Classification of States for Literal Neutrosophic and Plithogenic Markov Chains

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Abstract

In this paper we represent many classifications of neutrosophic and plithogenic Markov Chains states including absorbent states, inessential and essential states, recurrent states and communicated states. We prove that if a state (i) according to a neutrosophic Markov Chain with neutrosophic transition matrix $M_N = A + BI$ is classified as any of the previous classifications then it is also classified as the same classification in classical scene to two Markov Chains defined with transition matrices $A, A + B$ respectively. Also, we prove that if a state (i) according to a plithogenic Markov Chain with plithogenic transition matrix $M_P = A + BP_1 + CP_2$ is classified as any of the previous classifications then it is also classified as the same classification in classical scene to three Markov Chains defined with transition matrices $A, A + B, A + B + C$ respectively. Many theorems and solved examples are presented and solved successfully.

Keywords: Neutrosophic; Plithogenic; Markov Chain, Absorbent States; Inessential States; Essential States; Recurrent States; Communicated States.

1. Introduction

Neutrosophic logic and plithogenic logic are the most recent extensions to logic presented by F. Smarandache. [1]–[4]. Applications of neutrosophic logic in wide domain of science are recently raised including linear algebra, abstract algebra, cryptography, statistics, probability theory, queueing theory, artificial intelligence, etc. [5]–[16]. In probability theory, researchers presented an idea to defined neutrosophic random variables in the form $X_N = X + I\,; I^2 = I$ [17]. This definition was used in many other papers to generalize many concepts related to probability theory, see [18]–[20].

Concept of literal neutrosophic probability was introduced and studied well by M.B. Zeina and M. Abobala in [21]. Upon this new concept, many neutrosophic probability distributions were generated and extended from crisp logic to neutrosophic logic like neutrosophic exponential distribution, neutrosophic gamma distribution, neutrosophic Kumaraswamy distribution, neutrosophic Marshall Olkin classes of distributions [21]–[24]. M.B. Zeina et al presented a new generalization of literal neutrosophic probability theory to what is known by literal plithogenic probability theory and studied many applications of this new theory in many fields of probability theory [22], [23], [25]–[28].

Recently, concept of literal neutrosophic and plithogenic Markov Chains was presented by S. Massassati [28]. Neutrosophic and plithogenic transition matrices were presented in the form $M_N = A + BI\,; I^2 = I, M_P = A + BP_1 + CP_2\,; P_1^2 = P_2^2 = P_1P_2 = P_2P_1 = P_2$ where A, B, C are classical matrices fulfill some conditions. Many properties of these matrices were studied well and formed the mathematical form of Literal Neutrosophic and Plithogenic Markov

Doi: https://doi.org/10.54216/JNFS.080206
Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

Chains. In this paper we are going to complete this last study about Markov Chains and close a gap in literature about classifying states of Markov Chains in both Neutrosophic and Plithogenic concepts.

Definition 2.1 [28]

A collection of neutrosophic random variables $X_0, X_1, X_2, \dots$ satisfying:

$Pr\{X_{n+1} = i_{n+1}|X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\} = Pr\{X_{n+1} = i_{n+1}|X_n = i_n\}$

is called a literal neutrosophic markov chain if:

$$Pr\{X_{n+1} = i_{n+1}|X_n = i_n\} = p_1 + p_2 I; 0 \le p_1 \le 1, 0 \le p_1 + p_2 \le 1, I^2 = I.$$

Definition 2.2

A collection of Plithogenic random variables $X_0, X_1, X_2, \dots$ satisfying:

$Pr\{X_{n+1} = i_{n+1}|X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\} = Pr\{X_{n+1} = i_{n+1}|X_n = i_n\}$

is called a literal plithogenic markov chain if:

$$Pr\{X_{n+1} = i_{n+1}|X_n = i_n\} = p_0 + p_1 P_1 + p_2 P_2; 0 \le p_0 \le 1, 0 \le p_0 + p_1 \le 1, 0 \le p_0 + p_1 + p_2 \le 1, P_i^2 = P_i, P_i * P_j = P_j * P_i = P_{\max(i,j)}, i = 0,1,2, j = 0,1,2.$$

Theorem 2.1

A squared neutrosophic matrix:

$$M_N = A + BI = [a_{ij} + b_{ij} I]_{n\times n}$$

Is a neutrosophic Markov transition matrix if both $A, A + B$ are Markov transition matrices in classical sense.

Theorem 2.2

A squared plithogenic matrix:

$$M_P = A + BP_1 + CP_2 = [a_{ij} + b_{ij} P_1 + c_{ij} P_2]_{n\times n}$$

Is a plithogenic Markov transition matrix if $A, A + B, A + B + C$ are Markov transition matrices in classical sense.

2. Classification of neutrosophic Markov chains states

Let in the following $M_N = [m_{ij}]_{n\times n}$ be a neutrosophic Markov transition matrix

Definition 3.1

We say that state i is an absorbent state if:

$$m_{ii}^{(n)} = 1 \ \forall\, n \ge 1 ; \mathrm{i} = 1,2, \dots, n$$

Theorem 3.1

If the state i is absorbent according to the transition matrix $M_N = A + BI$ then it is absorbent in classical sense according to A and A + B.

Doi: https://doi.org/10.54216/JNFS.080206
Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

Proof:
We have:

$$M_N = \begin{bmatrix} m_{11} & \cdots & m_{1i} & \cdots & m_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ m_{n1} & \cdots & m_{ni} & \cdots & m_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11}I & \cdots & a_{1i}+b_{1i}I & \cdots & a_{1n}+b_{1n}I \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1}+b_{n1}I & \cdots & a_{ni}+b_{ni}I & \cdots & a_{nn}+b_{nn}I \end{bmatrix}$$

which means that:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \quad \& \quad A+B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1i}+b_{1i} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{ni}+b_{ni} & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

And this completes the proof.

Example 3.1:
Let:

$$M_N = \begin{bmatrix} 1 & 0 & 0 \\ 0.4+0.1I & 0 & 0.6-0.1I \\ 0.3-0.2I & 0 & 0.7+0.2I \end{bmatrix}$$

We can conclude that state (1) is absorbent since:

$$M_N^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.4+0.1I & 0 & 0.6-0.1I \\ 0.3-0.2I & 0 & 0.7+0.2I \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0.4+0.1I & 0 & 0.6-0.1I \\ 0.3-0.2I & 0 & 0.7+0.2I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.58-0.03I & 0 & 0.42+0.03I \\ 0.51-0.32I & 0 & 0.49+0.32I \end{bmatrix}$$

$$M_N^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.58-0.03I & 0 & 0.42+0.03I \\ 0.51-0.32I & 0 & 0.49+0.32I \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0.4+0.1I & 0 & 0.6-0.1I \\ 0.3-0.2I & 0 & 0.7+0.2I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.706-0.111I & 0 & 0.294+0.111I \\ 0.657-0.386I & 0 & 0.343+0.368I \end{bmatrix}$$

And so on for all the powers $n \geq 1$, hence state (1) is absorbent.

We also can conclude that M_N is equivalent to the matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0 & 0.6 \\ 0.3 & 0 & 0.7 \end{bmatrix}, A+B = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0 & 0.9 \end{bmatrix}$$

Where the first element of matrices A^n and $(A+B)^n$ is equal to 1 $\forall n \geq 1$, i.e. state (1) is an absorbent state of matrices A and A + B.

Definition 3.2

We say that state i is an inessential state if there exists some $k \in \mathbb{N}$ such that:

$$m_{ij}^{(k)} >_N 0$$

But:

$$m_{ji}^{(n)} = 0 \; \forall n \in \mathbb{N}$$

Where:

$$i \neq j \;\&\; i,j = 1,\ldots,n$$

Doi: https://doi.org/10.54216/JNFS.080206
Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

Otherwise, we say that state i is essential.

Theorem 3.2

If the state i is an inessential state according to the transition matrix $M_N = A + BI$ then it is inessential in classical sense according to A and A + B.

Proof

Since i is inessential then there exists some $k \in \mathbb{N}$ such that:

$$m_{ij}^{(k)} = \left(a_{ij} + b_{ij}I\right)^{(k)} >_N 0$$

Which means that:

$$\left(a_{ij} + b_{ij}\right)^{(k)} > 0 \ \& \ \left(a_{ij}\right)^{(k)} > 0$$

Also,

$$m_{ji}^{(n)} = \left(a_{ji} + b_{ji}I\right)^{(n)} = 0 \ \forall n \in \mathbb{N}$$

And this means that:

$$a_{ji} = 0, b_{ji} = 0$$

So:

$$a_{ji} + b_{ji} = 0$$

Finally, we conclude that i is inessential in classical sense according to A and A + B.

Example 3.2

Let:

$$M_N = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5+0.4I & 0.5-0.4I \\ 0 & 0 & 0 & 0.5+0.1I & 0 & 0.5-0.1I \\ 0 & 0 & 0 & 0.5-0.3I & 0.5+0.3I & 0 \end{bmatrix} \end{array}$$

We can see immediately that:

state (1) is inessential state because $m_{12} = 1 > 0 \ \& \ m_{j1}^{(n)} = 0 \ \forall \ n \in N$
state (2) is inessential state because $m_{23} = 1 > 0 \ \& \ m_{j2}^{(n)} = 0 \ \forall \ n \in N$
state (3) is inessential state because $m_{34} = 1 > 0 \ \& \ m_{j3}^{(n)} = 0 \ \forall \ n \in N$
state (4) is essential state because $m_{45} = 0.5 + 0.4I >_N 0 \ \& \ m_{54} = 0.5 + 0.1I >_N 0$
state (5) is essential state because $m_{56} = 0.5 - 0.1I >_N 0 \ \& \ m_{65} = 0.5 + 0.3I >_N 0$
state (6) is essential state because $m_{64} = 0.5 - 0.3I >_N 0 \ \& \ m_{46} = 0.5 - 0.4I >_N 0$

Definition 3.3

We say that state j is accessible state from state i if it fulfils:

$$m_{ij}^{(n)} > 0 \text{ for every } n \geq 1$$

Doi: https://doi.org/10.54216/JNFS.080206
Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

Theorem 3.3

If the state i is an accessible state from j according to the transition matrix $M_N = A + BI$ then it is accessible in classical sense according to A and A + B.

Proof

Since state i is accessible then it satisfies:

$$m_{ij}^{(n)} = \left(a_{ij} + b_{ij}I\right)^{(n)} >_N 0 \, for \, some \, n \in \mathbb{N}$$

Which means that:

$$a_{ij}{}^{(n)} >_N 0 \text{ and } \left(a_{ij} + b_{ij}\right)^{(n)} >_N 0$$

So, we conclude that i is accessible in classical sense according to A and A + B.

Example 3.3

Let

$$M_N = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 + 0.1I & 0 & 0.6 - 0.1I \\ 0.3 - 0.2I & 0 & 0.7 + 0.2I \end{bmatrix}$$

We can easily see that state (2) is not accessible from any other state, also, state (3) is not accessible from state (1), but state (1) is accessible from all states, state (3) is accessible from state (2).

These results also hold for matrices:

$$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0 & 0.9 \end{bmatrix} \quad and \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0 & 0.6 \\ 0.3 & 0 & 0.7 \end{bmatrix}$$

Definition 3.4

We call state (i) according to matrix M_N a recurrent state if there exists some $n \in \mathbb{N}$ such that $m_{ii}^{(n)} > 0$, elsewise we call it a non-recurrent state.

Theorem 3.4

If the state i is a recurrent state according to transition matrix $M_N = A + BI$ then it is recurrent in classical sense according to A and A + B.

Proof

Since state (i) is recurrent then there exists some $n \in \mathbb{N}$ such that $m_{ii}^{(n)} > 0$, hence:

$$m_{ii}^{(n)} = (a_{ii} + b_{ii}I)^{(n)} >_N 0$$

Which means that:

$$a_{ii}{}^{(n)} >_N 0 \; and \; (a_{ii} + b_{ii})^{(n)} >_N 0$$

Which completes the proof.

Doi: https://doi.org/10.54216/JNFS.080206
Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

Example 3.4

Let:

$$M_N = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 + 0.1I & 0 & 0.6 - 0.1I \\ 0.3 - 0.2I & 0 & 0.7 + 0.2I \end{bmatrix}$$

We notice that states (1) and (3) are recurrent states because:

$$m_{11} = 1 > 0, m_{33} = 0.3 - 0.2I >_N 0$$

Also, states (1) and (3) are recurrent according to matrices $A, A + B$ because:

$$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0 & 0.9 \end{bmatrix} \quad and \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0 & 0.6 \\ 0.3 & 0 & 0.7 \end{bmatrix}$$

Where:

$$a_{11} = 1 > 0, a_{33} = 0.7 > 0$$

$$a_{11} + b_{11} = 1 > 0\, , a_{33} + b_{33} = \ 0.9 > 0$$

Definition 3.5

We say that state (i), (j) are communicated according to matrix M_N if there exists $n \in \mathbb{N}$ such that:

$$m_{ij}^{(n)} > 0\, , m_{ji}^{(n)} > 0$$

And we write:

$$i \leftrightarrow j$$

Theorem 3.5

States (i), (j) are communicated according to matrix M_N iff it is communicated according to $A, A + B$.

Proof

Since (i), (j) are communicated according to matrix M_N then there exists $n \in \mathbb{N}$ such that:

$$m_{ij}^{(n)} > 0\, , m_{ji}^{(n)} > 0$$

Which means according to theorem 3.3 that states (i), (j) are accessible according to $A, A + B$.

3. Classification of Plithogenic Markov Chains States

Let in the following $M_P = [m_{ij}]_{n \times n}$ be a Plithogenic Markov transition matrix

Definition 4.1

We say that state i is an absorbent state if:

$$m_{ii}^{(n)} = 1 \; \forall \, n \geq 1\, ; \mathrm{i} = 1{,}2, \dots, n$$

Doi: https://doi.org/10.54216/JNFS.080206
Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

Theorem 4.1

If the state i is absorbent according to the transition matrix $M_P = A + BP_1 + CP_2$ then it is absorbent in classical sense according to A , A + B and A + B + C.

Proof:

We have:

$$M_P = \begin{bmatrix} m_{11} & \cdots & m_{1i} & \cdots & m_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ m_{n1} & \cdots & m_{ni} & \cdots & m_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11}P_1 + c_{11}P_2 & \cdots & a_{1i} + b_{1i}P_1 + c_{1i}P_2 & \cdots & a_{1n} + b_{1n}P_1 + c_{1n}P_2 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} + b_{n1}P_1 + c_{n1}P_2 & \cdots & a_{ni} + b_{ni}P_1 + c_{ni}P_2 & \cdots & a_{nn} + b_{nn}P_1 + c_{nn}P_2 \end{bmatrix}$$

which means that:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}, \quad A + B = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1i} + b_{1i} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} + b_{n1} & \cdots & a_{ni} + b_{ni} & \cdots & a_{nn} + b_{nn} \end{bmatrix}$$

$$A + B + C = \begin{bmatrix} a_{11} + b_{11} + c_{11} & \cdots & a_{1i} + b_{1i} + c_{1i} & \cdots & a_{1n} + b_{1n} + c_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} + b_{n1} + c_{n1} & \cdots & a_{ni} + b_{ni} + c_{ni} & \cdots & a_{nn} + b_{nn} + c_{nn} \end{bmatrix}$$

And this completes the proof.

Example 4.1:

Let:

$$M_P = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 + 0.6P_1 - 0.1P_2 & 0 & 0.7 - 0.6P_1 + 0.1P_2 \\ 0.2 - 0.1P_1 + 0.6P_2 & 0 & 0.8 + 0.1P_1 - 0.6P_2 \end{bmatrix}$$

We can conclude that state (1) is absorbent state since:

$$M_P^2 = M_P . M_P = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 + 0.6P_1 - 0.1P_2 & 0 & 0.7 - 0.6P_1 + 0.1P_2 \\ 0.2 - 0.1P_1 + 0.6P_2 & 0 & 0.8 + 0.1P_1 - 0.6P_2 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0.3 + 0.6P_1 - 0.1P_2 & 0 & 0.7 - 0.6P_1 + 0.1P_2 \\ 0.2 - 0.1P_1 + 0.6P_2 & 0 & 0.8 + 0.1P_1 - 0.6P_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0.44 + 0.47P_1 + 0.03P_2 & 0 & 0.56 - 0.47P_1 - 0.03P_2 \\ 0.36 - 0.17P_1 + 0.72P_2 & 0 & 0.64 + 0.17P_1 - 0.72P_2 \end{bmatrix}$$

$$M_P^3 = M_P^2 . M_P = \begin{bmatrix} 1 & 0 & 0 \\ 0.44 + 0.47P_1 + 0.03P_2 & 0 & 0.56 - 0.47P_1 - 0.03P_2 \\ 0.36 - 0.17P_1 + 0.72P_2 & 0 & 0.64 + 0.17P_1 - 0.72P_2 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0.3 + 0.6P_1 - 0.1P_2 & 0 & 0.7 - 0.6P_1 + 0.1P_2 \\ 0.2 - 0.1P_1 + 0.6P_2 & 0 & 0.8 + 0.1P_1 - 0.6P_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0.552 + 0.367P_1 + 0.063P_2 & 0 & 0.448 - 0.367P_1 - 0.063P_2 \\ 0.488 - 0.217P_1 + 0.702P_2 & 0 & 0.512 + 0.217P_1 - 0.702P_2 \end{bmatrix}$$

And so on for all the powers $n \geq 1$, hence state (1) is absorbent.

We also can conclude that M_P is equivalent to the matrices:

Doi: https://doi.org/10.54216/JNFS.080206
Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 0 & 0.7 \\ 0.2 & 0 & 0.8 \end{bmatrix}, A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0.9 & 0 & 0.1 \\ 0.1 & 0 & 0.9 \end{bmatrix} \text{ and } A + B + C = \begin{bmatrix} 1 & 0 & 0 \\ 0.8 & 0 & 0.2 \\ 0.7 & 0 & 0.3 \end{bmatrix}$$

Where the first element of matrices A^n , $(A+B)^n$ and $(A+B+C)^n$is equal to 1 $\forall n \geq 1$, i.e. state (1) is an absorbent state of matrices A , A +B and A + B + C.

Definition 4.2

We say that state i is an inessential state if there exists some $k \in \mathbb{N}$ such that:

$$m_{ij}^{(k)} >_P 0$$

But:

$$m_{ji}^{(n)} = 0 \; \forall n \in \mathbb{N}$$

Where:

$$i \neq j \; \& \; i, j = 1, \dots, n$$

Otherwise, we say that state i is essential.

Theorem 4.2

If the state i is an inessential state according to the transition matrix $M_P = A + BP_1 + CP_2$ then it is inessential in classical sense according to A , A + B and A + B + C.

Proof

Since i is inessential then there exists some $k \in \mathbb{N}$ such that:

$$m_{ij}^{(k)} = \left(a_{ij} + b_{ij}P_1 + c_{ij}P_2\right)^{(k)} >_P 0$$

Which means that:

$$\left(a_{ij} + b_{ij} + c_{ij}\right)^{(k)} > 0 \; \& \; \left(a_{ij} + b_{ij}\right)^{(k)} > 0, \left(a_{ij}\right)^{(k)} > 0$$

Also,

$$m_{ji}^{(n)} = \left(a_{ji} + b_{ji}P_1 + c_{ji}P_2\right)^{(n)} = 0 \; \forall n \in \mathbb{N}$$

And this means that:

$$a_{ji} = 0, b_{ji} = 0, c_{ji} = 0$$

So:

$$a_{ji} + b_{ji} = 0 \, , a_{ji} + b_{ji} + c_{ji} = 0$$

Finally, we conclude that i is inessential in classical sense according to A , A + B and A + B + C

Example 4.2

Let:

Doi: https://doi.org/10.54216/JNFS.080206
Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

$$M_P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 + 0.6P_1 - 0.1P_2 & 0.7 - 0.6P_1 + 0.1P_2 \\ 0 & 0 & 0 & 0.5 + 0.1P_1 - 0.2P_2 & 0 & 0.5 - 0.1P_1 + 0.2P_2 \\ 0 & 0 & 0 & 0.5 - 0.3P_1 + 0.1P_2 & 0.5 + 0.3P_1 - 0.1P_2 & 0 \end{bmatrix} \end{matrix}$$

We can see immediately that:

state (1) is inessential state because $m_{12} = 1 > 0$ & $m_{j1}^{(n)} = 0 \; \forall \, n \in N$
state (2) is inessential state because $m_{23} = 1 > 0$ & $m_{j2}^{(n)} = 0 \; \forall \, n \in N$
state (3) is inessential state because $m_{34} = 1 > 0$ & $m_{j3}^{(n)} = 0 \; \forall \, n \in N$
state (4) is essential state because $m_{45} = 0.3 + 0.6P_1 - 0.1P_2 >_P 0$ & $m_{54} = 0.5 + 0.1P_1 - 0.2P_2 >_P 0$
state (5) is essential state because $m_{56} = 0.5 - 0.1P_1 + 0.2P_2 >_P 0$ & $m_{65} = 0.5 + 0.3P_1 - 0.1P_2 >_P 0$
state (6) is essential state because $m_{64} = 0.5 - 0.3P_1 + 0.1P_2 >_P 0$ & $m_{46} = 0.7 - 0.6P_1 + 0.1P_2 >_P 0$

Definition 4.3

We say that state j is accessible state from state i if it fulfils:

$$m_{ij}^{(n)} > 0 \text{ for every } n \geq 1$$

Theorem 4.3

If the state i is an accessible state from j according to the transition matrix $M_P = A + BP_1 + CP_2$ then it is accessible in classical sense according to A , A + B and A + B + C.

Proof

Since state i is accessible then it satisfies:

$$m_{ij}^{(n)} = \left(a_{ij} + b_{ij}P_1 + c_{ij}P_2\right)^{(n)} >_P 0 \; for \; some \; n \in \mathbb{N}$$

Which means that:

$$a_{ij}{}^{(n)} >_P 0 \, , \left(a_{ij} + b_{ij}\right)^{(n)} >_P 0, \left(a_{ij} + b_{ij} + c_{ij}\right)^{(n)} >_P 0$$

So, we conclude that i is accessible in classical sense according to A , A + B and A + B + C.

Example 4.3

Let

$$M_P = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 + 0.6P_1 - 0.1P_2 & 0 & 0.7 - 0.6P_1 + 0.1P_2 \\ 0.2 - 0.1P_1 + 0.6P_2 & 0 & 0.8 + 0.1P_1 - 0.6P_2 \end{bmatrix}$$

We can easily see that state (2) is not accessible from any other state, also, state (3) is not accessible from state (1), but state (1) is accessible from all states, state (3) is accessible from state (2).

These results also hold for matrices:

$$A + B + C = \begin{bmatrix} 1 & 0 & 0 \\ 0.8 & 0 & 0.2 \\ 0.7 & 0 & 0.3 \end{bmatrix} \; and \quad A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0 & 0.9 \end{bmatrix} , \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0 & 0.6 \\ 0.3 & 0 & 0.7 \end{bmatrix}$$

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Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

Definition 4.4

We call state (i) according to matrix M_P a recurrent state if there exists some $n \in \mathbb{N}$ such that $m_{ii}^{(n)} > 0$, elsewise we call it a non-recurrent state.

Theorem 4.4

If the state i is a recurrent state according to transition matrix $M_P = A + BP_1 + CP_2$ then it is recurrent in classical sense according to A , A + B and A + B + C.

Proof

Since state (i) is recurrent then there exists some $n \in \mathbb{N}$ such that $m_{ii}^{(n)} > 0$, hence:

$$m_{ii}^{(n)} = (a_{ii} + b_{ii}P_1 + c_{ii}P_2)^{(n)} >_P 0$$

Which means that:

$$a_{ii}{}^{(n)} >_P 0 , (a_{ii} + b_{ii})^{(n)} >_P 0 \ and \ (a_{ii} + b_{ii} + c_{ii})^{(n)} >_P 0$$

Which completes the proof.

Example 4.4

Let:

$$M_P = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 + 0.6P_1 - 0.1P_2 & 0 & 0.7 - 0.6P_1 + 0.1P_2 \\ 0.2 - 0.1P_1 + 0.6P_2 & 0 & 0.8 + 0.1P_1 - 0.6P_2 \end{bmatrix}$$

We notice that states (1) and (3) are recurrent states because:

$$m_{11} = 1 > 0, m_{33} = 0.8 + 0.1P_1 - 0.6P_2 >_P 0$$

Also, states (1) and (3) are recurrent according to matrices $A, A + B$ and A + B + C because:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 0 & 0.7 \\ 0.2 & 0 & 0.8 \end{bmatrix}, A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0.9 & 0 & 0.1 \\ 0.1 & 0 & 0.9 \end{bmatrix} and \quad A + B + C = \begin{bmatrix} 1 & 0 & 0 \\ 0.8 & 0 & 0.2 \\ 0.7 & 0 & 0.3 \end{bmatrix}$$

Where:

$$a_{11} = 1 > 0, a_{33} = 0.8 > 0$$

$$a_{11} + b_{11} = 1 > 0 , a_{33} + b_{33} = 0.9 > 0$$

$$a_{11} + b_{11} + c_{11} = 1 > 0 , a_{33} + b_{33} + c_{33} = 0.3 > 0$$

Definition 4.5

We say that state (i), (j) are communicated according to matrix M_P if there exists $n \in \mathbb{N}$ such that:

$$m_{ij}^{(n)} > 0 , m_{ji}^{(n)} > 0$$

And we write:

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Received: November 17, 2023 Revised: January 19, 2024 Accepted: April 27, 2024

$$i \leftrightarrow j$$

Theorem 4.5

States (i), (j) are communicated according to matrix M_P iff it is communicated according to $A, A+B$and $A+B+C$.

Proof

Since (i), (j) are communicated according to matrix M_P then there exists $n \in \mathbb{N}$ such that:

$$m_{ij}^{(n)} > 0\,, m_{ji}^{(n)} > 0$$

Which means according to theorem 4.3 that states (i), (j) are accessible according to $A, A+B$ and $A+B+C$.

5. Conclusions and future research directions

In this paper we closed the gap of classifying states of a Markov Chain in both neutrosophic and plithogenic logic. We have well defined when a state is called to be absorbent, inessential, essential, accessible recurrent and communicated in literal neutrosophic and plithogenic environment. This research can be applied in cryptography, queuing theory, page ranking algorithms and many important problems in real-life.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Acknowledgments: Authors are very thankful to editors-in-chief and to reviewers for the valuable help and recommendations.

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