LRPS Method for Solving Linear Partial Differential Equations and Neutrosophic Differential Equations of Fractional Order with Numerical Solutions

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Abstract

In this work, fractional partial equations' and neutrosophic fractional partial equations analytical series solutions are presented, we consider the fractional derivative in the meaning of Caputo in these formulas. We offer a novel objective method the LRPS which is a strong instrument for precise analytically and numerical solutions to these problems by setting an excellent example, we stress precision, effectiveness, and application style, also we can find exact answers when there is a pattern between the series' parts; alternatively, we can only offer approximations. The Mathematica application is used to assess the numerical and graphical findings to make sure the solutions generated are accurate and that the approach can be modified to solve this kind of this problem. The findings obtained demonstrated that our current procedure is appropriate and efficient for resolving PDEs.

Keywords: LRPS Method; Caputo; Fractional Order; Numerical Solutions, neutrosophic differential equations.

1. Introduction

In the domains of economics, biochemistry, operations research, and other sciences, fractional derivatives could be effectively accustomed represent many widely different events [1,3,4,10,14] primary cause of this is because acceptable modelling of real-world problems depends on both the current time and a preceding chronological date, which may be accomplished via fractional calculus [5,6,13].

As a result, many science and engineering academics have focused on fractional differential equations (FDEs) while creating procedures for linear and nonlinear issues and talking about dynamical systems. Additionally, solutions to the FDEs have been studied using approximation and numerical techniques [8,9]. Many academics are interested in the subject of fractional initial value issues (FIVPs), an extension of standard initial value issues that can capture some characteristics of real-life situations more realistically than standard DEs. The concept behind the presence and durability of solutions to the structure of FIVPs has received a lot of attention [2,11,20].

In order to examine the answers to various types of FPDEs analytical and numerical methodologies have indeed been devised. To name a few, the pseudo spectral strategy, transforming homotopy assessment method, the dispersion of Wavelet transform method, the iterative expansion method, the deterioration of the Adomian method and the homotopy analysis method [7,15,19].

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The extension of RPS has been seen in several PDE, notably fractional PDE. Time-fractional diffraction PDE, formulae of KdV-Burgers and formulae of fractional Boussinesq are a few instances [1,12].

Lately, the LRPS method was employed. It was first presented and shown in [18]. By transforming the central problem into Laplace domain and generating new algebra formulas' resolutions, the LRPS approach combines Laplace transform method and the RPS method to provide accurate and results as Faster Power Set (FPS) approximations. Reversing Laplace for values obtained can then be used to address the identification challenge. The suggested Laplace expansion's unknown parameters may be expressed using the concept of limits, as opposed to the FRPS concept, which according to the derivation and necessitates the consuming computation of many fractional derivatives in stages to uncover solutions.

Neutrosophic differential equations are considered as a generalized case of classical differential solutions, they were studied by many authors in [21-22], they have presented both numerical and exact solutions for many different types of them.

In this paper we offer a novel objective method, the LRPS which is a strong instrument for precise analytically and numerical solutions to these problems. By setting an excellent example, we stress precision, effectiveness, and application style. We can find exact answers when there is a pattern between the series' parts; alternatively, we can only offer approximations.

2. Foundations of fractional calculus concepts

The fractional integrate with order is defined in a number of ways, and they are not all interchangeable. Riemann-definition Liouville's and Caputo's concept are the two used most types.

Definition 2.1 [24] The given definition of Mittag-Leffler formula:

$$E_{\beta}(z):=\sum_{\mathrm{j}=0}^{\infty}\frac{z^{j}}{\Gamma(\beta \mathrm{j}+1)},\beta>0 \tag{1}$$

Definition 2.2 [24] The integral of R-L time-fraction with $\alpha>0$ of the multivariable function $\psi(x,t)$,$t>0$ is defined by:

$$J_t^{\alpha}\psi(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\psi(x,\tau)d\tau, \tag{2}$$

Definition 2.3 [18] The derivative of Caputo time-fraction with $m-1<\alpha<m$ of the multivariable function $\psi(x,t)$ as mentioned:

$$D^{\alpha}\psi(x,t)=\begin{cases}J_t^{m-\alpha}\dfrac{\partial^m}{\partial t^m}\psi(x,t), & m-1<\alpha<m,\\ \dfrac{\partial^m}{\partial t^m}\psi(x,t), & \alpha=m.\end{cases} \tag{3}$$

Definition 2.4 [18] If the following improper integral:

$$\Psi(x,s)=\int_0^{\infty}e^{-st}\psi(x,t)dt, \tag{4}$$

exists for all s in a domain $D\subseteq\mathbb{C}$, then it is Laplace transform of $\psi(\mathrm{x},t)$, and indicated as $\mathcal{L}[\psi(x,t)](x,s)$.

Definition 2.5 [18] Using the inverse Laplace transform described below, the original function $\psi(x,t)$ may be recovered from the Laplace transform $\Psi(x,s)$:

$$\psi(\mathrm{x},t)=\int_{\zeta-i\infty}^{\zeta+i\infty}e^{st}\Psi(\mathrm{x},s)ds,\zeta=\mathrm{Re}(s), \tag{5}$$

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Lemma 2.6 [18] Suppose that $\Psi(x,s) = \mathcal{L}[\psi(x,t)]$, and $\Omega(x,s) = \mathcal{L}[\phi(x,t)]$ and $\eta, \lambda, \beta, \sigma$ are elements. The following characteristics are thus met:

1. Linearity: The Laplace transform and its inverse are linear operators, that is, for η , σ and $\lambda \in \mathbb{R}$, then
2. $\mathcal{L}[\eta\psi(x,t) + \lambda\phi(x,t)] = \eta\mathcal{L}[\psi(x,t)] + \lambda\mathcal{L}[\phi(x,t)] = \eta\Psi(x,s) + \lambda\Omega(x,s)$.
3. $\mathcal{L}^{-1}[\eta\Psi(x,s) + \lambda\Omega(x,s)] = \eta\mathcal{L}^{-1}[\Psi(x,s)] + \lambda\mathcal{L}^{-1}[\Omega(x,s)] = \eta\psi(x,t) + \phi(x,t)$.
4. Shifting:$\mathcal{L}[e^{\sigma t}\phi(x,t)] = \Psi(s-\sigma)$.
5. $\mathcal{L}[\psi(\beta x, \beta t)] = \frac{1}{\beta}\Psi\left(\frac{x}{\beta}, \frac{s}{\beta}\right), \beta > 0$.
6. $\lim_{s\to\infty} s\Psi(x,s) = \psi(x,0)$.
7. $\mathcal{L}[\partial_t^n\psi(x,t)] = s^n\Psi(x,s) - \sum_k^{n-1} s^{n-k-1}\, \partial_t^k\psi(x,0)$.

Definition 2.7[16] A series expansion illustration of the structure:

$$\sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + \cdots, \tag{6}$$

Where $t \geq t_0$, referred to as a series of fractional power (FPS, for short) regarding t_0, and $f_n(x)$ are functions of x.

Corollary 2.8 [18] Consider the $\Psi(x,s) = \mathcal{L}[\psi(\mathrm{x},\mathrm{t})]$ is as follows fractional Laurent expansion (FLE):

$$\Psi(x,s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}}, 0 < \alpha \leq 1, s > 0. \tag{7}$$

Then $\mathrm{f}_n(x) = (D_t^{n\alpha}\psi)(0)$.

Lemma 2.9 [18] The transform of inverse Laplace of FLE in Proposition 2.8 does have subsequent type:

$$\psi(x,t) = \sum_{n=0}^{\infty} \frac{D_t^{n\alpha}\psi(x,0)}{\Gamma(n\alpha+1)} t^{n\alpha}, 0 < \alpha \leq 1, t \geq 0. \tag{8}$$

3. Constructing LRPS Solution to Fractional Differential Equations

In this part, we provide a generic LRPS for FPDEs as following equation:

$$D_t^{\alpha}u(x,t) - \frac{u(x,t)}{2} = g(x,t). \tag{9}$$

*w*ith initial condition

$$u(x,0) = C(x). \tag{10}$$

The first step for that is to apply Laplace transform to system (9) to transfer it to Laplace space as follows:

$$\mathcal{L}\{D_t^{\alpha}u(x,t)\} - \frac{1}{2}\mathcal{L}\{u(x,t)\} = \mathcal{L}\{g(x,t)\}. \tag{11}$$

Considering that $U(x,s) = \mathcal{L}\{u(x,t)\}$, we obtain:

$$s^{\alpha}U(x,s) - s^{\alpha-1}C(x) - \frac{1}{2}U(x,s) = G(x,t). \tag{12}$$

Dividing Eq. (12) on s^{α}gives a new form to it as follows:

$$U(x,s) - \frac{C(x)}{s} + \frac{1}{2s^{\alpha}}U(x,s) - \frac{G(x,t)}{s^{\alpha}} = 0 \tag{13}$$

In the next step, we assume the solution of the algebraic system (3.8), $U(x,s)$ has expansions in fractional Lorient series form as follows:

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$$U(x,s)=\sum_{i=0}^{\infty}\frac{C_i(x)}{s^{1+i\alpha}},0<\alpha\leq 1,x\in I,s>0, \tag{14}$$

The first coefficient of (14), as follows:

$$U(x,s)=\frac{C(x)}{s}+\sum_{i=1}^{\infty}\frac{C_i(x)}{s^{1+i\alpha}},0<\alpha\leq 1,x\in I,s>0\,, \tag{15}$$

Now, we construction the Laplace residual functions (LRFs) of the algebraic system (13) as follows:

$$LRes_1(x,s)=U(x,s)-\frac{C(x)}{s}+\frac{1}{2s^{\alpha}}U(x,s)-\frac{G(x,t)}{s^{\alpha}}. \tag{16}$$

The k-th term of (15) is provided by:

$$U_k(x,s)=\frac{C(x)}{s}+\sum_{i=1}^{k}\frac{C_i(x)}{s^{1+i\alpha}},0<\alpha\leq 1,x\in I,s>0, \tag{17}$$

Hence, the kth-LRF can indeed be described as follows:

$$LRes_k(x,s)=U_k(x,s)-\frac{C(x)}{s}+\frac{1}{2s^{\alpha}}U_k(x,s)-\frac{G(x,t)}{s^{\alpha}} \tag{18}$$

It is clear that $\mathrm{Lim}_{k\to\infty}\mathrm{LRes}_k(s)=\mathrm{LRes}(s)$, $\mathrm{LRes}(s)=0$, and thus $s^k\mathrm{LRes}(s)=0$ for $s>0$ and $k=0,1,2,3,\ldots$. Thus, $\mathrm{Lim}_{s\to\infty}\left(s^k\mathrm{LRes}(s)\right)=0$ [18]:

$$\mathrm{Lim}_{s\to\infty}s^{k\alpha+1}\mathrm{LRes}_k(s)=0,\quad \alpha>0,\quad k=1,2,3,\ldots \tag{19}$$

The$1st$-LRF is:

$$LRes_1(x,s)=\frac{C(x)}{s}+\frac{C_1(x)}{s^{1+\alpha}}-\frac{C(x)}{s}+\frac{1}{2s^{\alpha}}\left(\frac{C(x)}{s}+\frac{C_1(x)}{s^{1+\alpha}}\right)-\frac{G(x,t)}{s^{\alpha}} \tag{20}$$

Multiply both sides of Eq. (20) by $s^{\alpha+1}$to get:

$$s^{\alpha+1}LRes_1(x,s)=C_1(x)+\frac{C(x)}{2}+\frac{C_1(x)}{s^{\alpha}}-sG(x,t)\,. \tag{21}$$

Taking the limit of both sides of (21) as $s\to\infty$, and considering the fact in (19) and solving the resulting algebraic system we obtain:

$$C_1(x)=\frac{2G'(x,t)-C(x)}{2}. \tag{22}$$

Where $\lim_{s\to\infty}sG(x,t)=G'(x,t)$

Finding the quantity of next unidentified parameter is straightforward $U_2(x,s)$ substitute kth-truncated series of (17), $U_2(x,s)=\frac{C(x)}{s}+\frac{C_1(x)}{s^{1+\alpha}}+\frac{C_2(x)}{s^{1+2\alpha}}$ into the kth-LRF, $LRes_k(\mathrm{x},\mathrm{s})$ of (18), to get the following:

$$LRes_2(x,s)=\frac{C(x)}{s}+\frac{2G'(x,t)-C(x)}{2s^{1+\alpha}}+\frac{C_2(x)}{s^{1+2\alpha}}-\frac{C(x)}{s}+\frac{1}{2s^{\alpha}}\left(\frac{C(x)}{s}+\frac{2G'(x,t)-C(x)}{2s^{1+\alpha}}+\frac{C_2(x)}{s^{1+2\alpha}}\right)-\frac{G(x,t)}{s^{\alpha}} \tag{23}$$

Again, multiply both sides of (23) by $s^{2\alpha+1}$to get:

$$s^{2\alpha+1}LRes_2(x,s)=C_2(x)+\frac{2G'(x,t)-C(x)}{4}+\frac{C_2(x)}{s^{\alpha}}-s^2G(x,t). \tag{24}$$

Taking the limit of both sides of (24) as $s\to\infty$, and considering the fact in (19) we may achieve the following abbreviated form by resolving the resultant algebraic equation:

$$C_2(x)=\frac{4G''(x,t)+C(x)-2G'(x,t)}{4}. \tag{25}$$

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Also to determine the third undetermined parameters' quantity $U_3(x,s)$ substitute kth-truncated series of (17), $U_3(x,s) = \frac{C(x)}{s} + \frac{C_1(x)}{s^{1+\alpha}} + \frac{C_2(x)}{s^{1+2\alpha}} + \frac{C_3(x)}{s^{1+3\alpha}}$ into the kth-LRF, $LRes_k(\mathrm{x,s})$ of (18), to get the following:

$$LRes_3(x,s) = \frac{C(x)}{s} + \frac{2G'(x,t) - C(x)}{2s^{1+\alpha}} + \frac{4G''(x,t) + C(x) - 2G'(x,t)}{4s^{1+2\alpha}} + \frac{C_3(x)}{s^{1+3\alpha}} - \frac{C(x)}{s} + \frac{1}{2s^{\alpha}}\left(\frac{C(x)}{s} + \frac{2G'(x,t) - C(x)}{2s^{1+\alpha}} + \frac{4G''(x,t) + C(x) - 2G'(x,t)}{4s^{1+2\alpha}} + \frac{C_3(x)}{s^{1+3\alpha}}\right) - \frac{G(x,t)}{s^{\alpha}}. \tag{26}$$

We can get easily:

$$C_3(x) = \frac{8G'''(x,t) - 4G''(x,t) - C(x) + 2G'(x,t)}{8}. \tag{27}$$

Therefore, Eq. (17) has the following chain form solution:

$$U(x,s) = \frac{C(x)}{s} + \frac{2G'(x,t) - C(x)}{2s^{1+\alpha}} + \frac{4G''(x,t) + C(x) - 2G'(x,t)}{4s^{1+2\alpha}} + \frac{8G'''(x,t) - 4G''(x,t) - C(x) + 2G'(x,t)}{8s^{1+3\alpha}} + \cdots. \tag{28}$$

Using the inverse of Laplace for (28) gives 4th-approximate solution to Eq. (3.5) and (3.6) in the following series form:

$$u(x,t) = \mathrm{C}(x) + \frac{2G'(x,t) - C(x)}{2\Gamma(1+\alpha)}t^{\alpha} + \frac{4G''(x,t) + C(x) - 2G'(x,t)}{4\Gamma(1+2\alpha)}t^{2\alpha} + \frac{8G'''(x,t) - 4G''(x,t) - C(x) + 2G'(x,t)}{8\Gamma(1+3\alpha)}t^{3\alpha} + \cdots. \tag{29}$$

4. Application

Application 4.1: Think about the below FPDEs:

$$D_t^{\alpha}u(x,t) - \frac{u(x,t)}{6} = 0, t \geq 0, 0 < \alpha \leq 1. \tag{30}$$

With initial condition:

$$u(x,0) = \sin(x)\,. \tag{31}$$

By comparing (30) and (31) with (9) and (10), we find that, $g(x,t) = 0$ and $C(x) = \sin(x)$.Thus, we can determine the 4th-approximate LRPS solution to the system (30) and (31). The first step for that is in order to implement the transform of Laplace to system (30) to transfer it to the Laplace space as follows:

$$\mathcal{L}\{D_t^{\alpha}u(x,t)\} - \frac{1}{6}\mathcal{L}\{u(x,t)\} = \mathcal{L}\{0\}. \tag{32}$$

considering that $U(x,s) = \mathcal{L}\{u(x,t)\}$ the following sentence/:

$$s^{\alpha}U(x,s) - s^{\alpha-1}\sin(x) - \frac{1}{6}U(x,s) = 0. \tag{33}$$

Dividing Eq. (33) on s^{α}gives a new form to it as follows:

$$U(x,s) - \frac{\sin(x)}{s} - \frac{1}{6s^{\alpha}}U(x,s) = 0\,. \tag{34}$$

We assume the solution of the algebraic system (34), $U(x,s)$ has expansions in fractional Lorient series form as following:

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$$U(x,s)=\sum_{i=0}^{\infty}\frac{f_i(x)}{s^{1+i\alpha}},0<\alpha\le 1,s>0, \tag{35}$$

The first coefficient of (35) is rewriteable as:

$$U(x,s)=\frac{\sin(x)}{s}+\sum_{i=1}^{\infty}\frac{f_i(x)}{s^{1+i\alpha}},0<\alpha\le 1,s>0. \tag{36}$$

Now, we construction the LRPS solution is defining the LRF of the algebraic system (34) as follows:

$$LRes_1(x,s)=U(x,s)-\frac{\sin(x)}{s}-\frac{1}{6s^{\alpha}}U(x,s), \tag{37}$$

If we considered formula (36)'s kth-truncated which is obtained as:

$$U_k(x,s)=\frac{\sin(x)}{s}+\sum_{i=1}^{k}\frac{f_i(x)}{s^{1+i\alpha}},0<\alpha\le 1,x\in I,s>0, \tag{38}$$

then we can define the kth-LRF as follows:

$$LRes_k(x,s)=U_k(x,s)-\frac{\sin(x)}{s}-\frac{1}{6s^{\alpha}}U_k(x,s), \tag{39}$$

Therefore, the solution of (39) in a series form is:

$$U_4(x,s)=\frac{\sin(x)}{s}+\frac{1}{6}\frac{\sin(x)}{s^{\alpha+1}}+\frac{1}{6}\frac{\sin(x)}{s^{2\alpha+1}}+\frac{1}{6}\frac{\sin(x)}{s^{3\alpha+1}}+\frac{1}{6}\frac{\sin(x)}{s^{4\alpha+1}} \tag{40}$$

Making use of inverse Laplace of formula (40) gives 5$^{\text{th}}$-approximate in the following series form:

$$u_4(x,t)=\sin(x)+\frac{1}{6}\frac{\sin(x)}{\Gamma(1+\alpha)}t^{\alpha}+\frac{1}{6}\frac{\sin(x)}{\Gamma(1+2\alpha)}t^{2\alpha}+\frac{1}{6}\frac{\sin(x)}{\Gamma(1+3\alpha)}t^{3\alpha}+\frac{1}{6}\frac{\sin(x)}{\Gamma(1+4\alpha)}t^{4\alpha}. \tag{41}$$

The accurate result is:

$$u(x,t)=\frac{1}{6}\frac{\sin(x)\,t^{\alpha}}{\Gamma[1+\alpha]}. \tag{42}$$

view some numerical results of the solution obtained in Eq. (41), but also to assess the correctness of technique's approximate result, we'll employ the following two forms of error:

$$Abs.Err.(x,t)=|\theta(x,t)-\theta_k(x,t)|, \tag{43}$$

and

$$Re.Err.(x,t)=\left|\frac{\theta(x,t)-\theta_k(x,t)}{\theta(x,t)}\right|, \tag{44}$$

where $\theta(x,t)$ is really exact result and $\theta_k(x,t)$ is an approximation result of order k.

Tables 4.1, 4.2, and 4.3 display some numerical results for the solution of Application 4.1 given in Eq. (41) at $\alpha=1$, $\alpha=0.75$, and $\alpha=0.5$, and the corresponding absolute and relative errors. The results indicate the accuracy of the 5th-approximate LRPS solutions obtained. It is clear from the results that the region of convergence varies with different α values so that the convergence region becomes smaller when α value decreases. The tables show that the convergent region is $\mathbb{R}\times[0.3,1.2]$when $\alpha=1$, and the

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convergent region is $\mathbb{R} \times [0.3,1.2]$ when $\alpha = 0.75$, whereas the convergent region is $\mathbb{R} \times [0.2,0.6]$ when $\alpha = 0.5$.

Figure 4.1 a, b, c, d, e, and f display the next graph at 5th and 10th estimate LRPS results; $u_5(x,t)$ and $u_{10}(x,t)$ are provided in (41) the exact result for $\alpha = 1$. These graphs illustrate the behavior of the solutions at $\alpha = 1$, $\alpha = 0.75$, and $\alpha = 0.5$, and confirm the region of convergence shown in the tables. Figures 4.1 illustrate the agreement as between fifth close approximation as well as the exact result of $\alpha = 1$, while all graphs depict the consistency of the behavior of results with varying parameters of α.

Table 1: Numerical comparisons between the exact value of $u(x,t)$ and the 5th-approximation of $u(x,t)$ at $\alpha = 1$

x	t	$u_5(x,t)$	$u(x,t)$	Re. Err.	Abs. Err.
0	0.3	0.0998334	0.0998334	1.50086×10^{-13}	4.43534×10^{-14}
3		0.0415807	0.0415807	2.80115×10^{-13}	4.41869×10^{-14}
6		-0.182163	-0.182163	2.56197×10^{-12}	4.30767×10^{-14}
0	0.8	0.198669	0.198669	2.98758×10^{-9}	2.14316×10^{-9}
3		-0.0583741	-0.0583741	3.50063×10^{-9}	2.14189×10^{-9}
6		-0.0830894	-0.0830894	4.24547×10^{-9}	2.09774×10^{-9}
0	1.2	0.29552	0.29552	1.97881×10^{-7}	1.84433×10^{-7}
3		-0.157746	-0.157746	2.29626×10^{-7}	1.85193×10^{-7}
6		0.0168139	0.0168139	2.12481×10^{-7}	1.82247×10^{-7}

Table 2: Numerical comparisons between the 5th-approximationand the 10th-approximation of $u(x,t)$ at $\alpha = 0.75$

x	t	$u_5(x,t)$	$u_{10}(x,t)$	Re. Err.	Abs. Err.
0	0.3	0.29552	0.29552	1.46652×10^{-7}	4.33386×10^{-8}
3		-0.157746	-0.157746	6.32345×10^{-7}	9.97497×10^{-8}
6		0.0168137	0.0168139	0.14324×10^{-4}	2.40842×10^{-7}
0	0.5	0.479427	0.479426	3.22202×10^{-6}	1.54472×10^{-6}
3		-0.350782	-0.350783	4.33204×10^{-6}	1.51961×10^{-6}
6		0.215115	0.21512	0.211673×10^{-4}	4.55352×10^{-6}
0	0.9	0.783421	0.783327	1.19787×10^{-4}	0.938326×10^{-4}
3		-0.687756	-0.687766	0.142139×10^{-4}	9.77584×10^{-6}
6		0.578327	0.57844	1.95679×10^{-4}	0.113189×10^{-3}

Table 3: Numerical comparisons between the 5th-approximation and the 10th-approximation of $u(x,t)$ at $\alpha = 0.5$

x	t	$u_5(x,t)$	$u_{10}(x,t)$	Re.Err.	Abs. Err.
0	0.2	0.198669	0.198669	1.27764×10^{-8}	2.53827×10^{-9}
3		-0.0583741	-0.0583741	1.71689×10^{-7}	1.00222×10^{-8}
6		-0.0830894	-0.0830894	2.69373×10^{-7}	2.2382×10^{-8}
0	0.4	0.389419	0.389418	8.32927×10^{-7}	3.24357×10^{-7}
3		-0.255541	-0.255541	1.87608×10^{-6}	4.79415×10^{-7}
6		0.116548	0.116549	1.09275×10^{-5}	1.27359×10^{-6}
0	0.6	0.564648	0.564642	9.78763×10^{-6}	5.52651×10^{-6}
3		-0.442517	-0.44252	8.1687×10^{-6}	3.61482×10^{-6}
6		0.311529	0.311541	0.40713×10^{-4}	0.126838×10^{-4}

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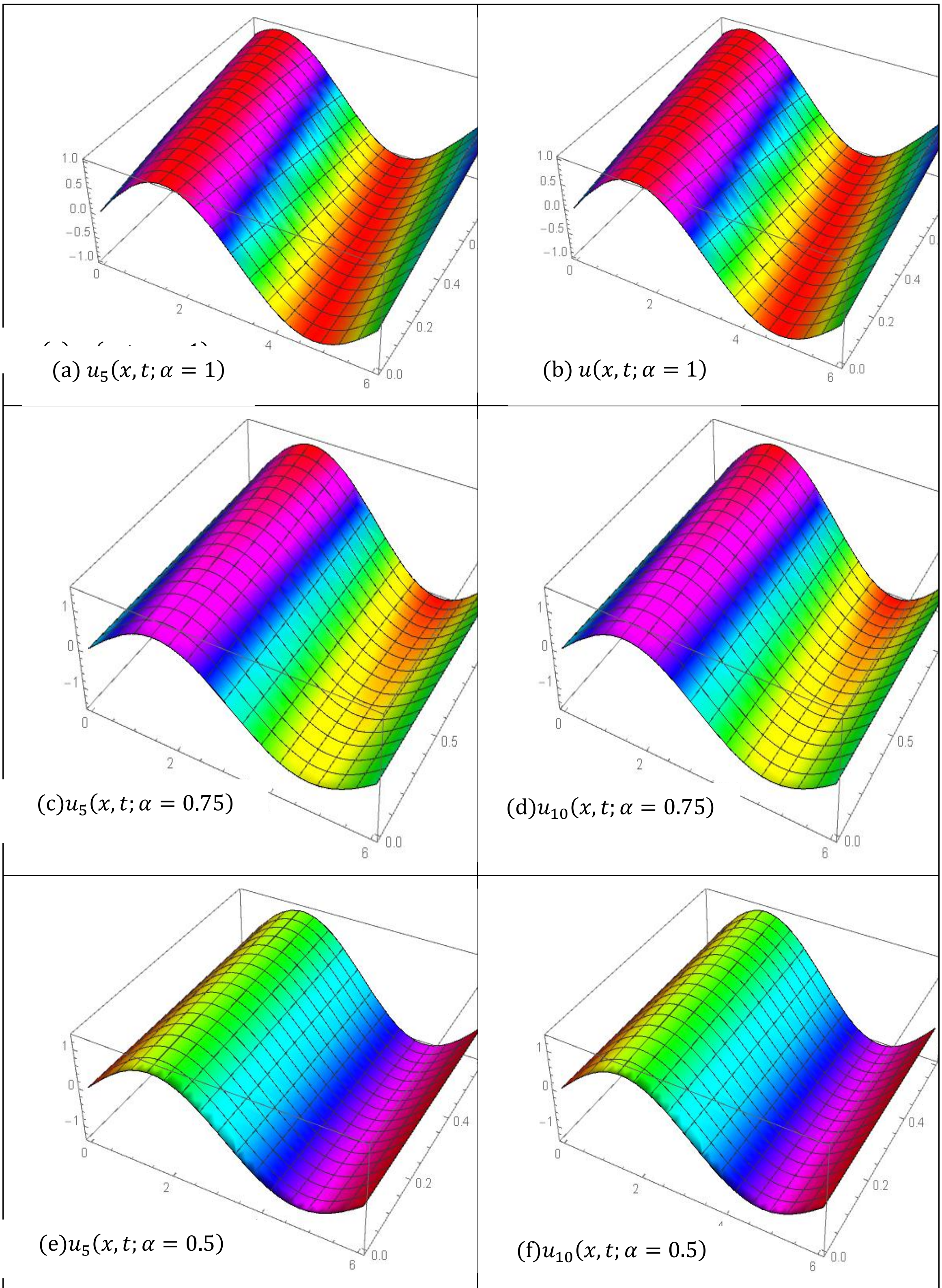

(a) $u_5(x, t; \alpha = 1)$ (b) $u(x, t; \alpha = 1)$

(c) $u_5(x, t; \alpha = 0.75)$ (d) $u_{10}(x, t; \alpha = 0.75)$

(e) $u_5(x, t; \alpha = 0.5)$ (f) $u_{10}(x, t; \alpha = 0.5)$

Figure 4.1: The surface graphs of the 5th and 10th approximations of $u(x, t)$, and the exact solution at different values of α.

Application 4.2: Think about the below neutrosophic FPDEs:

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$$D_t^{\alpha} u(x+yI,t) - \frac{u(x+yI,t)}{6} = 0, t \geq 0, 0 < \alpha \leq 1. \tag{31}$$

With initial condition:

$$u(x+yI,0) = \sin(x+yI)\,. \tag{32}$$

By comparing (31) and (32) with (9) and (10), we find that, $g(x+yI,t) = 0$ and $C(x) = \sin(x+yI)$.Thus, we can determine the 4th-approximate LRPS solution to the system (30) and (31). The first step for that is in order to implement the transform of Laplace to system (30) to transfer it to the Laplace space as follows:

$$\mathcal{L}\{D_t^{\alpha} u(x+yI,t)\} - \frac{1}{6}\mathcal{L}\{u(x+yI,t)\} = \mathcal{L}\{0\}. \tag{33}$$

considering that $U(x+yI,s) = \mathcal{L}\{u(x+yI,t)\}$ the following sentence/:

$$s^{\alpha} U(x+yI,s) - s^{\alpha-1}\sin(x+yI) - \frac{1}{6}U(x+yI,s) = 0. \tag{34}$$

Dividing Eq. (34) on s^{α}gives a new form to it as follows:

$$U(x+yI,s) - \frac{\sin(x+yI)}{s} - \frac{1}{6s^{\alpha}}U(x+yI,s) = 0\,. \tag{35}$$

We assume the solution of the algebraic system (35), $U(x+yI,s)$ has expansions in fractional Lorient series form as following:

$$U(x+yI,s) = \sum_{i=0}^{\infty} \frac{f_i(x+yI)}{s^{1+i\alpha}}, 0 < \alpha \leq 1, s > 0, \tag{36}$$

The first coefficient of (36) is rewriteable as:

$$U(x+yI,s) = \frac{\sin(x+yI)}{s} + \sum_{i=1}^{\infty} \frac{f_i(x+yI)}{s^{1+i\alpha}}, 0 < \alpha \leq 1, s > 0. \tag{37}$$

Now, we construction the LRPS solution is defining the LRF of the algebraic system (35) as follows:

$$LRes_1(x+yI,s) = U(x+yI,s) - \frac{\sin(x+yI)}{s} - \frac{1}{6s^{\alpha}}U(x+yI,s), \tag{38}$$

If we considered formula (37)'s kth-truncated which is obtained as:

$$U_k(x+yI,s) = \frac{\sin(x+yI)}{s} + \sum_{i=1}^{k} \frac{f_i(x+yI)}{s^{1+i\alpha}}, 0 < \alpha \leq 1, x+yI \in I, s > 0, \tag{39}$$

then we can define the kth-LRF as follows:

$$LRes_k(x+yI,s) = U_k(x+yI,s) - \frac{\sin(x+yI)}{s} - \frac{1}{6s^{\alpha}}U_k(x+yI,s), \tag{40}$$

Therefore, the solution of (40) in a series form is:

$$U_4(x+yI,s) = \frac{\sin(x+yI)}{s} + \frac{1}{6}\frac{\sin(x+yI)}{s^{\alpha+1}} + \frac{1}{6}\frac{\sin(x+yI)}{s^{2\alpha+1}} + \frac{1}{6}\frac{\sin(x+yI)}{s^{3\alpha+1}} + \frac{1}{6}\frac{\sin(x+yI)}{s^{4\alpha+1}} \tag{41}$$

Making use of inverse Laplace of formula (41) gives 5th-approximate in the following series form:

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$$u_4(x+yI,t) = \sin(x+yI) + \frac{1}{6}\frac{\sin(x+yI)}{\Gamma(1+\alpha)}t^{\alpha} + \frac{1}{6}\frac{\sin(x+yI)}{\Gamma(1+2\alpha)}t^{2\alpha} + \frac{1}{6}\frac{\sin(x+yI)}{\Gamma(1+3\alpha)}t^{3\alpha} + \frac{1}{6}\frac{\sin(x+yI)}{\Gamma(1+4\alpha)}t^{4\alpha}. \quad (42)$$

T
he accurate result is:

$$u(x+yI,t) = \frac{1}{6}\frac{\sin(x+yI)\,t^{\alpha}}{\Gamma[1+\alpha]}. \quad (43)$$

view some numerical results of the solution obtained in Eq. (42), but also to assess the correctness of technique's approximate result, we'll employ the following two forms of error:

$$Abs.Err.(x+yI,t) = |\theta(x+yI,t) - \theta_k(x+yI,t)|, \quad (44)$$

and

$$Re.Err.(x+yI,t) = \left|\frac{\theta(x+yI,t) - \theta_k(x+yI,t)}{\theta(x+yI,t)}\right|, \quad (45)$$

Table 3: Numerical comparisons between the exact value of $u(x+yI,t)$ and the 5th-approximation of $u(x+yI,t)$ at $\alpha = 1$

$x +$	t	$u_5(x + yI,t)$	$u(x+yI,t)$	Re. Err.	Abs. Err.
0	0.3	0.0998334	0.0998334	1.50086×10^{-13}	4.43534×10^{-14}
3		0.0415807	0.0415807	2.80115×10^{-13}	4.41869×10^{-14}
6		-0.182163	-0.182163	2.56197×10^{-12}	4.30767×10^{-14}
0	0.8	0.198669	0.198669	2.98758×10^{-9}	2.14316×10^{-9}
3		-0.0583741	-0.0583741	3.50063×10^{-9}	2.14189×10^{-9}
6		-0.0830894	-0.0830894	4.24547×10^{-9}	2.09774×10^{-9}
0	1.2	0.29552	0.29552	1.97881×10^{-7}	1.84433×10^{-7}
3		-0.157746	-0.157746	2.29626×10^{-7}	1.85193×10^{-7}
6		0.0168139	0.0168139	2.12481×10^{-7}	1.82247×10^{-7}

5. Conclusions

This study developed LRPS, a novel analytical-numerical method that is a good instrument for resolving FPDEs. The FPDEs are solved numerically using this approach in order to achieve precise results. We were successful in locating approximate methods using our suggested approach in shape of a fast- series convergence via simply calculable parts. The findings indicate that our recommended approach is very effective and provides estimated answers that are close to precise answers with acceptable error rates.

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