An Algorithm for Solving Nonlinear Third-Order Differential Equations Using Exponential Spline Functions

Ahmed R. Khlefha *[1]

[1] Department of Mathematics, College of Education, University of Sumer, Thi-Qar, Iraq.
Emails: arkdsh85@gmail.com

Abstract

This research dealt with the study of the boundary values associated with differential equations, which are of the non-linear and third-order type. A new algorithm was created that uses exponential spline functions to study and address boundary value problems of a general nature. We have demonstrated that the numerical approach used, which was built using exponential spline functions, gives us good and accurate results for these problems, which have been compared to existing numerical methods. We found that the proposed method is accurate and effective compared to other Spalline methods.

Keywords: Exponential spline functions; boundary value problems, absolute errors.

1. Introduction

Nonlinear boundary value problems play an important role in controlling many scientific phenomena and in various specializations in engineering and applied sciences. We have extensively studied higher-order non-linear boundary value problems because of their importance in mathematical applications and can be dealt with in practical applications. Therefore, boundary value problems have received much attention from scholars in the fields of mathematics, physics, and engineering, due to their importance in searching for and analyzing solutions. Obtaining an analytical solution to nonlinear boundary value problems is a difficult task and often cannot be achieved in many problems. To study, analyze and address such issues, we found a lot of research. However, many researchers have encountered a number of difficulties in finding solutions to various problems that exhibit nonlinear properties. The exponential spline function can be considered of great importance in modern studies in order to find solutions to boundary value problems. Specifically, many different methods have been widely applied to third-order boundary value problems, which are a type of ordinary differential equations.Includes references that address third-order boundary value problems through many numerical methods, including the finite difference method [1–3], The authors of the aforementioned research have studied several strategies, including the cubic spline approach [4–6], the quadratic spline function [7–9], the pentagonal spline technique [10–12], non-polynomial spline techniques [13–17], and other methods. These methods contain verification and approximation in solutions to third-order boundary value problems [18,19]. Their method relies on using ordinary cubic spline functions, such as what order is in the form of $\mathcal{T}_n = \mathcal{S}pan\{1, \hat{\eta}, \hat{\eta}^2\}$, and building consistency equations. This previously mentioned method showed second-order convergence, outperforming other previous methods containing finite difference, clustering, and regular splines. Many scholars have discussed extensively the equation of third-degree boundary value problems, as they have found and explored the use of non-polynomial spline functions in order to approximate the solution to these problems. Specifically, researchers in [20,21] studied and investigated a non-polynomial spline method that relies on quadratic functions. In their research, the researchers [20,21] used the quad-spline function, which can be denoted by $\mathcal{T}_n = \mathcal{S}pan\{1, \hat{\eta}, \hat{\eta}^2, \sin\tau\hat{\eta}, \cos\tau\hat{\eta}\}$. Likewise, the researchers in [22] used another type that differs from the quad-spline, which can be denoted by $\mathcal{T}_n =$

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024

$\mathcal{S}pan\{1,\hat{\eta},\hat{\eta}^2,\mathrm{e}^{(\tau\hat{\eta})},\sin\tau\hat{\eta}\}$. Accurate results and convergence were obtained through the use of the approach under discussion, enabling very easy and accurate approximations to be made. The authors [23] developed a unique method using non-polynomial quintic spline operates to solve boundary value problems (BVPs).They implemented a novel set of techniques to solve numerous third-order boundary value problems. Consequently, the proposed approach was used to test numerical instances, coupled with enhanced end conditions. The authors in [24] specifically focused on the creation of a numerical method using quartic spline functions to handle a specific type of third-order boundary value problems (BVPs).The researchers in [25] introduced a specific type of third-order boundary value problems (BVPs) to guide the numerical BVPs based on splines. The investigators in [26] successfully resolved the linear and nonlinear situations. The approach was customized using only four mesh points. They successfully resolved the instances of boundary value problems (BVPs) with distinct conditions. The approach exhibits convergence up to fourth order. A typical third-order boundary value problem (BVP) can be expressed in the following form:

$$v^{(3)}(\hat{\eta}_i)-\mathcal{K}\left(\hat{\eta},v(\hat{\eta}),v^{(1)}(\hat{\eta}),v^{(2)}(\hat{\eta})\right)=\phi(\eta)v^2,\quad \hat{\eta}\in[a,b] \tag{1}$$

With boundary value problems,

$$v(0)=\gamma_1,\quad v^{(1)}(0)=\gamma_2,\quad v^{(1)}(1)=\gamma_3. \tag{2}$$

when γ_i, with i ranging from 1 to 3, is a definite real constant. To obtain the estimates of the answers of the aforementioned of equations (1-2) using various exponenatial spline techniques, we employed exponential quartic spline functions with the following structure, $\mathcal{T}_n=\mathcal{S}pan\{1,\hat{\eta},\hat{\eta}^2,\hat{\eta}^3,\mathrm{e}^{(\tau\hat{\eta})},\mathrm{e}^{(-\tau\hat{\eta})}\}$,where τ is the free parameter.The paper is structured in the following manner . In Section two, we present a concise explanation of the mathematical derivation of this exponential spline function that is not a polynomial. We introduce the spline relations for the purpose of discretizing the provided in equation (1). Section three introduces our numerical approach for solving a set of non-linear third-order boundary-value problems. We discuss the formulation of boundary conditions, truncation error, and a specific class of the method. In Section four , we provide numerical evidence to compare and demonstrate the effectiveness of the methods. Our results show that our algorithm outperforms an iterative method.

2. Methodology Development

To construct the spline approximation for the fourth-order boundary value problem (1.2), the interval $[a,b]$ is partitioned into n equidistant subintervals. To accomplish this, we will introduce a collection of grid points $\hat{\eta}_i=a+ih, i=0,1,2,\dots,n-1,n$. In order to $a=\hat{\eta}_0$, $b=\hat{\eta}_n$ and $h=\frac{b-a}{n}$. Let $v(\hat{\eta})$ represent the precise solution of the equation (1), and v_i yi denote an estimate of $v(\hat{\eta}_i)$ that comes from the portion $E_i(\hat{\eta})$ of a combined spline function that passes through both points $(\hat{\eta}_i,v_i)$ and $(\hat{\eta}_{i+1},v_{i+1})$. The exponatail spline function $E_i(\hat{\eta})$ in the subinterval $[\hat{\eta}_i,\hat{\eta}_{i+1}], i=0,1,..n-1$ takes the form:

$$E_i(\hat{\eta})=\hat{a}_i e^{\tau(\hat{\eta}-\hat{\eta}_i)}+\hat{b}_i e^{-\tau(\hat{\eta}-\hat{\eta}_i)}+\hat{c}_i(\hat{\eta}-\hat{\eta}_i)^3+\hat{d}_i(\hat{\eta}-\hat{\eta}_i)^2+\hat{e}_i(\hat{\eta}-\hat{\eta}_i)+\hat{f}_i, \tag{3}$$

where a_i,b_i,c_i,d_i,e_i and f_i are arbitrary real and finite constants, and τ is a free parameter. In order to find the six coefficients of integration for equation (3) using $v_i,v_{i+1},v_{i+1},\psi_i,\psi_{i+1},\mathrm{Z}_i,\mathrm{Z}_{i+1},\mathcal{M}_i$. and $\mathcal{M}_{i+1}$. We define the interpolator conditions for the variable for $\hat{\eta}_i$ and $\hat{\eta}_{i+1}$

$$\left.\begin{aligned} E_i(\hat{\eta}_i)&=v_i, & E_i(\hat{\eta}_{i+1})&=v_{i+1}\\ E_i^{(2)}(\hat{\eta}_i)&=\psi_i, & E_i^{(2)}(\hat{\eta}_{i+1})&=\psi_{i+1},\\ E_i^{(3)}(\hat{\eta}_i)&=\mathrm{Z}_i, & E_i^{(3)}(\hat{\eta}_{i+1})&=\mathrm{Z}_{i+1},\\ E_i^{(4)}(\hat{\eta}_i)&=\mathcal{M}_i, & E_i^{(4)}(\hat{\eta}_{i+1})&=\mathcal{M}_{i+1}. \end{aligned}\right\}\ \text{for } i=0,1,..,n \tag{4}$$

A spline can be defined in terms of $v_i,v_{i+1},\psi_i,\psi_{i+1},\mathrm{Z}_i,\mathrm{Z}_{i+1},\mathcal{M}_i$ and $\mathcal{M}_{i+1}$, the coefficients presented in Equation (3) are computed as follows:

$$\hat{a}_i=\frac{h^4\left(\mathcal{M}_{i+1}-\mathcal{M}_i e^{\tilde{\theta}}\right)}{\tilde{\theta}^4\left(e^{\tilde{\theta}}-e^{\tilde{\theta}}\right)},$$

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024

$$\hat{b}_i = \frac{h^4\left(\mathcal{M}_i e^{\tilde{\theta}} - \mathcal{M}_{i+1}\right)}{2\tilde{\theta}^4\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)},$$

$$\hat{c}_i = \frac{\psi_{i+1} - \psi_i}{6h} - \frac{h^2(\mathcal{M}_{i+1} - \mathcal{M}_i)}{6h\tilde{\theta}^2},$$

$$\hat{d}_i = \frac{1}{2}\left(\psi_i + \frac{h^2\mathcal{M}_i}{\tilde{\theta}^2}\right),$$

$$\hat{e}_i = \frac{v_{i+1} - v_i}{h} - \frac{h^4(\mathcal{M}_i - \mathcal{M}_{i+1})}{h\tilde{\theta}^4} - \frac{h}{6}(\psi_{i+1} + 2\psi_i) - \frac{h^3}{6\tilde{\theta}^2}(\mathcal{M}_{i+1} + 2\mathcal{M}_i),$$

$$\hat{f}_i = y_i - \frac{h^4\mathcal{M}_i}{\tilde{\theta}^4}.$$

Where $\tilde{\theta} = \tau h$, and $i = 0,1,2,..,n$.

Applying the continuity criteria of the first and third derivatives at the given position $(\hat{\eta}_i, v_i)$, that is $E_{i-1}^{(j)}(\hat{\eta}_i) = E_i^{(j)}(\hat{\eta}_i)$, $j = 1,3$, we get the following relations for $i = 0,1,\dots,n$.

$$\frac{h^3}{\tilde{\theta}}\left(\mathcal{M}_{i-1}\frac{\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)}{2} + \frac{2\mathcal{M}_i\left(e^{\tilde{\theta}} + e^{\tilde{\theta}}\right) - \mathcal{M}_i\left(e^{\tilde{\theta}} + e^{\tilde{\theta}}\right)^2}{\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)}\right) + \frac{h}{2}\left(\psi_i - 3\psi_{i-1} + \frac{3h^2}{\tilde{\theta}^2}(\mathcal{M}_i - \mathcal{M}_{i-1})\right) + \frac{h^3\mathcal{M}_i}{\tilde{\theta}^2} + \frac{h^3\mathcal{M}_{i-1}}{\tilde{\theta}^4} +$$
$$\frac{h^3\mathcal{M}_i}{\tilde{\theta}^4} - \frac{h\psi_{i-1}}{6} - \frac{h\psi_{i-1}}{3} - \frac{h^3\mathcal{M}_i}{3\tilde{\theta}^2} - \frac{h^3\mathcal{M}_{i-1}}{3\tilde{\theta}^2} = \mathcal{M}_{i+1}\left(\frac{2h^3}{\tilde{\theta}^3\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)} - \frac{h^5}{6\tilde{\theta}^2} - \frac{h^3}{\tilde{\theta}^4}\right) - \mathcal{M}_i\left(\frac{h^3\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)}{2\tilde{\theta}^3} - \frac{2h^3}{6\tilde{\theta}^2} + \frac{h}{\tilde{\theta}^2}\right)$$
$$-\tfrac{1}{6}(2h\psi_i - h\psi_{i+1}) + \tfrac{1}{h}(v_{i+1} - v_i).$$
(5)

Upon simplification, Equation (5) results in:

$$\psi_{i+1} + 4\psi_i + \psi_{i+1} = \tfrac{6}{h^2}(v_{i+1} - 2v_i + v_{i-1}) - 6h^2(\omega_1\mathcal{M}_{i-1} + 2\delta_1\mathcal{M}_i + \omega_1\mathcal{M}_{i+1}).$$
(6)

where $\omega_1 = \frac{3h^3}{6\tilde{\theta}^2} - \frac{2h^3}{3\tilde{\theta}^2} + \frac{2h^2}{\tilde{\theta}^2\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)}$, $\delta_1 = \frac{3\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)}{2h\tilde{\theta}^3}$.

By utilising the regularity condition of the third derivative at the locations of intersection, we once again derive:

$$\frac{h\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)}{2\tilde{\theta}} - \frac{2h\mathcal{M}_i}{\tilde{\theta}\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)} + \frac{h\left(e^{\tilde{\theta}} + e^{\tilde{\theta}}\right)^2}{4\tilde{\theta}} + \frac{\psi_i}{h} - \frac{\psi_{i-1}}{h} + \frac{h\mathcal{M}_i}{\tilde{\theta}^2} - \frac{h\mathcal{M}_{i-1}}{\tilde{\theta}^2} = -\frac{h\mathcal{M}_{i+1}}{\tilde{\theta}\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)} + \frac{h\left(e^{\tilde{\theta}} + e^{\tilde{\theta}}\right)\mathcal{M}_i}{\tilde{\theta}\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)}$$
$$+\tfrac{\psi_{i+1}}{h} - \tfrac{\psi_i}{h} + \tfrac{\psi_{i-1}}{h} + \tfrac{h\mathcal{M}_{i+1}}{\tilde{\theta}^2} - \tfrac{h\mathcal{M}_i}{\tilde{\theta}^2}.$$
(7)

By reducing the complexity of Equation (7), we obtain:

$$\psi_{i+1} - 2\psi_i + \psi_{i+1} = \omega_2\mathcal{M}_{i-1} + 2\delta_2\mathcal{M}_i + \omega_2\mathcal{M}_{i+1}.$$
(8)

Where $\omega_2 = \frac{h^4}{\tilde{\theta}^3} - \frac{1}{k^2}$ and $\delta_2 = \frac{h^2}{\tilde{\theta}^2} - \frac{h^4\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)}{\tilde{\theta}^3}$.

By subtracting Equation (8) from Equation (9), we may derive the following:

$$\psi_i = \tfrac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) - h^2\left(\left(\omega_1 + \tfrac{\lambda_2}{6}\right)\mathcal{M}_{i+1} + 2\left(\delta_1 + \tfrac{\rho_2}{6}\right)\mathcal{M}_i + \left(\omega_1 + \tfrac{\omega_2}{6}\right)\mathcal{M}_{i-1}\right).$$
(9)

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024

By utilising the concept of continuous for the third derivatives and using Equation (9), we obtain the following kind of relationship:

$\mathrm{Z}_i = \frac{1}{h^3}(v_{i+2} - 3v_{i+1} + 3v_i - v_{i-1}) - h(q\mathcal{M}_{i+2} + (q_1 - q + \mu_1)\mathcal{M}_{i+1} + (q - q_1 + \mu_2)\mathcal{M}_i - q\mathcal{M}_{i-1})$. (10)

Where $q = \lambda_1 + \frac{\lambda_2}{6}, q_1 = 2\left(\delta_1 + \frac{\delta_2}{6}\right),\ \mu_1 = \frac{1}{\tilde{\theta}^2}\left(\tilde{\theta}\frac{2}{\left(e^{\tilde{\theta}} - e^{\tilde{\theta}}\right)} - 1\right)$ and $\mu_2 = \frac{1}{\tilde{\theta}^2}\left(1 - \tilde{\theta}\frac{\left(e^{\tilde{\theta}} + e^{\tilde{\theta}}\right)}{2}\right)$.

By utilizing equations (10) and (8), we derive the following relationship:

$(v_{i+2} - 2v_{i+1} + 2v_{i-1} - v_{i-2}) = \Sigma(\mathrm{Z}_{i+2} + \mathrm{Z}_{i-2}) + s\mathrm{Z}_i + \ell(\mathrm{Z}_{i+1} + \mathrm{Z}_{i-1})$.
(11)

Where $\Sigma = \left(\frac{h^3 q}{\mu_1 + \mu_2}\right)$, for $i = 2,3,\dots,n-2$.

3. Explanation of the Approach and Formulation of Boundary Conditions

The suggested differential equation corresponds to the given mesh point $\hat{\eta}_i$.
$v^{(3)}(\hat{\eta}_i) - \mathcal{K}\left(\hat{\eta}, v(\hat{\eta}), v^{(1)}(\hat{\eta}), v^{(2)}(\hat{\eta})\right) = \phi(\eta)v^2, \quad \hat{\eta} \in [a,b]$
(12)

subjected to boundary conditions (2), may be discretized by

$\mathrm{Z}_i + \mathcal{K}_i = \phi_i v_i^2$
(13)

Where $\mathrm{Z}_i = E_i^{(3)}(\hat{\eta}_i)$ and $\phi_i = \phi(x_i)$.

In equation (13) we get by applying the spline relation (11).

$$-(\mu_1 + \mu_2)v_{i-2} + (2\mu_1 - 4\mu_2)v_{i-1} + (-2\mu_1 + 4\mu_2 - h^2 q_1\mathcal{K}_{i+1})v_{i+1} + (-\mu_1 - \mu_2 - qh^2\mathcal{K}_{i+2})v_{i+2}$$

$-h^2(q\mathcal{K}_{i-2} + q_1\mathcal{K}_{i-1} + \rho\mathcal{K}_i) = -h^3(q\phi_{i-2}v_{i-2}^2 + q_1\phi_{i-1}v_{i-1}^2 + \rho\phi_i v_i^2 + q_1\phi_{i+1}v_{i+1}^2$.
(14)

In order to obtain a singular solution, it is necessary to establish additional equations that are connected to equation (14) and incorporate the specified boundary conditions. To obtain the calculation for the second-order boundary, we define the following:

$(\mu_1 + \mu_2)[v_1 - 3v_2 + 3v_3 - v_4] = h\left(-v_2^{(3)} - v_3^{(3)}\right) - h^3\left[\mu_1 v_3^{(3)} - \mu_2 v_2^{(3)}\right],\ i = 1,$
(15)

$(\mu_1 + \mu_2)[v_{n-3} - 3v_{n-2} + 3v_{n-1} - v_n] = h\left(-v_{n-1}^{(3)} - v_{n-1}^{(1)}\right) - h^3\left[\mu_1 v_{n-1}^{(3)} - \mu_2 v_{n-2}^{(3)}\right], i = n-1.$
(16)

for any choice of μ_1 and $\mu_2, \mu_1 + \mu_2 = \frac{1}{2}$. Using Equation (1) we have:

$(\mu_1 + \mu_2)[v_1 - 3v_2 - 3v_3 - v_4] - (h\mathcal{K}_2 + \mu_2 h^3\mathcal{K}_2 - h\mathcal{K}_3 + \mu_1 h^3\mathcal{K}_3) = [-h + \mu_2 h^3]\phi_2 v_i^2 -$

$-[h + \mu_1 h^3]\phi_3 v_i^2,$
(17)

$(\mu_1 + \mu_2)[v_{N-3} - 3v_{N-2} - 3v_{N-1} - v_N] - (h\mathcal{K}_{N-2} + \mu_2 h^3\mathcal{K}_{N-2} - h\mathcal{K}_{N-3} + \mu_1 h^3\mathcal{K}_{N-3})$

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024

$= [-h + \mu_2 h^3]\phi_{n-2}v_{n-2}^2 - [h + \mu_1 h^3]\phi_{n-1}v_{n-1}^2.$
(18)

In order to get the fourth-order boundary calculation, we construct the following definitions:

$(\mu_1 + \mu_2)[v_1 - 3v_2 + 3v_3 - v_4] \ = \frac{h}{2}\left(-v_2^{(3)} - v_3^{(3)}\right) - \frac{h^3}{8}\left[\mu_1 v_3^{(3)} - \mu_2 v_3^{(3)}\right],\ i = 1.$
(19)

$(\alpha + \beta)[v_{n-3} - 3v_{n-2} + 3v_{n-1} - v_n] = \frac{h}{2}\left(-v_{n-2}^{(3)} - v_{n-1}^{(3)}\right) - \frac{h^3}{8}\left[\mu_2 v_{n-1}^{(3)} - \mu_2 v_{n-2}^{(3)}\right],\ i = n-1.$
(20)

for any choice of μ_1 and $\mu_2, \mu_1 + \mu_2 = \frac{1}{2}$. Using Eqution(3) we have:

$(\mu_1 + \mu_2)[v_1 - 3v_2 - 3v_3 - v_4] - \left(\frac{h}{2}\mathcal{K}_2 + \mu_2\frac{h^3}{8}\mathcal{K}_2 - \frac{h}{2}\mathcal{K}_3 + \mu_1\frac{h^3}{8}\mathcal{K}_3\right) = \left[-\frac{h}{2} + \mu_2\frac{h^3}{8}\right]\phi_2 v_i^2$

$-\left[\frac{h}{2} + \mu_1\frac{h^3}{8}\right]\phi_3 v_i^2,$
(21)

$(\mu_1 + \mu_2)[v_{n-3} - 3v_{n-2} - 3v_{n-1} - v_n] - \left(\frac{h}{2}\mathcal{K}_{n-2} + \mu_2\frac{h^3}{8}\mathcal{K}_{n-2} - \frac{h}{2}\mathcal{K}_{n-1} + \mu_1\frac{h^3}{8}\mathcal{K}_{n-1}\right)$

$= \left[-\frac{h}{2} + \mu_2\frac{h^3}{8}\right]\phi_{n-2}v_{n-2}^2 - \left[\frac{h}{2} + \mu_1\frac{h^3}{8}\right]\phi_{n-1}v_{n-2}^2.$
(22)

By utilizing the Taylor series expansion on function (11) centered at the point $\hat{\eta}_i$, we may determine the precise value of the local truncation error in the following manner:

$\mathcal{T}_i = \left[\frac{1}{6}(9\mu_1 - 2\mu_2) - (4q + q_1)\right]h^4 v_i^{(4)} + \left(\frac{1}{1806}(33\mu_1 - 3\mu_2) - \frac{1}{12}(16q + q_1)\right)h^6 v_i^{(6)}$

$+\left[\frac{1}{313111}(6113\mu_1 + 27\mu_2) - \frac{1}{36}(4q + q_1)\right]h^8 v_i^{(8)} + O(h^9).$
(23)

4. Numerical Examples:

Now, we will examine four numerical instances to demonstrate the relative effectiveness of our strategy in scheme (14). All calculations are implemented by Maple 22.
Example1: We consider the following nonlinear boundary value problems :

$v^{(3)}(\hat{\eta}) - v(\eta) + \left(v^{(1)}(\hat{\eta})\right)^2 + \left(v(\eta)\right)^3 = \left(e^{\hat{\eta}}(\eta - \hat{\eta}^2) + e^{\hat{\eta}}(\hat{\eta}\ +\ 1)\right)^2 + 5e^{\hat{\eta}}(-2.\hat{\eta} +\ 1) - 8e^{\hat{\eta}}$

$v(0) = 0, v^{(1)}(0) = 1, v^{(1)}(1) = -e.$

With exact solution $v(\hat{\eta}) = e^{\hat{\eta}}(\eta - \hat{\eta}^2)$.

We solved this problem by considering different values of n $(n = 10)$, juxtaposing the outcomes with an accurate answer, and displaying the largest absolute errors produced from our computations in Table 1. Figure 1 depicts the comparison between the real solution and the numerical solution

Example 2: We consider the following nonlinear boundary value problems :

$v^{(3)}(\hat{\eta}) + v^{(2)}(\hat{\eta}) + \left(v^{(1)}(\hat{\eta})\right)^2$
$= e^{\widehat{3\eta}}\sin^3(\pi\hat{\eta})\ +\ (e^{\hat{\eta}}\sin(\pi\hat{\eta}) + \pi e^{\hat{\eta}}\cos^2(\pi\hat{\eta}))\ - 2e^{\hat{\eta}}\sin(\pi\hat{\eta})\ -\ 21e^{\hat{\eta}}\cos(\pi\hat{\eta})$

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024

$v(0) = 0, v^{(1)}(0) = 1, v^{(1)}(1) = -e.$

With exact solution $v(\hat{\eta}) = e^{\hat{\eta}} \sin(\pi\hat{\eta})$.

We solved this problem by considering different values of n $(n = 10)$, juxtaposing the outcomes with an accurate answer, and displaying the largest absolute errors produced from our computations in Table 2. Figure 2 depicts the comparison between the real solution and the numerical solution.

Table 1: Comparison between the exact solutions and proposed method for Example 1

ξ	Exact solution	Proposed method	Absolute errors
0.0	0.0000000000	0.0000000000	0.00000000
0.1	0.19346538262	0.19346538256	0.00000000
0.2	0.1954244413	0.1954244409	2.04×10^{-9}
0.3	0.2834703497	0.2834703489	2.82×10^{-9}
0.4	0.3580379275	0.3580379263	3.35×10^{-9}
0.5	0.4121803178	0.4121803167	2.66×10^{-9}
0.6	0.4373085120	0.4373085113	1.60×10^{-9}
0.7	0.4228880685	0.4228880677	1.89×10^{-9}
0.8	0.3560865485	0.3560865476	2.52×10^{-9}
0.9	0.2213642800	0.2213642811	4.96×10^{-9}
1.0	0.000000000	0.000000000	0.00000000

Table 2: Comparison between the exact solutions and proposed method for Example 2

ξ	Exact solution	Proposed method	Absolute errors
0.0	0.000000000	0.000000000	0.00000000
0.1	0.3413491904	0.3413491913	2.63×10^{-9}
0.2	0.7176077403	0.7176077432	4.04×10^{-9}
0.3	1.091679495	1.091679487	7.32×10^{-9}
0.4	1.418515628	1.418515616	8.45×10^{-9}
0.5	1.648720748	1.648720735	7.88×10^{-9}
0.6	1.733475228	1.733475216	6.92×10^{-9}
0.7	1.630478755	1.630478747	4.90×10^{-9}
0.8	1.310433132	1.310433129	2.28×10^{-9}
0.9	0.7634113915	0.7634113902	1.70×10^{-9}
1.0	0.0043292794	0.0043292794	0.0000000

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024

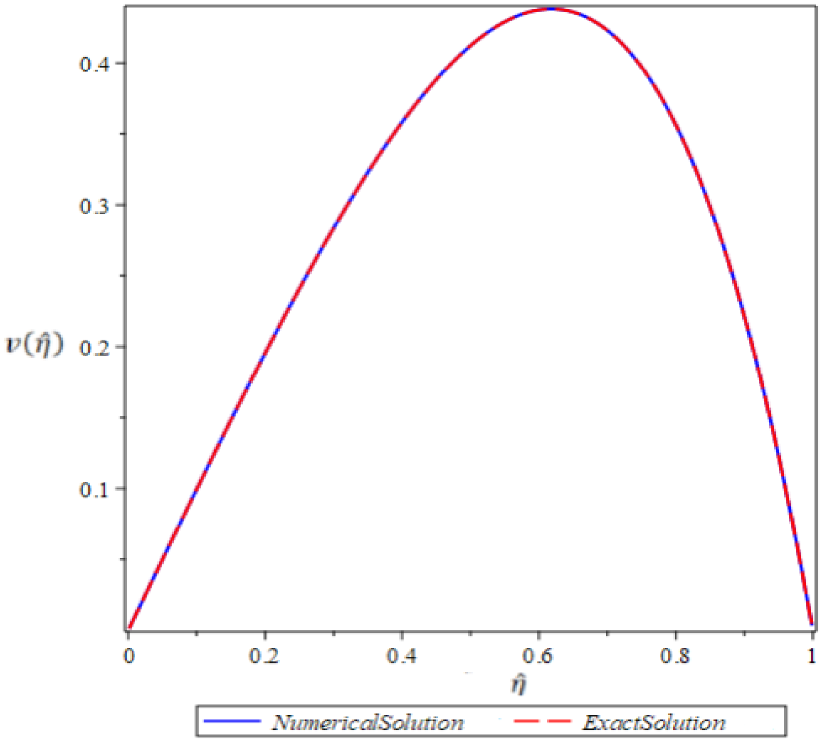

Figure 1: The exact and numerical solutions of Example 1

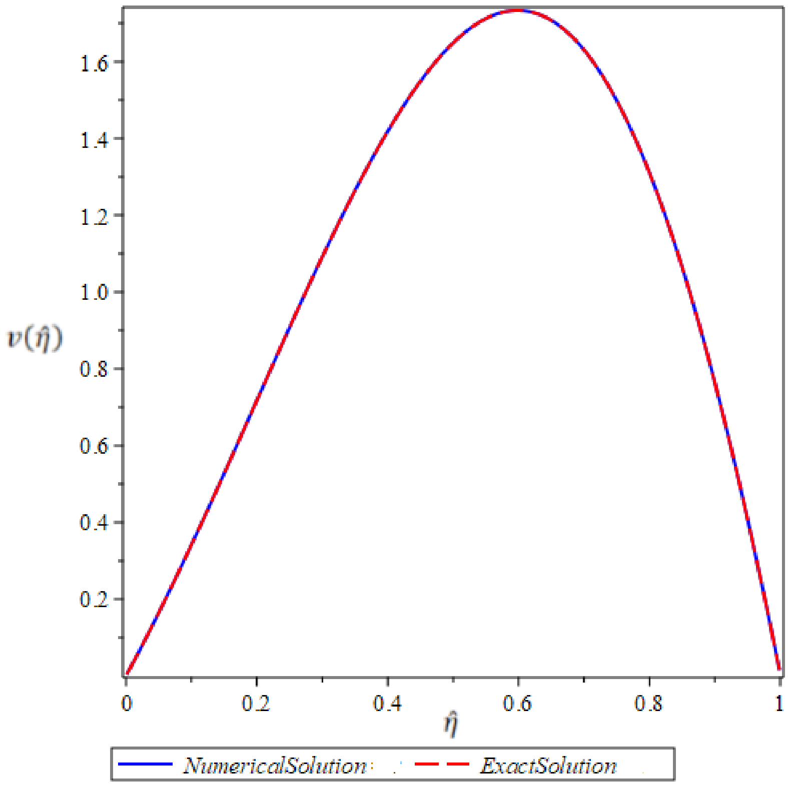

Figure 2: The exact and numerical solutions of Example 2

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024

5. Conclusion

In this study, the exponential spline function was used to generate numerical solutions to a set of third-order nonlinear boundary value problems. We used our mathematical model to study many numerical cases and obtained applicable results. The results are well presented through graphs and regular tables. When testing the numbers and tables with high accuracy, it became clear that the results on them show that the calculated order of convergence increases in relation to the method proposed by us. The proposed method can have great importance in many fields of applied sciences and engineering, especially in studying and treating boundary value problems of higher order and similar types.

Funding: “This research received no external funding”

Conflicts of Interest: “The authors declare no conflict of interest.”

References

[1] Al-Said EA. Numerical solutions for system of third-order boundary value problems. Int J Comput Math 2001;78:111–21. https://doi.org/10.1080/00207160108805100.

[2] Noor MA, Al-Said EA. Finite-difference method for a system of third-order boundary-value problems. J Optim Theory Appl 2002;112:627–37. https://doi.org/10.1023/a:1017972217727.

[3] Noor MA, Al-Said E, Noor KI. Finite difference method for solving a system of third-order boundary value problems. J Appl Math 2012;2012:1–10. https://doi.org/10.1155/2012/351764.

[4] Mamatha K, Phaneendra K. Solution of convection-diffusion problems using fourth order adaptive cubic spline method. J Math Comput Sci 2020;10:817–32. https://doi.org/10.28919/jmcs/4485.

[5] Mohanty RK, Kumar R, Setia N. Cubic spline approximation based on half-step discretization for 2D quasilinear elliptic equations. Int J Comput Methods Eng Sci Mech 2020;22:45–59. https://doi.org/10.1080/15502287.2020.1849444.

[6] Khandelwal P, Khan A, Sultana T. Discrete cubic spline technique for solving one-dimensional Bratu’s problem. Asian-European J Math 2022;15:2250011. https://doi.org/10.1142/s1793557122500115.

[7] Bica AM, Curilă D, Satmari Z. Improved error estimate and applications of the complete quartic spline. Gen Math 2019;27:71–83. https://doi.org/10.2478/gm-2019-0016.

[8] Wen W, Deng S, Liu T, Duan S, Huang F. An improved quartic B-spline based explicit time integration algorithm for structural dynamics. Eur J Mech 2022;91:104407. https://doi.org/10.1016/j.euromechsol.2021.104407.

[9] Vaid MK, Arora G. Quintic B-spline technique for numerical treatment of third order singular perturbed delay differential equation. Int J Math Eng Manag Sci 2019;4:1471. https://doi.org/10.33889/ijmems.2019.4.6-116.

[10] Huang W-F, Ren Y-X, Wang Q, Jiang X. High resolution finite volume scheme based on the quintic spline reconstruction on non-uniform grids. J Sci Comput 2018;74:1816–52.

[11] El Hajaji A, Serghini A, Melliani S, El Ghordaf J, Hilal K. A quintic spline collocation method for solving time-dependent convection-diffusion problems. Tatra Mt Math Publ 2021;80:15–34. https://doi.org/10.2478/tmmp-2021-0029.

[12] Khan A, Sultana T. Parametric quintic spline solution of third-order boundary value problems. Int J Comput Math 2012;89:1663–77. https://doi.org/10.1080/00207160.2012.689480.

[13] Sultana T, Khan A, Khandelwal P. A new non-polynomial spline method for solution of linear and non-linear third order dispersive equations. Adv Differ Equations 2018;2018:1–14. https://doi.org/10.1186/s13662-018-1763-z.

[14] Wakijira YA, Duressa GF, Bullo TA. Quintic non-polynomial spline methods for third order singularly perturbed boundary value problems. J King Saud Univ 2018;30:131–7. https://doi.org/10.1016/j.jksus.2017.01.008.

[15] Kumara Swamy D, Siva Prasad E, Soujanya G. Solution of convection-diffusion problems with singularity using variable mesh spline of third order. J Math Comput Sci 2020;10:1399–408.

[16] Chaurasia A, Gupta Y, Srivastava PC. A smooth approximation for non-linear second order

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024

boundary value problems using composite non-polynomial spline functions. Stud Univ Babes-Bolyai, Math 2020;65.
[17] Hamasalh FK, Headayat MA. The applications of non-polynomial spline to the numerical solution for fractional differential equations. AIP Conf. Proc., vol. 2334, AIP Publishing; 2021.
[18] Al-Said EA, Noor MA. Cubic splines method for a system of third-order boundary value problems. Appl Math Comput 2003;142:195–204.
[19] Said EA. A cubic spline method for solving a system of third order boundary value problems. Int J Pure Appl Math 2008;43:677–86.
[20] Khan MA, Tirmizi IA, Twizell EH. Non polynomial spline approach to the solution of a system of third-order boundary-value problems. Appl Math Comput 2005;168:152–63.
[21] Tirmizi IA, Khan MA. Quartic non-polynomial spline approach to the solution of a system of third-order boundary-value problems. J Math Anal Appl 2007;335:1095–104. https://doi.org/10.1016/j.jmaa.2007.02.046.
[22] Li L, Gao J. Non-polynomial spline method for a system of third order obstacle boundary value problems. J Math Sci Adv Appl 2014;30:49–69.
[23] Khan A, Sultana T. Non-polynomial quintic spline solution for the system of third order boundary-value problems. Numer Algorithms 2012;59:541–59. https://doi.org/10.1007/s11075-011-9503-4.
[24] Noor MA, Al-Said EE. Quartic splines solutions of third-order obstacle problems. Appl Math Comput 2004;153:307–16.
[25] Khan A, Aziz T. The numerical solution of third-order boundary-value problems using quintic splines. Appl Math Comput 2003;137:253–60. https://doi.org/10.1016/s0096-3003(02)00051-6.
[26] Salama AA, Mansour AA. Fourth-order finite-difference method for third-order boundary-value problems. Numer Heat Transf Part B 2005;47:383–401.

Doi: https://doi.org/10.54216/GJMSA.0100205
Received: October 18, 2023 Revised: January 22, 2024 Accepted: March 27, 2024