Product of rings based on neutrosophic sets

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Abstract

In this paper, we introduce the notion of the intrinsic product of neutrosophic sets, and some related properties are investigated. Characterizations of neutrosophic subrings, neutrosophic ideals, neutrosophic quasi-ideals, and neutrosophic bi-ideals are given.

Keywords: ring; neutrosophic subring; neutrosophic ideal; neutrosophic quasi-ideal; neutrosophic bi-ideal.

1 Introduction

Zadeh[11] first proposed the idea of fuzzy sets. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in.[1,3,4] The idea of intuitionistic fuzzy sets suggested by Atanassov[2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision-making.[7–9] The notion of neutrosophic sets was introduced by Smarandache[10] in 1999, which is a more general platform that extends the notions of classic sets, (intuitionistic) fuzzy sets, and interval-valued (intuitionistic) fuzzy sets. Neutrosophic set theory is applied to various parts which are referred to the site http://fs.unm.edu/neutrosophy.htm.

In this paper, we present the concept of the intrinsic product of neutrosophic sets and investigate some related properties. Characterizations of neutrosophic subrings, neutrosophic ideals, neutrosophic quasi-ideals, and neutrosophic bi-ideals are provided.

2 Preliminaries

Let X be a nonempty set. The neutrosophic set[2] on X is defined to be a structure

$$A := \{\langle x, \mu_A(x), \gamma_A(x), \psi_A(x)\rangle \mid x \in X\}, \tag{1}$$

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where $\mu_A : X \to [0,1]$ is a truth membership function, $\gamma_A : X \to [0,1]$ is an indeterminate membership function, and $\psi_A : X \to [0,1]$ is a false membership function. Simply put, the neutrosophic set in (1) is $A = (\mu_A, \gamma_A, \psi_A)$.

Definition 2.1. [6] A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ in a ring R is called a neutrosophic subring of R if

$$(\forall x, y \in R)\left(\begin{array}{l} \mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(x-y) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(x-y) \leq \max\{\psi_A(x), \psi_A(y)\} \end{array}\right),$$

$$(\forall x, y \in R)\left(\begin{array}{l} \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(xy) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(xy) \leq \max\{\psi_A(x), \psi_A(y)\} \end{array}\right).$$

Definition 2.2. [5] A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ in a ring R is called a neutrosophic left ideal of R if

$$(\forall x, y \in R)\left(\begin{array}{l} \mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(x-y) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(x-y) \leq \max\{\psi_A(x), \psi_A(y)\} \end{array}\right),$$

$$(\forall a, x \in R)\left(\begin{array}{l} \mu_A(ax) \geq \mu_A(x) \\ \gamma_A(ax) \geq \gamma_A(x) \\ \psi_A(ax) \leq \psi_A(x) \end{array}\right).$$

Definition 2.3. [5] A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ in a ring R is called a neutrosophic right ideal of R if

$$(\forall x, y \in R)\left(\begin{array}{l} \mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(x-y) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(x-y) \leq \max\{\psi_A(x), \psi_A(y)\} \end{array}\right),$$

$$(\forall a, x \in R)\left(\begin{array}{l} \mu_A(ax) \geq \mu_A(a) \\ \gamma_A(ax) \geq \gamma_A(a) \\ \psi_A(ax) \leq \psi_A(a) \end{array}\right).$$

Definition 2.4. [5] If $A = (\mu_A, \gamma_A, \psi_A)$ is both a neutrosophic right ideal and a neutrosophic left ideal of a ring R, then $A = (\mu_A, \gamma_A, \psi_A)$ is called a neutrosophic ideal of R.

3 Intrinsic products

Definition 3.1. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic sets in a ring R. The intrinsic product of $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ is defined to be the neutrosophic set $A * B = (\mu_{A*B}, \gamma_{A*B}, \psi_{A*B})$ in R given by

$$\mu_{A*B}(x) = \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_B(b_1), \mu_B(b_2), \ldots, \mu_B(b_m) \end{array}\right\}$$

$$\gamma_{A*B}(x) = \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_B(b_1), \gamma_B(b_2), \ldots, \gamma_B(b_m) \end{array}\right\}$$

$$\psi_{A*B}(x) = \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \max\left\{\begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_B(b_1), \psi_B(b_2), \ldots, \psi_B(b_m) \end{array}\right\}$$

if we can express $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$ for some $a_i, b_i \in R$ and for some positive integer m where each $a_ib_i \neq 0$. Otherwise, we define $A * B = 0$, that is, $\mu_{A*B}(x) = 0$, $\gamma_{A*B}(x) = 0$, and $\psi_{A*B}(x) = 1$.

Proposition 3.2. *Let $A = (\mu_A, \gamma_A, \psi_A)$, $B = (\mu_B, \gamma_B, \psi_B)$, and $C = (\mu_C, \gamma_C, \psi_C)$ be neutrosophic sets in a ring R. If $A \subseteq B$, then $A * C \subseteq B * C$ and $C * A \subseteq C * B$. Also, we have $A \circ B \subseteq A * B$, where $A \circ B = (\mu_{A\circ B}, \gamma_{A\circ B}, \psi_{A\circ B})$ is a neutrosophic set in R given by*

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$$\mu_{A\circ B}(x) = \begin{cases} \bigvee_{x=ab} \min\{\mu_A(a), \mu_B(b)\} & \text{if } x \text{ is factorizable in } R \\ 0 & \text{otherwise}, \end{cases}$$
$$\gamma_{A\circ B}(x) = \begin{cases} \bigvee_{x=ab} \min\{\gamma_A(a), \gamma_B(b)\} & \text{if } x \text{ is factorizable in } R \\ 0 & \text{otherwise}, \end{cases}$$
$$\psi_{A\circ B}(x) = \begin{cases} \bigwedge_{x=ab} \max\{\psi_A(a), \psi_B(b)\} & \text{if } x \text{ is factorizable in } R \\ 1 & \text{otherwise}. \end{cases}$$

Proof. Straightforward. □

Proposition 3.3. *For any neutrosophic sets $A = (\mu_A, \gamma_A, \psi_A)$, $B = (\mu_B, \gamma_B, \psi_B)$, and $C = (\mu_C, \gamma_C, \psi_C)$ in a ring R, we have $A * (B * C) = (A * B) * C$.*

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$, $B = (\mu_B, \gamma_B, \psi_B)$, and $C = (\mu_C, \gamma_C, \psi_C)$ be neutrosophic sets in a ring R and $x \in R$. We assume that x is expressible as $x = a_1b_1c_1 + a_2b_2c_2 + \cdots + a_mb_mc_m$, where $a_i, b_i, c_i \in R$ and $a_ib_ic_i \neq 0$. Then

$$\min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_B(b_1), \mu_B(b_2), \ldots, \mu_B(b_m), \\ \mu_C(c_1), \mu_C(c_2), \ldots, \mu_C(c_m) \end{array}\right\} \leq \min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_{B*C}(b_1c_1), \mu_{B*C}(b_2c_2), \ldots, \mu_{B*C}(b_mc_m) \end{array}\right\}$$
$$\leq \mu_{A*(B*C)}(x),$$

$$\min\left\{\begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_B(b_1), \gamma_B(b_2), \ldots, \gamma_B(b_m), \\ \gamma_C(c_1), \gamma_C(c_2), \ldots, \gamma_C(c_m) \end{array}\right\} \leq \min\left\{\begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_{B*C}(b_1c_1), \gamma_{B*C}(b_2c_2), \ldots, \gamma_{B*C}(b_mc_m) \end{array}\right\}$$
$$\leq \gamma_{A*(B*C)}(x),$$

$$\max\left\{\begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_B(b_1), \psi_B(b_2), \ldots, \psi_B(b_m), \\ \psi_C(c_1), \psi_C(c_2), \ldots, \psi_C(c_m) \end{array}\right\} \geq \max\left\{\begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_{B*C}(b_1c_1), \psi_{B*C}(b_2c_2), \ldots, \psi_{B*C}(b_mc_m) \end{array}\right\}$$
$$\geq \psi_{A*(B*C)}(x).$$

Now,

$$\mu_{(A*B)*C}(x) = \bigvee_{x=\sum_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l} \mu_{A*B}(a_1), \mu_{A*B}(a_2), \ldots, \mu_{A*B}(a_m), \\ \mu_C(b_1), \mu_C(b_2), \ldots, \mu_C(b_m) \end{array}\right\}$$

$$= \bigvee_{x=\sum_{\text{finite}} a_ib_i} \left(\bigvee_{a_1=\sum_{\text{finite}} a_{1i}b_{i1}} \left(\bigvee_{a_2=\sum_{\text{finite}} a_{2i}b_{i2}} \left(\cdots \left(\bigvee_{a_m=\sum_{\text{finite}} a_{mi}b_{im}} \min\left\{\begin{array}{l} p(a_1), \\ p(a_2), \\ \ldots \\ p(a_m), \\ p(b_1), \\ p(b) \end{array}\right\}\right) \cdots \right)\right)\right), \text{ where}$$

$$p(a_1) = \mu_A(a_{11}), \mu_A(a_{12}), \ldots, \mu_A(a_{1_{m_1}}),$$
$$p(a_2) = \mu_A(a_{21}), \mu_A(a_{22}), \ldots, \mu_A(a_{2_{m_2}}),$$
$$\vdots$$
$$p(a_m) = \mu_A(a_{m1}), \mu_A(a_{m2}), \ldots, \mu_A(a_{m_{m_m}})$$
$$p(b_1) = \mu_B(b_{11}), \mu_B(b_{11}), \ldots, \mu_B(b_{11}),$$
$$p(b) = \mu_C(b_1), \mu_C(b_2), \ldots, \mu_C(b_m)$$

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$\leq \mu_{A*(B*C)}(x)$,

$$\gamma_{(A*B)*C}(x) = \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min \left\{ \begin{array}{l} \gamma_{A*B}(a_1), \gamma_{A*B}(a_2), \ldots, \gamma_{A*B}(a_m), \\ \gamma_C(b_1), \gamma_C(b_2), \ldots, \gamma_C(b_m) \end{array} \right\}$$

$$= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \left(\bigvee_{a_1=\sum_{\text{finite}} a_{1i} b_{i1}} \left(\bigvee_{a_2=\sum_{\text{finite}} a_{2i} b_{i2}} \left(\cdots \left(\bigvee_{a_m=\sum_{\text{finite}} a_{mi} b_{im}} \min \left\{ \begin{array}{l} q(a_1), \\ q(a_2), \\ \ldots \\ q(a_m), \\ q(b_1), \\ q(b) \end{array} \right\} \right) \cdots \right) \right) \right), \text{ where}$$

$$q(a_1) = \gamma_A(a_{11}), \gamma_A(a_{12}), \ldots, \gamma_A(a_{1_{m_1}}),$$

$$q(a_2) = \gamma_A(a_{21}), \gamma_A(a_{22}), \ldots, \gamma_A(a_{2_{m_2}}),$$

$$\vdots$$

$$q(a_m) = \gamma_A(a_{m1}), \gamma_A(a_{m2}), \ldots, \gamma_A(a_{m_{m_m}})$$

$$q(b_1) = \gamma_B(b_{11}), \gamma_B(b_{11}), \ldots, \gamma_B(b_{11}),$$

$$q(b) = \gamma_C(b_1), \gamma_C(b_2), \ldots, \gamma_C(b_m)$$

$\leq \gamma_{A*(B*C)}(x)$,

$$\psi_{(A*B)*C}(x) = \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \max \left\{ \begin{array}{l} \psi_{A*B}(a_1), \psi_{A*B}(a_2), \ldots, \psi_{A*B}(a_m), \\ \psi_C(b_1), \psi_C(b_2), \ldots, \psi_C(b_m) \end{array} \right\}$$

$$= \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \left(\bigwedge_{a_1=\sum_{\text{finite}} a_{1i} b_{i1}} \left(\bigwedge_{a_2=\sum_{\text{finite}} a_{2i} b_{i2}} \left(\cdots \left(\bigwedge_{a_m=\sum_{\text{finite}} a_{mi} b_{im}} \min \left\{ \begin{array}{l} r(a_1), \\ r(a_2), \\ \ldots \\ r(a_m), \\ r(b_1), \\ r(b) \end{array} \right\} \right) \cdots \right) \right) \right), \text{ where}$$

$$r(a_1) = \psi_A(a_{11}), \psi_A(a_{12}), \ldots, \psi_A(a_{1_{m_1}}),$$

$$r(a_2) = \psi_A(a_{21}), \psi_A(a_{22}), \ldots, \psi_A(a_{2_{m_2}}),$$

$$\vdots$$

$$r(a_m) = \psi_A(a_{m1}), \psi_A(a_{m2}), \ldots, \psi_A(a_{m_{m_m}})$$

$$r(b_1) = \psi_B(b_{11}), \psi_B(b_{11}), \ldots, \psi_B(b_{11}),$$

$$r(b) = \psi_C(b_1), \psi_C(b_2), \ldots, \psi_C(b_m)$$

$\leq \psi_{A*(B*C)}(x)$.
Hence, $A * (B * C) \subseteq (A * B) * C$. Similarly, $(A * B) * C \subseteq A * (B * C)$. If x is not expressible as $x = a_1b_1c_1 + a_2b_2c_2 + \cdots + a_mb_mc_m$, then $\mu_{(A*B)*C}(x) = 0 = \mu_{A*(B*C)}(x)$, $\gamma_{(A*B)*C}(x) = 0 = \gamma_{A*(B*C)}(x)$, and $\psi_{(A*B)*C}(x) = 1 = \psi_{A*(B*C)}(x)$. Hence, $A * (B * C) = (A * B) * C$. □

Theorem 3.4. *Let $A = (\mu_A, \gamma_A, \psi_A)$, $B = (\mu_B, \gamma_B, \psi_B)$, and $C = (\mu_C, \gamma_C, \psi_C)$ be neutrosophic sets in a ring R. Then*

(1) $A * (B + C) \subseteq (A * B) + (A * C)$, and the equality is valid if

$$(\forall x \in R) \left(\begin{array}{l} \mu_C(0) = \mu_B(0) \geq \max\{\mu_B(x), \mu_C(x)\} \\ \gamma_C(0) = \gamma_B(0) \geq \max\{\gamma_B(x), \gamma_C(x)\} \\ \psi_C(0) = \psi_B(0) \leq \min\{\psi_B(x), \psi_C(x)\} \end{array} \right),$$

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(2) $(A+B)*C \subseteq (A*C)+(B*C)$*, and the equality is valid if*

$$(\forall x \in R)\left(\begin{array}{l}\mu_C(0)=\mu_B(0)\geq \max\{\mu_B(x),\mu_C(x)\}\\ \gamma_C(0)=\gamma_B(0)\geq \max\{\gamma_B(x),\gamma_C(x)\}\\ \psi_C(0)=\psi_B(0)\leq \min\{\psi_B(x),\psi_C(x)\}\end{array}\right).$$

Proof. (1) Let $x \in R$. If x is expressible in the form

$$x = a_1(u_1+v_1)+a_2(u_2+v_2)+\cdots+a_m(u_m+v_m),$$

where no term is zero, then

$$\begin{aligned}(\mu_{A*B}+\mu_{A*B})(x) &\geq \min\{\mu_{A*B}(\sum_{i=1}^{m}a_iu_i),\mu_{A*C}(\sum_{i=1}^{m}a_iv_i)\}\\ &\geq \min\left\{\begin{array}{l}\min\{\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\mu_B(u_1),\mu_B(u_2),\ldots,\mu_B(u_m)\},\\ \min\{\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\mu_B(v_1),\mu_C(v_2),\ldots,\mu_C(v_m)\}\end{array}\right\}\\ &= \min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\min\{\mu_B(u_1),\mu_C(v_1)\},\\ \min\{\mu_B(u_2),\mu_C(v_2)\},\ldots,\min\{\mu_B(u_m),\mu_C(v_m)\}\end{array}\right\},\end{aligned}$$

$$\begin{aligned}(\gamma_{A*B}+\gamma_{A*B})(x) &\geq \min\{\gamma_{A*B}(\sum_{i=1}^{m}a_iu_i),\gamma_{A*C}(\sum_{i=1}^{m}a_iv_i)\}\\ &\geq \min\left\{\begin{array}{l}\min\{\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\gamma_B(u_1),\gamma_B(u_2),\ldots,\gamma_B(u_m)\},\\ \min\{\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\gamma_B(v_1),\gamma_C(v_2),\ldots,\gamma_C(v_m)\}\end{array}\right\}\\ &= \min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\min\{\gamma_B(u_1),\gamma_C(v_1)\},\\ \min\{\gamma_B(u_2),\gamma_C(v_2)\},\ldots,\min\{\gamma_B(u_m),\gamma_C(v_m)\}\end{array}\right\},\end{aligned}$$

$$\begin{aligned}(\psi_{A*B}+\psi_{A*B})(x) &\leq \max\{\psi_{A*B}(\sum_{i=1}^{m}a_iu_i),\psi_{A*C}(\sum_{i=1}^{m}a_iv_i)\}\\ &\leq \max\left\{\begin{array}{l}\max\{\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\psi_B(u_1),\psi_B(u_2),\ldots,\psi_B(u_m)\},\\ \max\{\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\psi_B(v_1),\psi_C(v_2),\ldots,\psi_C(v_m)\}\end{array}\right\}\\ &= \max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\max\{\psi_B(u_1),\psi_C(v_1)\},\\ \max\{\psi_B(u_2),\psi_C(v_2)\},\ldots,\max\{\psi_B(u_m),\psi_C(v_m)\}\end{array}\right\}.\end{aligned}$$

Then

$$\mu_{A*(B+C)}(x) = \bigvee_{x=\sum_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m)\\ \mu_{B+C}(b_1),\mu_{B+C}(b_2),\ldots,\mu_{B+C}(b_m)\end{array}\right\}$$

$$= \bigvee_{\substack{x=\\ \sum_{\text{finite}} a_{1i}b_{i1}}}\left(\bigvee_{\substack{b_1=\\ u_1+v_1}}\left(\bigvee_{\substack{b_2=\\ u_2+v_2}}\left(\cdots\left(\bigvee_{\substack{b_m=\\ u_m+v_m}}\min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \min\{\mu_B(u_1),\mu_C(v_1)\},\ldots,\\ \min\{\mu_B(u_m),\mu_C(v_m)\}\end{array}\right\}\right)\cdots\right)\right)\right)$$

$\leq \mu_{(A*B)+(A*C)}(x)$,

$$\gamma_{A*(B+C)}(x) = \bigvee_{x=\sum_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m)\\ \gamma_{B+C}(b_1),\gamma_{B+C}(b_2),\ldots,\gamma_{B+C}(b_m)\end{array}\right\}$$

$$= \bigvee_{\substack{x=\\ \sum_{\text{finite}} a_{1i}b_{i1}}}\left(\bigvee_{\substack{b_1=\\ u_1+v_1}}\left(\bigvee_{\substack{b_2=\\ u_2+v_2}}\left(\cdots\left(\bigvee_{\substack{b_m=\\ u_m+v_m}}\min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \min\{\gamma_B(u_1),\gamma_C(v_1)\},\ldots,\\ \min\{\gamma_B(u_m),\gamma_C(v_m)\}\end{array}\right\}\right)\cdots\right)\right)\right)$$

$\leq \gamma_{(A*B)+(A*C)}(x)$,

$$\psi_{A*(B+C)}(x) = \bigwedge_{x=\sum_{\text{finite}} a_ib_i} \max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m)\\ \psi_{B+C}(b_1),\psi_{B+C}(b_2),\ldots,\psi_{B+C}(b_m)\end{array}\right\}$$

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$$= \bigwedge_{\substack{x=\\ \sum_{\text{finite}} a_{1i}b_{i1}}} \left(\bigwedge_{\substack{b_1=\\ u_1+v_1}} \left(\bigwedge_{\substack{b_2=\\ u_2+v_2}} \left(\cdots \left(\bigwedge_{\substack{b_m=\\ u_m+v_m}} \max \left\{ \begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \max\{\psi_B(u_1), \psi_C(v_1)\}, \ldots, \\ \max\{\psi_B(u_m), \psi_C(v_m)\} \end{array} \right\} \right) \cdots \right) \right) \right)$$

$\leq \psi_{(A*B)+(A*C)}(x)$.

If x is not expressible in the form $x = a_1(u_1+v_1)+a_2(u_2+v_2)+\cdots+a_m(u_m+v_m)$, then $\mu_{A*(B+C)}(x) = 0 = \mu_A(A*B)+(A*C)(x)$, $\gamma_{A*(B+C)}(x) = 0 = \gamma_A(A*B)+(A*C)(x)$, and $\psi_{A*(B+C)}(x) = 1 = \psi_A(A*B)+(A*C)(x)$. Now, suppose that $\mu_C(0) = \mu_B(0) \geq \max\{\mu_B(x), \mu_C(x)\}$, $\gamma_C(0) = \gamma_B(0) \geq \max\{\gamma_B(x), \gamma_C(x)\}$, and $\psi_C(0) = \psi_B(0) \leq \min\{\psi_B(x), \psi_C(x)\}$ for all $x \in R$. Let $x \in R$ and suppose that x is expressible in the form $x = \sum_{i=1}^{m} a_ib_i + \sum_{j=1}^{n} c_jd_j$, where $a_ib_i \neq 0$ and $c_jd_j \neq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then

$$x = \sum_{i=1}^{m} a_i(b_i+0) + \sum_{j=1}^{n} c_j(d_j+0)$$

and so

$$\begin{aligned} \mu_{A*(B+C)}(x) &= \mu_{A*(B+C)}\left(\sum_{i=1}^{m} a_ib_i + \sum_{j=1}^{n} c_jd_j\right) \\ &\geq \min\left\{ \begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_A(c_1), \mu_A(c_2), \ldots, \mu_A(c_n), \\ \mu_{B+C}(b_1), \mu_{B+C}(b_2), \ldots, \mu_{B+C}(b_m), \\ \mu_{B+C}(d_1), \mu_{B+C}(d_2), \ldots, \mu_{B+C}(d_n) \end{array} \right\} \\ &\geq \min\left\{ \begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_A(c_1), \mu_A(c_2), \ldots, \mu_A(c_n), \\ \mu_B(b_1), \mu_C(0), \mu_B(b_2), \mu_C(0), \ldots, \mu_B(b_m), \mu_C(0), \\ \mu_B(0), \mu_C(d_1), \mu_B(0), \mu_C(d_2), \ldots, \mu_B(0), \mu_C(d_n) \end{array} \right\} \\ &\geq \min\left\{ \begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_A(c_1), \mu_A(c_2), \ldots, \mu_A(c_n), \\ \mu_B(b_1), \mu_B(b_2), \ldots, \mu_B(b_m), \\ \mu_C(d_1), \mu_C(d_2), \ldots, \mu_C(d_n) \end{array} \right\}, \end{aligned}$$

$$\begin{aligned} \gamma_{A*(B+C)}(x) &= \gamma_{A*(B+C)}\left(\sum_{i=1}^{m} a_ib_i + \sum_{j=1}^{n} c_jd_j\right) \\ &\geq \min\left\{ \begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_A(c_1), \gamma_A(c_2), \ldots, \gamma_A(c_n), \\ \gamma_{B+C}(b_1), \gamma_{B+C}(b_2), \ldots, \gamma_{B+C}(b_m), \\ \gamma_{B+C}(d_1), \gamma_{B+C}(d_2), \ldots, \gamma_{B+C}(d_n) \end{array} \right\} \\ &\geq \min\left\{ \begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_A(c_1), \gamma_A(c_2), \ldots, \gamma_A(c_n), \\ \gamma_B(b_1), \gamma_C(0), \gamma_B(b_2), \gamma_C(0), \ldots, \gamma_B(b_m), \gamma_C(0), \\ \gamma_B(0), \gamma_C(d_1), \gamma_B(0), \gamma_C(d_2), \ldots, \gamma_B(0), \gamma_C(d_n) \end{array} \right\} \\ &\geq \min\left\{ \begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_A(c_1), \gamma_A(c_2), \ldots, \gamma_A(c_n), \\ \gamma_B(b_1), \gamma_B(b_2), \ldots, \gamma_B(b_m), \\ \gamma_C(d_1), \gamma_C(d_2), \ldots, \gamma_C(d_n) \end{array} \right\}, \end{aligned}$$

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$$\psi_{A*(B+C)}(x) = \psi_{A*(B+C)}\left(\sum_{i=1}^{m} a_i b_i + \sum_{j=1}^{n} c_j d_j\right)$$
$$\leq \max\left\{\begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_A(c_1), \psi_A(c_2), \ldots, \psi_A(c_n), \\ \psi_{B+C}(b_1), \psi_{B+C}(b_2), \ldots, \psi_{B+C}(b_m), \\ \psi_{B+C}(d_1), \psi_{B+C}(d_2), \ldots, \psi_{B+C}(d_n) \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_A(c_1), \psi_A(c_2), \ldots, \psi_A(c_n), \\ \psi_B(b_1), \psi_C(0), \psi_B(b_2), \psi_C(0), \ldots, \psi_B(b_m), \psi_C(0), \\ \psi_B(0), \psi_C(d_1), \psi_B(0), \psi_C(d_2), \ldots, \psi_B(0), \psi_C(d_n) \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_A(c_1), \psi_A(c_2), \ldots, \psi_A(c_n), \\ \psi_B(b_1), \psi_B(b_2), \ldots, \psi_B(b_m), \\ \psi_C(d_1), \psi_C(d_2), \ldots, \psi_C(d_n) \end{array}\right\}.$$

It follows that

$$\mu_{(A*B)+(A*C)}(x) = \bigvee_{x=u+v} \min\{\mu_{A*B}(u), \mu_{A*C}(v)\}$$
$$= \bigvee_{x=u+v}\left(\bigvee_{u=\sum_{i=1}^{m} a_i b_i}\left(\bigvee_{v=\sum_{j=1}^{n} c_i d_i} \min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_A(c_1), \mu_A(c_2), \ldots, \mu_A(c_n), \\ \mu_B(b_1), \mu_B(b_2), \ldots, \mu_B(b_m), \\ \mu_C(d_1), \mu_C(d_2), \ldots, \mu_C(d_n) \end{array}\right\}\right)\right)$$
$$\leq \mu_{A*(B+C)}(x),$$

$$\gamma_{(A*B)+(A*C)}(x) = \bigvee_{x=u+v} \min\{\gamma_{A*B}(u), \gamma_{A*C}(v)\}$$
$$= \bigvee_{x=u+v}\left(\bigvee_{u=\sum_{i=1}^{m} a_i b_i}\left(\bigvee_{v=\sum_{j=1}^{n} c_i d_i} \min\left\{\begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_A(c_1), \gamma_A(c_2), \ldots, \gamma_A(c_n), \\ \gamma_B(b_1), \gamma_B(b_2), \ldots, \gamma_B(b_m), \\ \gamma_C(d_1), \gamma_C(d_2), \ldots, \gamma_C(d_n) \end{array}\right\}\right)\right)$$
$$\leq \gamma_{A*(B+C)}(x),$$

$$\psi_{(A*B)+(A*C)}(x)$$
$$= \bigwedge_{x=u+v} \max\{\psi_{A*B}(u), \psi_{A*C}(v)\}$$
$$= \bigwedge_{x=u+v}\left(\bigwedge_{u=\sum_{i=1}^{m} a_i b_i}\left(\bigwedge_{v=\sum_{j=1}^{n} c_i d_i} \min\left\{\begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_A(c_1), \psi_A(c_2), \ldots, \psi_A(c_n), \\ \psi_B(b_1), \psi_B(b_2), \ldots, \psi_B(b_m), \\ \psi_C(d_1), \psi_C(d_2), \ldots, \psi_C(d_n) \end{array}\right\}\right)\right)$$
$$\leq \psi_{A*(B+C)}(x).$$

(2) Similar to the proof of (1). □

Lemma 3.5. *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic set in a ring R, then $A * A \subseteq A$ if and only if it satisfies the following conditions:*

$$\min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_A(b_1), \mu_A(b_2), \ldots, \mu_A(b_m) \end{array}\right\} \leq \mu_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m),$$

$$\min\left\{\begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_A(b_1), \gamma_A(b_2), \ldots, \gamma_A(b_m) \end{array}\right\} \leq \gamma_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m),$$

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$$\max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1),\psi_A(b_2),\ldots,\psi_A(b_m)\end{array}\right\}\geq\psi_A(a_1b_1+a_2b_2+\cdots+a_mb_m),$$

where $a_i, b_i \in R$

Proof. Straightforward. □

Theorem 3.6. *For a neutrosophic set* $A = (\mu_A, \gamma_A, \psi_A)$ *in a ring* R*, the following are equivalent:*

(1) $A = (\mu_A, \gamma_A, \psi_A)$ *is a neutrosophic subring of* R*,*

(2) $A = (\mu_A, \gamma_A, \psi_A)$ *is a neutrosophic subgroup of the additive group* $(R, +)$ *and* $A * A \subseteq A$*.*

Proof. (1)⇒(2): The condition $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\gamma_A(x-y) \geq \min\{\gamma_A(x), \gamma_A(y)\}$, and $\psi_A(x-y) \leq \max\{\psi_A(x), \psi_A(y)\}$ is equivalent to saying that $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of the additive group $(R, +)$. Let $a_i, b_i \in R$ where $i = 1, 2, \ldots, m$. Then

$$\begin{aligned}\min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \mu_A(b_1),\mu_A(b_2),\ldots,\mu_A(b_m)\end{array}\right\}&\leq\min\{\mu_A(a_1b_1),\mu_A(a_2b_2),\ldots,\mu_A(a_mb_m)\}\\&\leq\mu_a(a_1b_1+a_2b_2+\cdots+a_mb_m),\end{aligned}$$

$$\begin{aligned}\min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \gamma_A(b_1),\gamma_A(b_2),\ldots,\gamma_A(b_m)\end{array}\right\}&\leq\min\{\gamma_A(a_1b_1),\gamma_A(a_2b_2),\ldots,\gamma_A(a_mb_m)\}\\&\leq\gamma_a(a_1b_1+a_2b_2+\cdots+a_mb_m),\end{aligned}$$

$$\begin{aligned}\max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1),\psi_A(b_2),\ldots,\psi_A(b_m)\end{array}\right\}&\geq\max\{\psi_A(a_1b_1),\psi_A(a_2b_2),\ldots,\psi_A(a_mb_m)\}\\&\geq\psi_a(a_1b_1+a_2b_2+\cdots+a_mb_m),\end{aligned}$$

which shows from Lemma 3.5 that $A * A \subseteq A$.
(2)⇒(1): Applying Lemma 3.5 and the definition of neutrosophic subring, we have the desired result. □

Lemma 3.7. *For any neutrosophic set* $A = (\mu_A, \gamma_A, \psi_A)$ *in a ring* R*, the following are equivalent:*

(1) $A \circ A \subseteq A$*,*

(2) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$*,* $\gamma_A(xy) \geq \min\{\gamma_A(x), \gamma_A(y)\}$*, and* $\psi_A(xy) \leq \max\{\psi_A(x), \psi_A(y)\}$ *for all* $x, y \in R$*.*

Proof. Straightforward. □

Theorem 3.8. *If* $A = (\mu_A, \gamma_A, \psi_A)$ *and* $B = (\mu_B, \gamma_B, \psi_B)$ *are neutrosophic subrings of a ring* R *such that* $A * B = B * A$*, then* $A * B$ *is a neutrosophic subring of* R*.*

Proof. For any $x, y \in R$, let $x = \sum_{i=1}^{m} a_ib_i$ and $y = \sum_{j=1}^{n} c_jd_j$. Then

$$x - y = \sum_{i=1}^{m} a_ib_i + \sum_{j=1}^{n}(-c_j)d_j,$$

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and so

$$\mu_{A*B}(x-y) = \mu_{A*B}\left(\sum_{i=1}^{m} a_i b_i + \sum_{j=1}^{n}(-c_j)d_j\right) \geq \min\left\{\begin{array}{l}\mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m),\\ \mu_A(-c_1), \mu_A(-c_2), \ldots, \mu_A(-c_n),\\ \mu_A(b_1), \mu_A(b_2), \ldots, \mu_A(b_n),\\ \mu_A(d_1), \mu_A(d_2), \ldots, \mu_A(d_n)\end{array}\right\},$$

$$\gamma_{A*B}(x-y) = \gamma_{A*B}\left(\sum_{i=1}^{m} a_i b_i + \sum_{j=1}^{n}(-c_j)d_j\right) \geq \min\left\{\begin{array}{l}\gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m),\\ \gamma_A(-c_1), \gamma_A(-c_2), \ldots, \gamma_A(-c_n),\\ \gamma_A(b_1), \gamma_A(b_2), \ldots, \gamma_A(b_n),\\ \gamma_A(d_1), \gamma_A(d_2), \ldots, \gamma_A(d_n)\end{array}\right\},$$

$$\psi_{A*B}(x-y) = \psi_{A*B}\left(\sum_{i=1}^{m} a_i b_i + \sum_{j=1}^{n}(-c_j)d_j\right) \leq \max\left\{\begin{array}{l}\psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m),\\ \psi_A(-c_1), \psi_A(-c_2), \ldots, \psi_A(-c_n),\\ \psi_A(b_1), \psi_A(b_2), \ldots, \psi_A(b_n),\\ \psi_A(d_1), \psi_A(d_2), \ldots, \psi_A(d_n)\end{array}\right\}.$$

It follows that

$$\begin{aligned}\mu_{A*B}(x-y) &\geq \bigvee_{x=\sum_{\text{finite}} a_i b_i}\left(\bigvee_{y=\sum_{\text{finite}} c_j d_j} \min\left\{\begin{array}{l}\mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m),\\ \mu_A(-c_1), \mu_A(-c_2), \ldots, \mu_A(-c_n),\\ \mu_A(b_1), \mu_A(b_2), \ldots, \mu_A(b_n),\\ \mu_A(d_1), \mu_A(d_2), \ldots, \mu_A(d_n)\end{array}\right\}\right)\\ &= \min\{\mu_{A*B}(x), \mu_{A*B}(y)\},\end{aligned}$$

$$\begin{aligned}\gamma_{A*B}(x-y) &\geq \bigvee_{x=\sum_{\text{finite}} a_i b_i}\left(\bigvee_{y=\sum_{\text{finite}} c_j d_j} \min\left\{\begin{array}{l}\gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m),\\ \gamma_A(-c_1), \gamma_A(-c_2), \ldots, \gamma_A(-c_n),\\ \gamma_A(b_1), \gamma_A(b_2), \ldots, \gamma_A(b_n),\\ \gamma_A(d_1), \gamma_A(d_2), \ldots, \gamma_A(d_n)\end{array}\right\}\right)\\ &= \min\{\gamma_{A*B}(x), \gamma_{A*B}(y)\},\end{aligned}$$

$$\begin{aligned}\psi_{A*B}(x-y) &\leq \bigwedge_{x=\sum_{\text{finite}} a_i b_i}\left(\bigwedge_{y=\sum_{\text{finite}} c_j d_j} \max\left\{\begin{array}{l}\psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m),\\ \psi_A(-c_1), \psi_A(-c_2), \ldots, \psi_A(-c_n),\\ \psi_A(b_1), \psi_A(b_2), \ldots, \psi_A(b_n),\\ \psi_A(d_1), \psi_A(d_2), \ldots, \psi_A(d_n)\end{array}\right\}\right)\\ &= \max\{\psi_{A*B}(x), \psi_{A*B}(y)\},\end{aligned}$$

so that $A * B$ is a neutrosophic subgroup of $(R, +)$. Since $A * B = B * A$ by assumption, it follows from Propositions 3.2 and 3.3 and Theorem 3.6 (2) that

$$\begin{aligned}(A*B)\circ(A*B) &\subseteq (A*B)*(A*B) = A*(B*A)*B\\ &= A*(A*B)*B = (A*A)*(B*B)\\ &\subseteq A*B,\end{aligned}$$

so from Lemma 3.7 that

$$\mu_A(ab) \geq \min\{\mu_{A*B}(a), \mu_{A*B}(b)\},$$
$$\gamma_A(ab) \geq \min\{\gamma_{A*B}(a), \gamma_{A*B}(b)\},$$
$$\psi_A(ab) \geq \min\{\psi_{A*B}(a), \psi_{A*B}(b)\}$$

for all $a, b \in R$. Hence, $A * B$ is a neutrosophic subring of R. □

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Lemma 3.9. *A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ in a ring R is a neutrosophic left (resp., right) ideal of R if and only if*

(1) $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\gamma_A(x-y) \geq \min\{\gamma_A(x), \gamma_A(y)\}$, *and* $\psi_A(x-y) \leq \max\{\psi_A(x), \psi_A(y)\}$,

(2) $1_\sim * A \subseteq A$ *(resp.,* $A * 1_\sim \subseteq A$*).*

Proof. Suppose that $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic left ideal of R. It is clear that condition (1) holds. Let $x \in R$. We assume that x is expressible as $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$, where $a_i, b_i \in R$ and $a_ib_i \neq 0$. Then

$$\begin{aligned}
\mu_{1_\sim * A}(x) &= \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\left\{ \begin{array}{l} 1(a_1), 1(a_2), \ldots, 1(a_m), \\ \mu_A(b_1), \mu_A(b_2), \ldots, \mu_A(b_m) \end{array} \right\} \\
&\leq \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\{1, \mu_A(a_1b_1), \mu_A(a_2b_2), \ldots, \mu_A(a_mb_m)\} \\
&= \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\{\mu_A(a_1b_1), \mu_A(a_2b_2), \ldots, \mu_A(a_mb_m)\} \\
&\leq \mu_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m) \\
&= \mu_A(x),
\end{aligned}$$

$$\begin{aligned}
\gamma_{1_\sim * A}(x) &= \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\left\{ \begin{array}{l} 1(a_1), 1(a_2), \ldots, 1(a_m), \\ \gamma_A(b_1), \gamma_A(b_2), \ldots, \gamma_A(b_m) \end{array} \right\} \\
&\leq \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\{1, \gamma_A(a_1b_1), \gamma_A(a_2b_2), \ldots, \gamma_A(a_mb_m)\} \\
&= \bigvee_{\substack{x=\sum a_ib_i \\ \text{finite}}} \min\{\gamma_A(a_1b_1), \gamma_A(a_2b_2), \ldots, \gamma_A(a_mb_m)\} \\
&\leq \gamma_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m) \\
&= \gamma_A(x),
\end{aligned}$$

$$\begin{aligned}
\psi_{1_\sim * A}(x) &= \bigwedge_{\substack{x=\sum a_ib_i \\ \text{finite}}} \max\left\{ \begin{array}{l} 0(a_1), 0(a_2), \ldots, 0(a_m), \\ \psi_A(b_1), \psi_A(b_2), \ldots, \psi_A(b_m) \end{array} \right\} \\
&\geq \bigwedge_{\substack{x=\sum a_ib_i \\ \text{finite}}} \max\{0, \psi_A(a_1b_1), \psi_A(a_2b_2), \ldots, \psi_A(a_mb_m)\} \\
&= \bigwedge_{\substack{x=\sum a_ib_i \\ \text{finite}}} \max\{\psi_A(a_1b_1), \psi_A(a_2b_2), \ldots, \psi_A(a_mb_m)\} \\
&\geq \psi_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m) \\
&= \psi_A(x),
\end{aligned}$$

which shows that $1_\sim * A \subseteq A$. We remark that x is not expressible as $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$, then $\mu_{1_\sim * A}(x) = 0 \leq \mu_A(x)$, $\gamma_{1_\sim * A}(x) = 0 \leq \gamma_A(x)$, and $\psi_{1_\sim * A}(x) = 1 \geq \psi_A(x)$. Therefore, condition (2) is valid.

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Conversely, assume that (1) and (2) hold. Let $x, y \in R$. By using condition (2), we have

$$\begin{aligned}\mu_A(xy) &\geq \mu_{1\sim * A}(xy)\\ &= \bigvee_{xy=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}1(a_1), 1(a_2), \ldots, 1(a_m),\\ \mu_A(b_1), \mu_A(b_2), \ldots, \mu_A(b_m)\end{array}\right\}\\ &\geq \mu_A(1(x), \mu_A(y))\\ &= \mu_A(y),\end{aligned}$$

$$\begin{aligned}\gamma_A(xy) &\geq \gamma_{1\sim * A}(xy)\\ &= \bigvee_{xy=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}1(a_1), 1(a_2), \ldots, 1(a_m),\\ \gamma_A(b_1), \gamma_A(b_2), \ldots, \gamma_A(b_m)\end{array}\right\}\\ &\geq \gamma_A(1(x), \gamma_A(y))\\ &= \gamma_A(y),\end{aligned}$$

$$\begin{aligned}\psi_A(xy) &\geq \psi_{1\sim * A}(xy)\\ &= \bigvee_{xy=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}0(a_1), 0(a_2), \ldots, 0(a_m),\\ \psi_A(b_1), \psi_A(b_2), \ldots, \psi_A(b_m)\end{array}\right\}\\ &\leq \psi_A(0(x), \psi_A(y))\\ &= \psi_A(y).\end{aligned}$$

This means that $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic left ideal of R. In a similar way, we can prove that $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic right ideal of R. □

Theorem 3.10. *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic right ideal of a ring R and $B = (\mu_B, \gamma_B, \psi_B)$ is a neutrosophic left ideal of R, then $A * B \subseteq A \cap B$.*

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic right ideal of a ring R and $B = (\mu_B, \gamma_B, \psi_B)$ be a neutrosophic left ideal of R and $x \in R$. We assume that x is expressible as $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$, where $a_i, b_i \in R$ and $a_ib_i \neq 0$. Then

$$\begin{aligned}\min\left\{\begin{array}{l}\mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m),\\ \mu_A(b_1), \mu_A(b_2), \ldots, \mu_A(b_n)\end{array}\right\} &\leq \min\{\mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m)\}\\ &\leq \min\{\mu_A(a_1b_1), \mu_A(a_2b_2), \ldots, \mu_A(a_mb_m)\}\\ &\leq \mu_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m),\end{aligned}$$

$$\begin{aligned}\min\left\{\begin{array}{l}\gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m),\\ \gamma_A(b_1), \gamma_A(b_2), \ldots, \gamma_A(b_n)\end{array}\right\} &\leq \min\{\gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m)\}\\ &\leq \min\{\gamma_A(a_1b_1), \gamma_A(a_2b_2), \ldots, \gamma_A(a_mb_m)\}\\ &\leq \gamma_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m),\end{aligned}$$

$$\begin{aligned}\max\left\{\begin{array}{l}\psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m),\\ \psi_A(b_1), \psi_A(b_2), \ldots, \psi_A(b_n)\end{array}\right\} &\geq \min\{\psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m)\}\\ &\geq \min\{\psi_A(a_1b_1), \psi_A(a_2b_2), \ldots, \psi_A(a_mb_m)\}\\ &\geq \psi_A(a_1b_1 + a_2b_2 + \cdots + a_mb_m).\end{aligned}$$

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It follows that

$$\begin{aligned}\mu_{A*B}(x) &= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \mu_A(b_1),\mu_A(b_2),\ldots,\mu_A(b_m)\end{array}\right\}\\ &\leq \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \mu_A(b_1),\mu_A(b_2),\ldots,\mu_A(b_m)\end{array}\right\}\\ &\leq \mu_A(a_1b_1+a_2b_2+\cdots+a_mb_m)\\ &= \mu_A(x),\end{aligned}$$

$$\begin{aligned}\gamma_{A*B}(x) &= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \gamma_A(b_1),\gamma_A(b_2),\ldots,\gamma_A(b_m)\end{array}\right\}\\ &\leq \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \gamma_A(b_1),\gamma_A(b_2),\ldots,\gamma_A(b_m)\end{array}\right\}\\ &\leq \gamma_A(a_1b_1+a_2b_2+\cdots+a_mb_m)\\ &= \gamma_A(x),\end{aligned}$$

$$\begin{aligned}\psi_{A*B}(x) &= \bigwedge_{x=\sum\limits_{\text{finite}} a_ib_i} \max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1),\psi_A(b_2),\ldots,\psi_A(b_m)\end{array}\right\}\\ &\geq \bigwedge_{x=\sum\limits_{\text{finite}} a_ib_i} \max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1),\psi_A(b_2),\ldots,\psi_A(b_m)\end{array}\right\}\\ &\geq \psi_A(a_1b_1+a_2b_2+\cdots+a_mb_m)\\ &= \psi_A(x).\end{aligned}$$

Hence, $\mu_{A*B}(x) \leq \mu_A(x)$, $\gamma_{A*B}(x) \leq \gamma_A(x)$, and $\psi_{A*B}(x) \geq \psi_A(x)$. We remark that x is not expressible as $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$, then $\mu_{A*B}(x) = 0 \leq \mu_A(x)$, $\gamma_{A*B}(x) = 0 \leq \gamma_A(x)$, and $\psi_{A*B}(x) = 1 \geq \psi_A(x)$. Thus, $\mu_{A*B}(x) \leq \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \wedge \mu_B)(x)$, $\gamma_{A*B}(x) \leq \min\{\gamma_A(x), \gamma_B(x)\} = (\gamma_A \wedge \gamma_B)(x)$, and $\psi_{A*B}(x) \geq \max\{\psi_A(x), \psi_B(x)\} = (\psi_A \vee \psi_B)(x)$. Therefore, $A * B \subseteq A \cap B$. □

Theorem 3.11. *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic left ideal of a ring R and $B = (\mu_B, \gamma_B, \psi_B)$ is a neutrosophic right ideal of R, then $A * B$ is a neutrosophic ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic left ideal of a ring R and $B = (\mu_B, \gamma_B, \psi_B)$ be a neutrosophic right ideal of R. The first part of the Theorem 3.8 shows that $A * B$ is a neutrosophic subgroup of $(R, +)$. For any $x, y \in R$, we assume that x is expressible as $x = a_1b_1 + a_2b_2 + \cdots + a_mb_m$, where $a_i, b_i \in R$ and $a_ib_i \neq 0$. Then $xy = a_1(b_1y) + a_2(b_2y) + \cdots + a_m(b_my)$, and so

$$\begin{aligned}\min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \mu_A(b_1),\mu_A(b_2),\ldots,\mu_A(b_m)\end{array}\right\} &\leq \min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \mu_A(b_1y),\mu_A(b_2y),\ldots,\mu_A(b_my)\end{array}\right\}\\ &\leq \mu_{A*B}(xy),\end{aligned}$$

$$\begin{aligned}\min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \gamma_A(b_1),\gamma_A(b_2),\ldots,\gamma_A(b_m)\end{array}\right\} &\leq \min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \gamma_A(b_1y),\gamma_A(b_2y),\ldots,\gamma_A(b_my)\end{array}\right\}\\ &\leq \gamma_{A*B}(xy),\end{aligned}$$

$$\begin{aligned}\max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1),\psi_A(b_2),\ldots,\psi_A(b_m)\end{array}\right\} &\geq \max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1y),\psi_A(b_2y),\ldots,\psi_A(b_my)\end{array}\right\}\\ &\geq \psi_{A*B}(xy).\end{aligned}$$

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It follows that

$$\begin{aligned}\mu_{A*B}(x) &= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \mu_A(b_1),\mu_A(b_2),\ldots,\mu_A(b_m)\end{array}\right\}\\ &\leq \left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \mu_A(b_1),\mu_A(b_2),\ldots,\mu_A(b_m)\end{array}\right\}\\ &\leq \left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_2),\ldots,\mu_A(a_m),\\ \mu_A(b_1y),\mu_A(b_2y),\ldots,\mu_A(b_my)\end{array}\right\}\\ &= \mu_{A*B}(xy),\end{aligned}$$

$$\begin{aligned}\gamma_{A*B}(x) &= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \gamma_A(b_1),\gamma_A(b_2),\ldots,\gamma_A(b_m)\end{array}\right\}\\ &\leq \left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \gamma_A(b_1),\gamma_A(b_2),\ldots,\gamma_A(b_m)\end{array}\right\}\\ &\leq \left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_2),\ldots,\gamma_A(a_m),\\ \gamma_A(b_1y),\gamma_A(b_2y),\ldots,\gamma_A(b_my)\end{array}\right\}\\ &= \gamma_{A*B}(xy),\end{aligned}$$

$$\begin{aligned}\psi_{A*B}(x) &= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1),\psi_A(b_2),\ldots,\psi_A(b_m)\end{array}\right\}\\ &\geq \left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1),\psi_A(b_2),\ldots,\psi_A(b_m)\end{array}\right\}\\ &\geq \left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_2),\ldots,\psi_A(a_m),\\ \psi_A(b_1y),\psi_A(b_2y),\ldots,\psi_A(b_my)\end{array}\right\}\\ &= \psi_{A*B}(xy).\end{aligned}$$

Hence, $\mu_{A*B}(x) \leq \mu_A(xy)$, $\gamma_{A*B}(x) \leq \gamma_A(xy)$, and $\psi_{A*B}(x) \geq \psi_A(xy)$. Similarly, we get $\mu_{A*B}(y) \leq \mu_A(xy)$, $\gamma_{A*B}(y) \leq \gamma_A(xy)$, and $\psi_{A*B}(y) \geq \psi_A(xy)$. Thus, $A*B$ is a neutrosophic ideal of R. □

Definition 3.12. A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ in a ring R is called a neutrosophic quasi-ideal of R if

$$(\forall x, y \in R)\left(\begin{array}{l}\mu_A(x-y) \geq \min\{\mu_A(x),\mu_A(y)\}\\ \gamma_A(x-y) \geq \min\{\gamma_A(x),\gamma_A(y)\}\\ \psi_A(x-y) \leq \max\{\psi_A(x),\psi_A(y)\}\end{array}\right),$$

$$(A*1_\sim)\cap(1_\sim * A) \subseteq A.$$

Example 3.13. Let $R = \left\{0 = \begin{pmatrix}0&0\\0&0\end{pmatrix}, a = \begin{pmatrix}1&0\\0&0\end{pmatrix}, b = \begin{pmatrix}0&1\\0&0\end{pmatrix}, c = \begin{pmatrix}1&1\\0&0\end{pmatrix}\right\}$. Then $R = \{0, a, b, c\}$ is a ring of matrices under matrix addition and multiplication modulo 2. We define a neutrosophic set $A =$

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

$(\mu_A, \gamma_A, \psi_A)$ as follows: It is easy to verify that $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic quasi-ideal of R, but it is not a neutrosophic right ideal of R.

Definition 3.14. A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ in a ring R is called a neutrosophic bi-ideal of R if

$$(\forall x, y \in R)\left(\begin{array}{l}\mu_A(x-y) \geq \min\{\mu_A(x),\mu_A(y)\}\\ \gamma_A(x-y) \geq \min\{\gamma_A(x),\gamma_A(y)\}\\ \psi_A(x-y) \leq \max\{\psi_A(x),\psi_A(y)\}\end{array}\right),$$

$$A*A \subseteq A \text{ and } A*1_\sim * A \subseteq A.$$

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R	0	1	2	3
μ_A	0.8	0.1	0.3	0.2
γ_A	0.7	0.0	0.2	0.1
ψ_A	0.4	0.9	0.6	0.8

Example 3.15. Let $R = \{0, a, b, c\}$ be a ring with the following tables: We define a neutrosophic set $A =$

$-$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	0
c	0	b	0	a

$(\mu_A, \gamma_A, \psi_A)$ as follows: It is easy to verify that $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic bi-ideal of R, but it is

X	0	1	2	3
μ_A	1.0	0.6	0.0	0.0
γ_A	0.7	0.0	0.2	0.1
ψ_A	0.0	0.2	1.0	1.0

not a neutrosophic quasi-ideal of R.

The proof of the following two lemmas is straightforward and hence omitted.

Lemma 3.16. *Every neutrosophic left (right, two-sided) ideal of a ring R is a neutrosophic quasi-ideal of R.*

Lemma 3.17. *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic right ideal of a ring R and $B = (\mu_B, \gamma_B, \psi_B)$ is a neutrosophic left ideal of R, then $A \cap B$ is a neutrosophic quasi-ideal of R.*

Theorem 3.18. *Every neutrosophic quasi-ideal of a ring R is a neutrosophic bi-ideal of R.*

Proof. Let A be any neutrosophic quasi-ideal of R. Then $A*A = (A*A)\cap(A*A) \subseteq (A*1_\sim)\cap(1_\sim*A) \subseteq A$. Since $A * 1_\sim * A \subseteq A * (1_\sim * 1_\sim) \subseteq (A * 1_\sim)$ and $A * 1_\sim * A \subseteq (1_\sim * 1_\sim) * A \subseteq (1_\sim * A)$, we have $1_\sim * A * 1_\sim \subseteq (A * 1_\sim) \cap (1_\sim * A) \subseteq A$. Hence, $A * B$ is a neutrosophic bi-ideal of R. □

The converse of Theorem 3.18 is not true in general. For example, the neutrosophic bi-ideal $B = (\mu_B, \gamma_B, \psi_B)$ described in Example 3.15 is not a neutrosophic quasi-ideal of R.

Theorem 3.19. *The intrinsic product of two neutrosophic quasi-ideals of a ring R is a neutrosophic bi-ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be any two neutrosophic quasi-ideals of R. From Theorem 3.18, it follows that B is a neutrosophic bi-ideal of R. Then $A * 1_\sim * A \subseteq A$ and so

$$(A * B) * (A * B) = (A * B * A) * B \subseteq (A * 1_\sim * A) * B \subseteq A * B$$

and

$$\begin{aligned}(A * B) * 1_\sim(A * B) &= A * (B * (A * 1_\sim) * B) \\ &\subseteq A * (B * (1_\sim * 1_\sim) * B) \\ &\subseteq A * (B * 1_\sim * B) \\ &\subseteq A * B.\end{aligned}$$

Hence, $A * B$ is a neutrosophic bi-ideal of R. □

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4 Regular rings

A ring R is said to be regular if for each element x of R, there exists an element $a \in R$ such that $x = xax$. A ring R is regular if and only if $AB = A \cap B$ whenever A is a right ideal of R and B is a left ideal of R. The corresponding neutrosophic ideals nicely simulate this basic concept.

For a subset X of a ring R, we denote $\widetilde{X} = \{\langle x, \mu_{\widetilde{X}}(x), \gamma_{\widetilde{X}}(x), \psi_{\widetilde{X}}(x)\rangle \mid x \in R\}$ defined by $\mu_{\widetilde{X}}(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise,} \end{cases}$ $\gamma_{\widetilde{X}}(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise,} \end{cases}$ $\psi_{\widetilde{X}}(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$ for all $x \in R$. For the sake of simplicity, we shall use the symbol $\widetilde{X} = (\mu_{\widetilde{X}}, \gamma_{\widetilde{X}}, \mu_{\widetilde{X}})$ for $\widetilde{X} = \{\langle x, \mu_{\widetilde{X}}(x), \gamma_{\widetilde{X}}(x), \psi_{\widetilde{X}}(x)\rangle \mid x \in R\}$.

Lemma 4.1. *Let A be a nonempty subset of R. Then*

(1) A is a subring of R if and only if $\widetilde{A}$ is a neutrosophic subring of R,

(2) A is a left (right) ideal of R if and only if $\widetilde{A}$ is a neutrosophic left (right) ideal of R,

(3) A is a quasi-ideal of R if and only if $\widetilde{A}$ is a neutrosophic quasi-ideal of R.

Proof. Straightforward. □

Theorem 4.2. *A ring R is regular if and only if $A * B = A \cap B$ for every neutrosophic right ideal $A = (\mu_A, \gamma_A, \psi_A)$ of R and every neutrosophic left ideal $B = (\mu_B, \gamma_B, , \psi_B)$ of R.*

Proof. Suppose that R is a regular ring. By Theorem 3.10, we have $A * B \subseteq A \cap B$. To prove the opposite inclusion, let $x \in R$. Since R is a regular ring, there exists $y \in R$ such that $x = xyx$. Hence,

$$\begin{aligned}\mu_{A*B}(x) &= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{c}\mu_A(a_1), \mu_A(a_1), \ldots, \mu_A(a_1), \\ \mu_A(a_1), \mu_A(a_1), \ldots, \mu_A(a_1)\end{array}\right\} \\ &\geq \min\{\mu_A(xy), \mu_A(x)\} \\ &\geq \min\{\mu_A(x), \mu_A(x)\} \\ &= (\mu_A \wedge \mu_B)(x),\end{aligned}$$

$$\begin{aligned}\gamma_{A*B}(x) &= \bigvee_{x=\sum_{\text{finite}} a_i b_i} \min\left\{\begin{array}{c}\gamma_A(a_1), \gamma_A(a_1), \ldots, \gamma_A(a_1), \\ \gamma_A(a_1), \gamma_A(a_1), \ldots, \gamma_A(a_1)\end{array}\right\} \\ &\geq \min\{\gamma_A(xy), \gamma_A(x)\} \\ &\geq \min\{\gamma_A(x), \gamma_A(x)\} \\ &= (\gamma_A \wedge \gamma_B)(x),\end{aligned}$$

$$\begin{aligned}\psi_{A*B}(x) &= \bigwedge_{x=\sum_{\text{finite}} a_i b_i} \max\left\{\begin{array}{c}\psi_A(a_1), \psi_A(a_1), \ldots, \psi_A(a_1), \\ \psi_A(a_1), \psi_A(a_1), \ldots, \psi_A(a_1)\end{array}\right\} \\ &\leq \max\{\psi_A(xy), \psi_A(x)\} \\ &\leq \max\{\psi_A(x), \psi_A(x)\} \\ &= (\psi_A \vee \psi_B)(x).\end{aligned}$$

Hence, $\mu_{A*B}(x) \geq (\mu_A \wedge \mu_B)(x)$, $\gamma_{A*B}(x) \geq (\gamma_A \wedge \gamma_B)(x)$, and $\psi_{A*B}(x) \leq (\psi_A \vee \psi_B)(x)$ for all $x \in R$. It follows that $A * B \supseteq A \cap B$. Therefore, $A * B = A \cap B$.

Conversely, suppose that $A * B = A \cap B$ whenever A is a neutrosophic right ideal and B is a neutrosophic left ideal of R. Let U be a right ideal and V be a left ideal of R. By Lemma 4.1 (2), $\widetilde{U} = (\mu_{\widetilde{U}}, \gamma_{\widetilde{U}}, \psi_{\widetilde{U}})$ is a

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neutrosophic right ideal and $\widetilde{V} = (\mu_{\widetilde{V}}, \gamma_{\widetilde{V}}, \psi_{\widetilde{U}})$ is a neutrosophic left ideal of R. Hence, $\widetilde{V} \cap \widetilde{V} = \widetilde{V} * \widetilde{V}$. We always that $UV \subseteq U \cap V$. We show that $U \cap V \subseteq UV$. Let $x \in U \cap V$. Since

$$\begin{aligned}\mu_{\widetilde{U}*\widetilde{V}}(x) &= \bigvee_{x=\sum\limits_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l} \mu_{\widetilde{U}}(a_1), \mu_{\widetilde{U}}(a_2), \ldots, \mu_{\widetilde{U}}(a_m), \\ \mu_{\widetilde{U}}(b_1), \mu_{\widetilde{U}}(b_1), \ldots, \mu_{\widetilde{U}}(b_m) \end{array}\right\} \\ &= (\mu_{\widetilde{U}} \wedge \mu_{\widetilde{U}})(x) \\ &= \min\{\mu_{\widetilde{U}}(x), \mu_{\widetilde{U}}(x)\} \\ &= \min\{1, 1\} = 1,\end{aligned}$$

$$\begin{aligned}\gamma_{\widetilde{U}*\widetilde{V}}(x) &= \bigvee_{x=\sum\limits_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l} \gamma_{\widetilde{U}}(a_1), \gamma_{\widetilde{U}}(a_2), \ldots, \gamma_{\widetilde{U}}(a_m), \\ \gamma_{\widetilde{U}}(b_1), \gamma_{\widetilde{U}}(b_1), \ldots, \gamma_{\widetilde{U}}(b_m) \end{array}\right\} \\ &= (\gamma_{\widetilde{U}} \wedge \gamma_{\widetilde{U}})(x) \\ &= \min\{\gamma_{\widetilde{U}}(x), \gamma_{\widetilde{U}}(x)\} \\ &= \min\{1, 1\} = 1,\end{aligned}$$

$$\begin{aligned}\psi_{\widetilde{U}*\widetilde{V}}(x) &= \bigwedge_{x=\sum\limits_{\text{finite}} a_i b_i} \max\left\{\begin{array}{l} \psi_{\widetilde{U}}(a_1), \psi_{\widetilde{U}}(a_2), \ldots, \psi_{\widetilde{U}}(a_m), \\ \psi_{\widetilde{U}}(b_1), \psi_{\widetilde{U}}(b_1), \ldots, \psi_{\widetilde{U}}(b_m) \end{array}\right\} \\ &= (\psi_{\widetilde{U}} \vee \psi_{\widetilde{U}})(x) \\ &= \max\{\psi_{\widetilde{U}}(x), \psi_{\widetilde{U}}(x)\} \\ &= \max\{0, 0\} = 0,\end{aligned}$$

there must exist $a_i, b_i \in R$, for which $\mu_{\widetilde{U}}(a_i) = \mu_{\widetilde{U}}(b_i) = 1$, $\gamma_{\widetilde{U}}(a_i) = \gamma_{\widetilde{U}}(b_i) = 1$, and $\psi_{\widetilde{U}}(a_i) = \psi_{\widetilde{U}}(b_i) = 1$. This implies that $x = \sum\limits_{\text{finite}} a_i b_i \in UV$. Accordingly, $U \cap V \subseteq UV$. Hence, R is a ring. □

Corollary 4.3. *For a ring R, the following statements are true:*

(1) if R is a commutative ring such that $A * B = A \cap B$ for all neutrosophic ideals A, B of R, then R is a regular ring,

(2) if R is a regular ring, then every neutrosophic ideal of R is idempotent,

(3) if R is a commutative ring such that $A^2 = A$ for all neutrosophic ideal of R, then R is a regular ring.

Proof. Straightforward. □

Theorem 4.4. *For a ring R, the following statements are equivalent:*

(1) R is a regular ring,

(2) $A = A * 1_{\sim} * A$ for every neutrosophic bi-ideal A of R,

(3) $A = A * 1_{\sim} * A$ for every neutrosophic quasi-ideal A of R.

Proof. (1)⇒(2): Let $A = (\mu_A, \gamma_A)$ be any neutrosophic bi-ideal of R. Since A is a neutrosophic bi-ideal of R, we have $A * 1_{\sim} * A \subseteq A$. For the reverse inclusion, let $a \in R$. Then, since R is regular, there exists $x \in R$

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such that $a = axa$. Then

$$\begin{aligned}
\mu_{A*1_\sim *A}(x) &= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_1),\ldots,\mu_A(a_1),\\ \mu_{1_\sim *A}(b_1),\mu_{1_\sim *A}(b_2),\ldots,\mu_{1_\sim *A}(b_m)\end{array}\right\}\\
&\geq \min\{\mu_A(a),\mu_{1_\sim *A}(xa)\}\\
&= \min\left\{\mu_A(a), \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\mu_A(a_1),\mu_A(a_1),\ldots,\mu_A(a_1),\\ \mu_{1_\sim *A}(b_1),\mu_{1_\sim *A}(b_2),\ldots,\mu_{1_\sim *A}(b_m)\end{array}\right\}\right\}\\
&\geq \min\{\mu_A(a),\min\{1(x),\mu_A(a)\}\}\\
&= \min\{\mu_A(a),\min\{1,\mu_A(a)\}\}\\
&= \mu_A(a),
\end{aligned}$$

$$\begin{aligned}
\gamma_{A*1_\sim *A}(x) &= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_1),\ldots,\gamma_A(a_1),\\ \gamma_{1_\sim *A}(b_1),\gamma_{1_\sim *A}(b_2),\ldots,\gamma_{1_\sim *A}(b_m)\end{array}\right\}\\
&\geq \min\{\gamma_A(a),\gamma_{1_\sim *A}(xa)\}\\
&= \min\left\{\gamma_A(a), \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\gamma_A(a_1),\gamma_A(a_1),\ldots,\gamma_A(a_1),\\ \gamma_{1_\sim *A}(b_1),\gamma_{1_\sim *A}(b_2),\ldots,\gamma_{1_\sim *A}(b_m)\end{array}\right\}\right\}\\
&\geq \min\{\gamma_A(a),\min\{1(x),\gamma_A(a)\}\}\\
&= \min\{\gamma_A(a),\min\{1,\gamma_A(a)\}\}\\
&= \gamma_A(a),
\end{aligned}$$

$$\begin{aligned}
\psi_{A*1_\sim *A}(x) &= \bigwedge_{x=\sum\limits_{\text{finite}} a_ib_i} \max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_1),\ldots,\psi_A(a_1),\\ \psi_{1_\sim *A}(b_1),\psi_{1_\sim *A}(b_2),\ldots,\psi_{1_\sim *A}(b_m)\end{array}\right\}\\
&\leq \max\{\psi_A(a),\psi_{1_\sim *A}(xa)\}\\
&= \max\left\{\psi_A(a), \bigwedge_{x=\sum\limits_{\text{finite}} a_ib_i} \max\left\{\begin{array}{l}\psi_A(a_1),\psi_A(a_1),\ldots,\psi_A(a_1),\\ \psi_{1_\sim *A}(b_1),\psi_{1_\sim *A}(b_2),\ldots,\psi_{1_\sim *A}(b_m)\end{array}\right\}\right\}\\
&\leq \max\{\psi_A(a),\max\{0(x),\psi_A(a)\}\}\\
&= \max\{\psi_A(a),\max\{0,\psi_A(a)\}\}\\
&= \psi_A(a).
\end{aligned}$$

This shows that $A * 1_\sim * A \subseteq A$. Hence, $A = A * 1_\sim * A$.
(2)⇒(3): Since any neutrosophic quasi-ideal of R is a neutrosophic bi-ideal of R by Theorem 3.18, the implication (2)⇒(3) is valid.
(3)⇒(1): Let Q be any quasi-ideal of R, and a any element of Q. By Lemma 4.1 (3), $\widetilde{Q}$ is a neutrosophic quasi-ideal of R. Then

$$\begin{aligned}
\mu_{\widetilde{Q}*1_\sim *\widetilde{Q}}(a) &= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\mu_{\widetilde{Q}}(a_1),\mu_{\widetilde{Q}}(a_2),\ldots,\mu_{\widetilde{Q}}(a_m),\\ \mu_{1_\sim *\widetilde{Q}}(b_1),\mu_{1_\sim *\widetilde{Q}}(b_2),\ldots,\mu_{1_\sim *\widetilde{Q}}(b_m)\end{array}\right\}\\
&= \mu_{\widetilde{Q}}(a)\\
&= 1,
\end{aligned}$$

$$\begin{aligned}
\gamma_{\widetilde{Q}*1_\sim *\widetilde{Q}}(a) &= \bigvee_{x=\sum\limits_{\text{finite}} a_ib_i} \min\left\{\begin{array}{l}\gamma_{\widetilde{Q}}(a_1),\gamma_{\widetilde{Q}}(a_2),\ldots,\gamma_{\widetilde{Q}}(a_m),\\ \gamma_{1_\sim *\widetilde{Q}}(b_1),\gamma_{1_\sim *\widetilde{Q}}(b_2),\ldots,\gamma_{1_\sim *\widetilde{Q}}(b_m)\end{array}\right\}\\
&= \gamma_{\widetilde{Q}}(a)\\
&= 1,
\end{aligned}$$

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$$\begin{aligned}\psi_{\widetilde{Q}*1_{\sim}*\widetilde{Q}}(a) &= \bigwedge_{x=\sum\limits_{\text{finite}} a_i b_i} \max\left\{\begin{array}{l}\psi_{\widetilde{Q}}(a_1), \psi_{\widetilde{Q}}(a_2), \ldots, \psi_{\widetilde{Q}}(a_m), \\ \psi_{1_{\sim}*\widetilde{Q}}(b_1), \psi_{1_{\sim}*\widetilde{Q}}(b_2), \ldots, \psi_{1_{\sim}*\widetilde{Q}}(b_m)\end{array}\right\} \\ &= \psi_{\widetilde{Q}}(a) \\ &= 0.\end{aligned}$$

This implies that there exist $a_i, b_i \in R$ such that $\mu_{\widetilde{Q}}(a_i) = \mu_{1_{\sim}*\widetilde{Q}}(b_i) = 1$, $\gamma_{\widetilde{Q}}(a_i) = \gamma_{1_{\sim}*\widetilde{Q}}(b_i) = 1$, and $\psi_{\widetilde{Q}}(a_i) = \psi_{1_{\sim}*\widetilde{Q}}(b_i) = 0$ with $a = \sum\limits_{\text{finite}} a_i b_i$. Since

$$1 = \mu_{1_{\sim}*\widetilde{Q}}(b_i) = \bigvee_{b_i=\sum\limits_{\text{finite}} p_i q_i} \min\left\{\begin{array}{l}1(p_1), 1(p_2), \ldots, 1(p_m), \\ \mu_{\widetilde{Q}}(q_1), \mu_{\widetilde{Q}}(q_2), \ldots, \mu_{\widetilde{Q}}(q_m)\end{array}\right\},$$

$$1 = \gamma_{1_{\sim}*\widetilde{Q}}(b_i) = \bigvee_{b_i=\sum\limits_{\text{finite}} p_i q_i} \min\left\{\begin{array}{l}1(p_1), 1(p_2), \ldots, 1(p_m), \\ \gamma_{\widetilde{Q}}(q_1), \gamma_{\widetilde{Q}}(q_2), \ldots, \gamma_{\widetilde{Q}}(q_m)\end{array}\right\},$$

$$0 = \psi_{1_{\sim}*\widetilde{Q}}(b_i) = \bigwedge_{b_i=\sum\limits_{\text{finite}} p_i q_i} \max\left\{\begin{array}{l}0(p_1), 0(p_2), \ldots, 0(p_m), \\ \psi_{\widetilde{Q}}(q_1), \psi_{\widetilde{Q}}(q_2), \ldots, \psi_{\widetilde{Q}}(q_m)\end{array}\right\},$$

there exist $p_1, q_i \in R$ such that $p_i \in R$ and $q_i \in Q$ with $b_i = \sum\limits_{\text{finite}} p_i q_i$. Hence, $a_i, q_i \in Q$ and $p_i \in R$. Then

$$a = \sum_{\text{finite}} a_i b_i = \sum_{\text{finite}} a_i \left(\sum_{\text{finite}} p_i q_i\right) = \sum_{\text{finite}} a_i (p_i q_i) \in QRQ,$$

and so $a \in QRQ$. Thus, $Q \subseteq QRQ$. On the other hand, Q is a quasi-ideal of R, $QRQ \subseteq QR \cap RQ \subseteq Q$, and so $Q = QRQ$. Then it follows that R is regular. □

Theorem 4.5. *For a ring R, the following statements are equivalent:*

(1) R is regular,

(2) $A \cap B = B * A * B$ for every neutrosophic ideal A of R and every neutrosophic bi-ideal B of R,

(3) $A \cap B = B * A * B$ for every neutrosophic ideal A of R and every neutrosophic quasi-ideal B of R.

Proof. (1)⇒(2): Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be any neutrosophic ideal and any neutrosophic bi-ideal of R, respectively. Then $B * A * B \subseteq B * 1_{\sim} * B \subseteq B$ and $B * A * B \subseteq 1_{\sim} * A * 1_{\sim} \subseteq A$. Thus, $B * A * B \subseteq A \cap B$. In order to see that the converse inclusion holds, let a be any element of R. Then, since R is regular, there exists an element x in R such that $a = axa (= axaxa)$. Since A is a neutrosophic ideal of R, $\mu_A(xax) \leq \mu_A(xa) \leq \mu_A(a)$, $\gamma_A(xax) \leq \gamma_A(xa) \leq \gamma_A(a)$, and $\psi_A(xax) \geq \psi_A(xa) \geq \psi_A(a)$, so

$$\begin{aligned}\mu_{B*A*B}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}\mu_B(a_1), \mu_B(a_1), \ldots, \mu_B(a_1), \\ \mu_{A*B}(b_1), \mu_{A*B}(b_2), \ldots, \mu_{A*B}(b_m)\end{array}\right\} \\ &\geq \min\{\mu_B(a), \mu_{A*B}(xaxa)\} \\ &= \min\left\{\mu_B(a), \bigvee_{xaxa=\sum\limits_{\text{finite}} p_i q_i} \min\left\{\begin{array}{l}\mu_A(p_1), \mu_A(p_2), \ldots, \mu_A(p_m), \\ \mu_B(q_1), \mu_B(q_2), \ldots, \mu_B(q_m)\end{array}\right\}\right\} \\ &\geq \min\{\mu_B(a), \min\{\mu_A(xax), \mu_B(a)\}\} \\ &= \min\{\mu_B(a), \min\{\mu_A(a), \mu_B(a)\}\} \\ &= \min\{\mu_B(a), \mu_a(a)\} \\ &= (\mu_B \wedge \mu_A)(a),\end{aligned}$$

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$$\begin{aligned}
\gamma_{B*A*B}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{ \begin{array}{l} \gamma_B(a_1), \gamma_B(a_1), \ldots, \gamma_B(a_1), \\ \gamma_{A*B}(b_1), \gamma_{A*B}(b_2), \ldots, \gamma_{A*B}(b_m) \end{array} \right\} \\
&\geq \min\{\gamma_B(a), \gamma_{A*B}(xaxa)\} \\
&= \min\left\{ \gamma_B(a), \bigvee_{xaxa=\sum\limits_{\text{finite}} p_i q_i} \min\left\{ \begin{array}{l} \gamma_A(p_1), \gamma_A(p_2), \ldots, \gamma_A(p_m), \\ \gamma_B(q_1), \gamma_B(q_2), \ldots, \gamma_B(q_m) \end{array} \right\} \right\} \\
&\geq \min\{\gamma_B(a), \min\{\gamma_A(xax), \gamma_B(a)\}\} \\
&= \min\{\gamma_B(a), \min\{\gamma_A(a), \gamma_B(a)\}\} \\
&= \min\{\gamma_B(a), \gamma_a(a)\} \\
&= (\gamma_B \wedge \gamma_A)(a),
\end{aligned}$$

$$\begin{aligned}
\psi_{B*A*B}(a) &= \bigwedge_{a=\sum\limits_{\text{finite}} a_i b_i} \max\left\{ \begin{array}{l} \psi_B(a_1), \psi_B(a_1), \ldots, \psi_B(a_1), \\ \psi_{A*B}(b_1), \psi_{A*B}(b_2), \ldots, \psi_{A*B}(b_m) \end{array} \right\} \\
&\leq \max\{\psi_B(a), \psi_{A*B}(xaxa)\} \\
&= \max\left\{ \psi_B(a), \bigwedge_{xaxa=\sum\limits_{\text{finite}} p_i q_i} \max\left\{ \begin{array}{l} \psi_A(p_1), \psi_A(p_2), \ldots, \psi_A(p_m), \\ \psi_B(q_1), \psi_B(q_2), \ldots, \psi_B(q_m) \end{array} \right\} \right\} \\
&\leq \max\{\psi_B(a), \max\{\psi_A(xax), \psi_B(a)\}\} \\
&= \max\{\psi_B(a), \max\{\psi_A(a), \psi_B(a)\}\} \\
&= \max\{\psi_B(a), \psi_a(a)\} \\
&= (\psi_B \wedge \psi_A)(a).
\end{aligned}$$

Hence, $A \cap B \supseteq B * A * B$. Therefore, $A \cap B = B * A * B$.
(2)⇒(3): Since any neutrosophic quasi-ideal of R is a neutrosophic bi-ideal of R by Theorem 3.18, the implication (2)⇒(3) is valid.
(3)⇒(1): Let U and V be any ideal and any quasi-ideal of R. By Lemma 4.1 (3), $\widetilde{U}$ is a neutrosophic ideal and $\widetilde{V}$ is a neutrosophic quasi-ideal of R. Let $a \in U \cap V$. Then

$$\begin{aligned}
\mu_{\widetilde{V}*\widetilde{U}*\widetilde{V}}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{ \begin{array}{l} \mu_{\widetilde{V}}(a_1), \mu_{\widetilde{V}}(a_2), \ldots, \mu_{\widetilde{V}}(a_m), \\ \mu_{\widetilde{U}*\widetilde{V}}(b_1), \mu_{\widetilde{U}*\widetilde{V}}(b_2), \ldots, \mu_{\widetilde{U}*\widetilde{V}}(b_m) \end{array} \right\} \\
&= (\mu_{\widetilde{U}} \wedge \mu_{\widetilde{U}})(a) \\
&= \min\{\mu_{\widetilde{U}}(a), \mu_{\widetilde{U}}(a)\} \\
&= \min\{1, 1\} \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
\gamma_{\widetilde{V}*\widetilde{U}*\widetilde{V}}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{ \begin{array}{l} \gamma_{\widetilde{V}}(a_1), \gamma_{\widetilde{V}}(a_2), \ldots, \gamma_{\widetilde{V}}(a_m), \\ \gamma_{\widetilde{U}*\widetilde{V}}(b_1), \gamma_{\widetilde{U}*\widetilde{V}}(b_2), \ldots, \gamma_{\widetilde{U}*\widetilde{V}}(b_m) \end{array} \right\} \\
&= (\gamma_{\widetilde{U}} \wedge \gamma_{\widetilde{U}})(a) \\
&= \min\{\gamma_{\widetilde{U}}(a), \gamma_{\widetilde{U}}(a)\} \\
&= \min\{1, 1\} \\
&= 1,
\end{aligned}$$

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$$\begin{aligned}\psi_{\widetilde{V}*\widetilde{U}*\widetilde{V}}(a) &= \bigwedge_{a=\sum\limits_{\text{finite}} a_i b_i} \max\left\{\begin{array}{l} \psi_{\widetilde{V}}(a_1), \psi_{\widetilde{V}}(a_2), \ldots, \psi_{\widetilde{V}}(a_m), \\ \psi_{\widetilde{U}*\widetilde{V}}(b_1), \psi_{\widetilde{U}*\widetilde{V}}(b_2), \ldots, \psi_{\widetilde{U}*\widetilde{V}}(b_m) \end{array}\right\} \\ &= (\psi_{\widetilde{U}} \wedge \psi_{\widetilde{U}})(a) \\ &= \max\{\psi_{\widetilde{U}}(a), \psi_{\widetilde{U}}(a)\} \\ &= \max\{0, 0\} \\ &= 0.\end{aligned}$$

This implies that there exist $a_i, b_i \in R$ such that $\mu_{\widetilde{V}}(a_i) = \mu_{\widetilde{V}*\widetilde{U}*\widetilde{V}}(b_i) = 1$, $\gamma_{\widetilde{V}}(a_i) = \gamma_{\widetilde{V}*\widetilde{V}}(b_i) = 1$, and $\psi_{\widetilde{V}}(a_i) = \psi_{\widetilde{V}*\widetilde{V}}(b_i) = 0$ with $a = \sum\limits_{\text{finite}} a_i b_i$. Since

$$1 = \mu_{\widetilde{U}*\widetilde{V}}(b_i) = \bigvee_{b_i=\sum\limits_{\text{finite}} p_i q_i} \min\left\{\begin{array}{l} \mu_{\widetilde{U}}(p_1), \mu_{\widetilde{U}}(p_2), \ldots, \mu_{\widetilde{U}}(p_m), \\ \mu_{\widetilde{V}}(q_1), \mu_{\widetilde{V}}(q_2), \ldots, \mu_{\widetilde{V}}(q_m) \end{array}\right\},$$

$$1 = \gamma_{\widetilde{U}*\widetilde{V}}(b_i) = \bigvee_{b_i=\sum\limits_{\text{finite}} p_i q_i} \min\left\{\begin{array}{l} \gamma_{\widetilde{U}}(p_1), \gamma_{\widetilde{U}}(p_2), \ldots, \gamma_{\widetilde{U}}(p_m), \\ \gamma_{\widetilde{V}}(q_1), \gamma_{\widetilde{V}}(q_2), \ldots, \gamma_{\widetilde{V}}(q_m) \end{array}\right\},$$

$$0 = \psi_{\widetilde{U}*\widetilde{V}}(b_i) = \bigwedge_{b_i=\sum\limits_{\text{finite}} p_i q_i} \max\left\{\begin{array}{l} \psi_{\widetilde{U}}(p_1), \psi_{\widetilde{U}}(p_2), \ldots, \psi_{\widetilde{U}}(p_m), \\ \psi_{\widetilde{V}}(q_1), \psi_{\widetilde{V}}(q_2), \ldots, \psi_{\widetilde{V}}(q_m) \end{array}\right\},$$

there exist $p_1, q_i \in R$ such that $p_i \in u$ and $q_i \in v$ with $b_i = \sum\limits_{\text{finite}} p_i q_i$. Hence, $a_i, q_i \in v$ and $p_i \in u$. Then

$$a = \sum_{\text{finite}} a_i b_i = \sum_{\text{finite}} a_i \left(\sum_{\text{finite}} p_i q_i\right) = \sum_{\text{finite}} a_i (p_i q_i) \in VUV$$

and so $a \in VUV$. Thus, $U \cap V = VUV$. Then it follows that R is regular. □

Theorem 4.6. *For a ring R, the following statements are equivalent:*

(1) R is regular,

(2) $A \cap B = B * A * B$ for every neutrosophic ideal A of R and every neutrosophic bi-ideal B of R,

(3) $A \cap B = B * A * B$ for every neutrosophic ideal A of R and every neutrosophic quasi-ideal B of R,

(4) $B \cap C \subseteq B * C$ for every neutrosophic left ideal C and every neutrosophic bi-ideal B of R,

(5) $B \cap C \subseteq B * C$ for every neutrosophic left ideal C and every neutrosophic quasi-ideal B of R.

Proof. (1)⇒(2): Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be any neutrosophic right ideal and any neutrosophic bi-ideal of R, respectively. Let $a \in R$. Then, since R is regular, there exists $x \in R$ such that $a = axa$. Then

$$\begin{aligned}\mu_{A*B}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l} \mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_B(b_1), \mu_B(b_2), \ldots, \mu_B(b_m) \end{array}\right\} \\ &\geq \min\{\mu_B(ax), \mu_B(a)\} \\ &\geq \min\{\mu_A(a), \mu_B(a)\} \\ &= (\mu_A \wedge \mu_B)(a),\end{aligned}$$

$$\begin{aligned}\gamma_{A*B}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l} \gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_B(b_1), \gamma_B(b_2), \ldots, \gamma_B(b_m) \end{array}\right\} \\ &\geq \min\{\gamma_B(ax), \gamma_B(a)\} \\ &\geq \min\{\gamma_A(a), \gamma_B(a)\} \\ &= (\gamma_A \wedge \gamma_B)(a),\end{aligned}$$

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$$\begin{aligned}\psi_{A*B}(a) &= \bigwedge_{a=\sum\limits_{\text{finite}} a_i b_i} \max\left\{\begin{array}{l}\psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_B(b_1), \psi_B(b_2), \ldots, \psi_B(b_m)\end{array}\right\} \\ &\leq \max\{\psi_B(ax), \psi_B(a)\} \\ &\leq \max\{\psi_A(a), \psi_B(a)\} \\ &= (\psi_A \wedge \psi_B)(a),\end{aligned}$$

and so $A \cap B \subseteq A * B$.
(2)⇔(3)⇔(4)⇔(5): Straightforward.
(5)⇒(1): Since any neutrosophic left ideal of R is a neutrosophic quasi-ideal, it follows from Theorem 4.2 that R is regular. □

Theorem 4.7. *For a ring R, the following statements are equivalent:*

(1) R is regular,

(2) $A \cap B \cap C \subseteq A * B * C$ for every neutrosophic right ideal A, every neutrosophic left ideal B, and every neutrosophic bi-ideal C of R,

(3) $A \cap B \cap C \subseteq A * B * C$ for every neutrosophic right ideal A, every neutrosophic left ideal B, and every neutrosophic quasi-ideal C of R.

Proof. (1)⇒(2): Let $A = (\mu_A, \gamma_A, \psi_A)$, $B = (\mu_B, \gamma_B, \psi_B)$, and $C = (\mu_C, \gamma_C, \psi_C)$ be any neutrosophic right ideal and any neutrosophic left ideal, and neutrosophic bi-ideal of R, respectively. Let a be any element of R. Then, since R is regular, there exists an element x in R such that $a = axa (= axaxa)$. Then

$$\begin{aligned}\mu_{A*B*C}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}\mu_A(a_1), \mu_A(a_2), \ldots, \mu_A(a_m), \\ \mu_{B*C}(b_1), \mu_{B*C}(b_2), \ldots, \mu_{B*C}(b_m)\end{array}\right\} \\ &\geq \min\{\mu_B(ax), \mu_{B*C}(axa)\} \\ &= \min\left\{\mu_A(ax), \bigvee_{axa=\sum\limits_{\text{finite}} p_i q_i} \min\left\{\begin{array}{l}\mu_B(p_1), \mu_B(p_2), \ldots, \mu_B(p_m), \\ \mu_C(q_1), \mu_C(q_2), \ldots, \mu_C(q_m)\end{array}\right\}\right\} \\ &\geq \min\{\mu_A(ax), \min\{\mu_B(a), \mu_C(xa)\}\} \\ &= \min\{\mu_A(a), \min\{\mu_B(a), \mu_C(a)\}\} \\ &= (\mu_A \wedge \mu_B \wedge \mu_C)(a),\end{aligned}$$

$$\begin{aligned}\gamma_{A*B*C}(a) &= \bigvee_{a=\sum\limits_{\text{finite}} a_i b_i} \min\left\{\begin{array}{l}\gamma_A(a_1), \gamma_A(a_2), \ldots, \gamma_A(a_m), \\ \gamma_{B*C}(b_1), \gamma_{B*C}(b_2), \ldots, \gamma_{B*C}(b_m)\end{array}\right\} \\ &\geq \min\{\gamma_B(ax), \gamma_{B*C}(axa)\} \\ &= \min\left\{\gamma_A(ax), \bigvee_{axa=\sum\limits_{\text{finite}} p_i q_i} \min\left\{\begin{array}{l}\gamma_B(p_1), \gamma_B(p_2), \ldots, \gamma_B(p_m), \\ \gamma_C(q_1), \gamma_C(q_2), \ldots, \gamma_C(q_m)\end{array}\right\}\right\} \\ &\geq \min\{\gamma_A(ax), \min\{\gamma_B(a), \gamma_C(xa)\}\} \\ &= \min\{\gamma_A(a), \min\{\gamma_B(a), \gamma_C(a)\}\} \\ &= (\gamma_A \wedge \gamma_B \wedge \gamma_C)(a),\end{aligned}$$

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$$\begin{aligned}
\psi_{A*B*C}(a) &= \bigwedge_{a=\sum_{\text{finite}} a_i b_i} \max\left\{ \begin{array}{l} \psi_A(a_1), \psi_A(a_2), \ldots, \psi_A(a_m), \\ \psi_{B*C}(b_1), \psi_{B*C}(b_2), \ldots, \psi_{B*C}(b_m) \end{array} \right\} \\
&\leq \max\{\psi_B(ax), \psi_{B*C}(axa)\} \\
&= \max\left\{ \psi_A(ax), \bigwedge_{axa=\sum_{\text{finite}} p_i q_i} \max\left\{ \begin{array}{l} \psi_B(p_1), \psi_B(p_2), \ldots, \psi_B(p_m), \\ \psi_C(q_1), \psi_C(q_2), \ldots, \psi_C(q_m) \end{array} \right\} \right\} \\
&\leq \max\{\psi_A(ax), \max\{\psi_B(a), \psi_C(xa)\}\} \\
&= \max\{\psi_A(a), \max\{\psi_B(a), \psi_C(a)\}\} \\
&= (\psi_A \wedge \psi_B \wedge \psi_C)(a).
\end{aligned}$$

(2)⇒(3): Straightforward.
(3)⇒(1): Let $A = (\mu_A, \gamma_A, \psi_A)$ and $C = (\mu_C, \gamma_C, \psi_C)$ be any neutrosophic right ideal and any neutrosophic bi-ideal of R, respectively. Then, since R itself is a fuzzy quasi-ideal of R, we have $A \cap C = A \cap R \cap C \subseteq A * 1_{\sim} * C \subseteq A * C$. It follows from Theorem 4.2 that R is regular. □

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