}\frac{((1+\sigma)^n-\sigma)\,(\log t)^n}{n!\,n\,\dot{\zeta}(n+1)}$ is a right continuous function that is non-decreasing and tends to ∞ and $\dot{\pi}_{\overline{p\sigma}}(1) = 0$ and it satisfies the Chebyshev bound

$$\dot{\pi}_{\overline{p\sigma}}(t) \ll \frac{t}{\log t},\ \text{if}\ t \geq 1\ \text{and}\ \sigma \in [-1,0].$$

We now apply [2.1.3 Theorem] with $H = \dot{\pi}_{\overline{p\sigma}}$. Then there exists a sequence of real numbers that are positive $\dot{P} = \{b_j\}_j^{\infty}$ satisfying

$$1 < b_1 \leq\ b_2 \leq \cdots$$

such that

$$\left|\dot{\pi}_p(t) - H(t)\right| \leq 2$$

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and that

$$\left|\sum_{b_j\le t} b_j^{-ix} - \int_1^t u^{-ix}\, dH(u)\right| \ll \sqrt{t} + \sqrt{\frac{t\log(|x|+1)}{\log(t+1)}} \text{, for any } t \ge 1 \text{ and for any real } x.$$

The terms by definition $\dot{\Pi}_p(t) := \sum_{n=1}^{\infty} \frac{1}{n}\, \dot{\pi}_p\left(t^{\frac{1}{n}}\right) = \sum_{n=1}^{\left\lfloor\frac{\log t}{\log b_1}\right\rfloor} \frac{1}{n}\, \dot{\pi}_p\left(t^{\frac{1}{n}}\right)$ become identically zero when $n > \frac{\log t}{\log b_1}$. Thus, we find

$$\dot{\Pi}_p(t) = \sum_{n=1}^{\left\lfloor\frac{\log t}{\log b_1}\right\rfloor} \frac{1}{n}\left(H\left(t^{\frac{1}{n}}\right) + O(1)\right) = \sum_{n=1}^{\left\lfloor\frac{\log t}{\log b_1}\right\rfloor} \frac{1}{n} H\left(t^{\frac{1}{n}}\right) + O\left(\sum_{n=1}^{\left\lfloor\frac{\log t}{\log b_1}\right\rfloor} \frac{1}{n}\right).$$

This means that

$$\begin{aligned}\dot{\Pi}_p(t) &= \sum_{n=1}^{\left\lfloor\frac{\log t}{\log b_1}\right\rfloor} \sum_{r=1}^{\infty} \frac{1}{n} \frac{((1+\sigma)^r-\sigma)\left(\log t^{\frac{1}{n}}\right)^r}{r!\, r\, \dot{\zeta}(r+1)} + O(\log\log t)\\ &= \sum_{n=1}^{\left\lfloor\frac{\log t}{\log b_1}\right\rfloor} \sum_{r=1}^{\infty} \frac{1}{n^{(r+1)}} \frac{((1+\sigma)^r-\sigma)\,(\log t)^r}{r!\, r\, \dot{\zeta}(r+1)} + O(\log\log t)\\ &= \sum_{r=1}^{\infty} \dot{\zeta}(r+1) \frac{((1+\sigma)^r-\sigma)\,(\log t)^r}{r!\, r\, \dot{\zeta}(r+1)} + O(\log\log t)\\ &= \dot{\Pi}_{\overline{p\sigma}}(t) + O(\log\log t).\end{aligned}$$

Taking ($\sigma = 0$ or $\sigma = -1$) and according to $\dot{\psi}_p(t) = \int_1^t \log u \left(\dot{\Pi}_p(u)\right)' du$, we fined

$$\dot{\psi}_{\overline{p\sigma}}(t) = t - \log t - 1\,, t \ge 1,$$

resulting in

$$\begin{aligned}\dot{\psi}_{\mathbf{p}}(t) &= \int_1^t \log u \left(\dot{\Pi}_p(u)\right)' du = \int_1^t \log u \left(\dot{\Pi}_{\overline{p\sigma}}(u)\right)' du + \int_1^t \log u \left(\dot{\Pi}_p(u) - \dot{\Pi}_{\overline{p\sigma}}(u)\right)' du\\ &= t - \log t - 1 + (\dot{\Pi}_p(t) - \dot{\Pi}_{\overline{p\sigma}}(t)) \log t - \int_1^t \frac{\dot{\Pi}_p(u) - \dot{\Pi}_{\overline{p\sigma}}(u)}{u}\, du\\ &= t + O(\log t \log\log t) + O\left(\int_1^t \frac{(\log\log u)}{u}\, du\right)\\ &= t + O(\log t \log\log t).\end{aligned}$$

This gives that

$$\dot{\psi}_p(t) = t + O(t^{\beta}), \text{ for some } \beta \in \left(0, \frac{1}{2}\right).$$

Consequently, we can now use [2.1.4 Theorem] to get the following:

$$\dot{N}_P(t) = \rho t + O\left(t\, e^{-\frac{\sqrt{1-\beta}}{4}(\log t \log\log t)^{\frac{1}{2}}}\right), \text{ for some positive constants } \rho > 0.$$

Now, by using the bound of the error term of $\dot{N}_P(t)$, we find

$$\dot{\zeta}_p(\theta + ix) \ll (x^q) \text{ for some } q > 0, \text{ for } 1 - \frac{1}{4}\sqrt{\frac{\sqrt{1-\beta}\,\log x}{x}} \ll \theta, \text{ as follows:}$$

Take

$$h(t) = \frac{\sqrt{1-\beta}}{4}(\log t \log\log t)^{\frac{1}{2}}, \ (t \ge e) \text{ for some } \beta \in \left(0, \frac{1}{2}\right).$$

This means that, $h(t)$ is strictly increasing, positive function tending to infinity and

$$h(e^t) = \frac{\sqrt{1-\beta}}{4}(\log e^t\ \log\log e^t)^{\frac{1}{2}} = \frac{\sqrt{1-\beta}}{4}(t \log t)^{\frac{1}{2}}.$$

Thus,

$$\frac{h(e^t)}{t} = \frac{1}{4}\sqrt{\frac{(1-\beta)\log t}{t}} \text{ decreases.}$$

For $\left(h(t)\right)' = \frac{\frac{\sqrt{1-\beta}}{4}(1+\log\log t)}{(2t)\sqrt{(\log t \log\log t)}}$.

This means, $\left(h(t)\right)' = o\left(\frac{1}{t}\right)$ [since, $\lim_{t\to\infty} \frac{\frac{\sqrt{1-\beta}}{4}(1+\log\log t)\, t}{(2t)\sqrt{(\log t \log\log t)}} = 0$].

By using [2.2. Theorem in [10]], we get,

$\dot{\zeta}_p(\theta + ix) \ll (x^q)$, for $1 - \frac{1}{4}\sqrt{\frac{(1-\beta)\log x}{x}} \ll \theta$, for some $q > 0$. ©

3.2.7. Corollary Suppose that $\dot{\psi}_p(t) = t + \left(t^{\beta-\frac{1}{2}}\, e^{\frac{1}{2}(\log t)^{1-\beta}}\right)$, for some $0 < \beta < \frac{1}{2}$. Then there exist $q > 0$ such that

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$$\dot{\zeta}_p(\theta + ix) \ll (x^q) \text{ as } x \to \infty \text{ in this region where } 1 - \frac{1}{4}\sqrt{\frac{(1-\beta)\log x}{x}} \ll \theta.$$

Proof. Let $\dot{\psi}_p(t) = t + O(t^{\beta - \frac{1}{2}} e^{\frac{1}{2}(\log t)^{1-\beta}})$, $0 < \beta < \frac{1}{2}$. This means that

$$\begin{aligned}\dot{\psi}_p(t) - t &\le c_1\left(\frac{1}{\sqrt{t}}\, e^{\beta(\log t)}\, e^{\frac{1}{2}(\log t)^{1-\beta}}\right), \text{ for some } c_1 > 0\\ &= c_1\left(\frac{1}{\sqrt{t}}\, e^{\beta(\log t) + \frac{1}{2}(\log t)^{1-\beta}}\right),\\ &\le c_1\left(\frac{1}{\sqrt{t}}\, e^{\beta(\log t) + \frac{1}{2}(\log t)}\right)\\ &= c_1\left(\frac{1}{\sqrt{t}}\, e^{(\beta + \frac{1}{2})(\log t)}\right)\\ &= c_1 t^{\frac{-1}{2} + \left(\beta + \frac{1}{2}\right)}.\end{aligned}$$

Thus, $\dot{\psi}_p(t) = t + O(t^{\beta})$.

According to [3.2.6. Theorem], we find there exist $q > 0$ such that

$$\dot{\zeta}_p(\theta + ix) \ll (x^q) \text{ as } x \to \infty \text{ in this region where } 1 - \frac{1}{4}\sqrt{\frac{(1-\beta)\log x}{x}} \ll \theta.$$

4. Neutrosophic and σ

In this section, we will clarify the relationship between (σ) and the indeterminate part of the neutrosophic measure. Suppose that $t \ge 1$ and $\dot{\pi}_{\overline{p\sigma}}(t) = \sum_{n=1}^{\infty} \frac{((1+\sigma)^n - \sigma)(\log t)^n}{n!\, n\, \dot{\zeta}(n+1)}$ such that

$$A = \left| \sum_{\substack{p \le t\\ p \in \dot{P}}} 1 - \sum_{n=1}^{\infty} \frac{((1+\sigma)^n - \sigma)(\log t)^n}{n!\, n\, \dot{\zeta}(n+1)} \right|,$$

for some sequence of real numbers that are positive $\dot{P} = \{b_j\}_j^{\infty}$ satisfying

$$1 < b_1 \le b_2 \le \cdots .$$

We don't know exactly the quantity of A, but only that it belongs to [0, z]. for some z.
This quantity can be interpreted as a neutrosohic measure

$$\left| \sum_{\substack{p \le t\\ p \in \dot{P}}} 1 - \sum_{n=1}^{\infty} \frac{((1+\sigma)^n - \sigma)(\log t)^n}{n!\, n\, \dot{\zeta}(n+1)} \right| = 0 + I,$$

where 0 is the determinate part of this quantity and its indeterminate part $I \in [0,z]$. Its anti-part is 0.
According to [3.2.5. Lemma, 2.1.3 Theorem], we find that $z = 2$ if $t \ge 1$ and $\sigma \in [-1,0]$. However, this does not mean that there are no values of $\sigma \notin [-1,0]$ that would make "A belongs to [0,2]". It suggests instead that values for σ lying within the range [-1,0] allow us to use certain important previous theories, ensuring the achievement and correctness of our results.

5. Conclusion

The aim of this work is to present concrete examples of an outer generalized prime system involving the Beurling zeta function, and to create a discrete generalized prime system under a specified condition in order to provide a new upper bound for the Beurling zeta function. This work lays the groundwork for future research aiming to open a new window for integrating number theory with the field of topology. This effort contributes to generating new ideas and concepts that can be generalized using fuzzy sets, neutrosophic sets, and soft sets.

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