A Study on First and Second Order Bipolar Fuzzy Topological Spaces and Crisp Topological Spaces and Analyzing the Connections Between Them

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Abstract

In our previous paper we discussed about the concept of SOBPFS, SOBPFT and its mathematical modelling in medical diagnosis. In this paper, the detailed study about SOBPFT accordance with FOBPFT and crisp topological spaces are analysed and also some natural examples of SOBPFT are provided. In third section, the connections between FOBPFT and SOBPFT under five different cases are discussed. And last section tells that, from a crisp topology τ on X there exists three different SOBPFT denoted by $\widehat{\omega(\tau)}$, $\widehat{\omega_*(\tau)}$ and $\widehat{\omega_\varepsilon(\tau)}$ and from a SOBPFT on X there exists three crisp topologies denoted by $i(\hat{\tau}_{\mathfrak{B}})$, $i^*(\hat{\tau}_{\mathfrak{B}})$ and $i_\varepsilon(\hat{\tau}_{\mathfrak{B}})$.

Keywords: Fuzzy set (FS); fuzzy topology (FT); First order bipolar-fuzzy set (FOBPFS); first order bipolar-fuzzy topological spaces (FOBPFT); second order bipolar fuzzy set (SOBPFS); second order bipolar fuzzy topological spaces (SOBPFT).

1. Introduction

Fuzzy set theory was introduced by Zadeh [16] in 1965 which revolutionalized the field of mathematics and artificial intelligence by providing a framework to handle uncertainty and vagueness in decision-making processes. Whereas crisp topology deals with crisp distinctions between elements. (i.e..,) being either fully in or fully out of a set . The important contributions to the study of fuzzy set theory were made by Zimmerman [19] (1996), Yager [15] (1980), Chang [2] (1968), Lowen [9] (1976), Gougen [3] (1967), Gottwald [4] (1993), Hohle [5] (1978), Kaufmann [7] (1975) and many others. In 1968, based on fuzzy sets Chang introduced the concept of fuzzy topological spaces and defined some basic concepts such as open fuzzy sets, closed fuzzy sets, neighbourhood of a fuzzy set, interior of a fuzzy set, fuzzy continuity and fuzzy compactness. In order to study deeper into the structure of fuzzy topological spaces, in 1976 Lowen modified the concept of fuzzy topological spaces of Chang. Also, the author introduced two functors $\tilde{\omega}$ and $\tilde{\iota}$ to establish the connection between fuzzy topological spaces and topological spaces. In 1994, Zhang [18] introduced the notion of bipolar fuzziness. In 2019, Kim [8] et al., introduced the concept of bipolar fuzzy topology and defined some basic concepts such as bipolar fuzzy point, bipolar fuzzy base, bipolar fuzzy subbase, bipolar fuzzy subspace, bipolar fuzzy quotient space, bipolar fuzzy neighbourhood, bipolar fuzzy initial topology, bipolar fuzzy continuity and bipolar fuzzy compactness and obtained some basic properties of each concepts. In 1975, Zadeh [17] introduced the concept of fuzzy set of type 2 (second order fuzzy set) as an extension of a fuzzy set. A detailed study of second order fuzzy sets was done by Mizumuto and Tanaka [10,11] (1976, 1981). Norwich and Turksen [13,14] (1981,1984)

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

have used the concept of second order fuzzy sets in their stochastic fuzzy model. In 2007, Kalaichelvi [6] introduced the concept of second order fuzzy topological spaces using fuzzy sets of type 2 defined by Zadeh (1975) and studied some second order fuzzy structures. In 2024, Muthamizhselvi and Vijayalakshmi [12] introduced the new concept of second order bipolar fuzzy topology and its application in medical diagnosis.

2. Preliminary definitions

Definition : 2.1

Let **X** be an arbitrary nonempty set. Let $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$. A **FS** in **X** is a map from **X** into I**.**

Definition : 2.2

A subset $\tau \subset I^X$ is said to **FT** X iff

(i) The **0** & **1** belong to τ

(ii) $f_\lambda \in \tau \ \forall \lambda \in \Lambda$ implies $\bigvee_{\lambda \in \Lambda} f_\lambda \in \tau$

(iii) $f, g \in \tau$ implies $f \wedge g \in \tau$

Then (X, τ) is called **FT space**.

Definition : 2.3

A pair $A_{bp} = \left(A_{bp}^{+}, A_{bp}^{-}\right)$ is called **FOBPFS** in X, where $A_{bp}^{+}: X \to [0,1]$ and $A_{bp}^{-}: X \to [-1,0]$.

Definition : 2.4

A pair $\widehat{A}_{bp} = \left(\widehat{A}_{bp}^{+}, \widehat{A}_{bp}^{-}\right)$ is called a **SOBPFS** in X where $\widehat{A}_{bp}^{+}: X \to [0,1]^{[0,1]}$ such that$\widehat{A}_{bp}^{+}(x)(\alpha) \in [0,1]$ and $\widehat{A}_{bp}^{-}: X \to [-1,0]^{[0,1]}$ such that $\widehat{A}_{bp}^{-}(x)(\alpha) \in [-1,0]$, where $\alpha \in I$ & $x \in X$.

Definition : 2.5

A collection $\hat{\tau}_{\mathfrak{B}}$ SOBPFS on X defines a **SOBPFT** on X iff

(i) $\hat{0}_{bp}, \hat{1}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) $\left(\widehat{A}_{bp}\right)_\lambda \in \hat{\tau}_{\mathfrak{B}}$, for each $\lambda \in \Lambda$ implies $\bigcup_{\lambda \in \Lambda}\left(\widehat{A}_{bp}\right)_\lambda \in \hat{\tau}_{\mathfrak{B}}$.

(iii) $\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$, for each $i = 1$ to m implies that $\bigcap_{i=1}^{n}\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$.

The pair $(X, \hat{\tau}_{\mathfrak{B}})$ is called a SOBPFT.

3. Connections between first order bipolar fuzzy and second order bipolar fuzzy topological spaces

Theorem : 3.1

Every FOBPFT $\tau_{\mathfrak{B}} = \{(A_{bp})_\lambda / \lambda \in \Lambda\}$ on X defines a SOBPFT (Lowen) $\hat{\tau}_{\mathfrak{B}} = \{(\widehat{A}_{bp})_\lambda / (A_{bp})_\lambda \in \tau_{\mathfrak{B}}\}$ on X, where $\left(\widehat{A}_{bp}^{+}\right)_\lambda (x)(\alpha) = \left(A_{bp}^{+}\right)_\lambda(x)$ and $\left(\widehat{A}_{bp}^{-}\right)_\lambda(x)(\alpha) = \left(A_{bp}^{-}\right)_\lambda(x)$ for every $x \in X$ and for every $\alpha \in I$. The correspondence $\tau_{\mathfrak{B}} \to \hat{\tau}_{\mathfrak{B}}$ is denoted as $\mathbb{C}_1$.

Proof:

To prove $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT (Lowen) on X. By the definition of $\left(\widehat{A}_{bp}\right)_\lambda$, the correspondence $\left(A_{bp}\right)_\lambda \to \left(\widehat{A}_{bp}\right)_\lambda$ is one-one

(i) Since $0_{bp}, 1_{bp}, \alpha_{bp} \in \tau_{\mathfrak{B}}$, $\hat{0}_{bp}, \hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

(ii) To prove :-$\hat{\tau}_{\mathfrak{B}}$ is closed with respect to arbitrary union

Given $\left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda_0 \subseteq \Lambda$.

To prove: $\left(\cup_{\lambda\in\Lambda_0}\left(\widehat{A}_{bp}\right)_{\lambda}\right) \in \hat{\tau}_{\mathfrak{B}}$

$\left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$ for $\lambda \in \Lambda_0 \subseteq \Lambda$ implies $\left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}$, for $\lambda \in \Lambda_0 \subseteq \Lambda$

implies $\cup_{\lambda\in\Lambda_0}\left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}$

implies $\cup_{\lambda\in\Lambda_0}\widehat{\left(A_{bp}\right)_{\lambda}} \in \hat{\tau}_{\mathfrak{B}}$

implies $\left(\vee_{\lambda\in\Lambda_0}\widehat{\left({A_{bp}}^{+}\right)_{\lambda}}, \wedge_{\lambda\in\Lambda_0}\widehat{\left({A_{bp}}^{-}\right)_{\lambda}}\right) \in \hat{\tau}_{\mathfrak{B}}$

Consider

For every $x \in X$ and for every $\alpha \in I$

$$\left(\vee_{\lambda\in\Lambda_0}\widehat{\left({A_{bp}}^{+}\right)_{\lambda}}\right)(x)(\alpha) = \left(\vee_{\lambda\in\Lambda_0}\left({A_{bp}}^{+}\right)_{\lambda}\right)(x)$$

$$= \vee_{\lambda\in\Lambda_0}\left(\left({A_{bp}}^{+}\right)_{\lambda}(x)\right)$$

$$= \vee_{\lambda\in\Lambda_0}\left(\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}(x)(\alpha)\right)$$

$$= \left(\vee_{\lambda\in\Lambda_0}\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}\right)(x)(\alpha)$$

Therefore $\left(\vee_{\lambda\in\Lambda_0}\widehat{\left({A_{bp}}^{+}\right)_{\lambda}}\right) = \vee_{\lambda\in\Lambda_0}\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}$ and

$$\left(\wedge_{\lambda\in\Lambda_0}\widehat{\left({A_{bp}}^{-}\right)_{\lambda}}\right)(x)(\alpha) = \left(\wedge_{\lambda\in\Lambda_0}\left({A_{bp}}^{-}\right)_{\lambda}\right)(x)$$

$$= \wedge_{\lambda\in\Lambda_0}\left(\left({A_{bp}}^{-}\right)_{\lambda}(x)\right)$$

$$= \wedge_{\lambda\in\Lambda_0}\left(\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}(x)(\alpha)\right)$$

$$= \left(\wedge_{\lambda\in\Lambda_0}\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}\right)(x)(\alpha)$$

Therefore $\left(\wedge_{\lambda\in\Lambda_0}\widehat{\left({A_{bp}}^{-}\right)_{\lambda}}\right) = \left(\wedge_{\lambda\in\Lambda_0}\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}\right)$

implies $\cup_{\lambda\in\Lambda_0}\widehat{\left(A_{bp}\right)_{\lambda}} = \left(\cup_{\lambda\in\Lambda_0}\left(\widehat{A}_{bp}\right)_{\lambda}\right)$

Therefore $\left(\cup_{\lambda\in\Lambda_0}\left(\widehat{A}_{bp}\right)_{\lambda}\right) \in \hat{\tau}_{\mathfrak{B}}$

(i) To prove :- $\hat{\tau}_{\mathfrak{B}}$ is closed with respect to finite intersection

Given $\left(\widehat{A}_{bp}\right)_{i} \in \hat{\tau}_{\mathfrak{B}}$, for $i = 1,2,\ldots m$

To prove: $\left(\cap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_{i}\right) \in \hat{\tau}_{\mathfrak{B}}$

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

$(\widehat{A}_{bp})_i \in \hat{\tau}_{\mathfrak{B}}$, for $i = 1,2, \dots m$ implies $(A_{bp})_i \in \tau_{\mathfrak{B}}$,for $i = 1,2, \dots m$
implies $\cap_{i=1}^{m}(A_{bp})_i \in \tau_{\mathfrak{B}}$

implies $\cap_{i=1}^{m}\widehat{(A_{bp})_i} \in \hat{\tau}_{\mathfrak{B}}$

implies $\left(\wedge_{i=1}^{m}\widehat{({A_{bp}}^{+})_i}, \vee_{i=1}^{m}\widehat{({A_{bp}}^{-})_i}\right) \in \hat{\tau}_{\mathfrak{B}}$
Consider,
For every $x \in X$ and for every $\alpha \in I$

$$\left(\wedge_{i=1}^{m}\widehat{({A_{bp}}^{+})_i}\right)(x)(\alpha) = \left(\wedge_{i=1}^{m}({A_{bp}}^{+})_i\right)(x)$$

$$= \wedge_{i=1}^{m}\left(({A_{bp}}^{+})_i(x)\right)$$

$$= \wedge_{i=1}^{m}\left((\widehat{A}_{bp}{}^{+})_i(x)(\alpha)\right)$$

$$= \left(\wedge_{i=1}^{m}(\widehat{A}_{bp}{}^{+})_i\right)(x)(\alpha)$$

Therefore $\left(\wedge_{i=1}^{m}\widehat{({A_{bp}}^{+})_i}\right) = \wedge_{i=1}^{m}(\widehat{A}_{bp}{}^{+})_i$ and

$$\left(\vee_{i=1}^{m}\widehat{({A_{bp}}^{-})_i}\right)(x)(\alpha) = \left(\vee_{i=1}^{m}({A_{bp}}^{-})_i\right)(x)$$

$$= \vee_{i=1}^{m}\left(({A_{bp}}^{-})_i(x)\right)$$

$$= \vee_{i=1}^{m}\left((\widehat{A}_{bp}{}^{-})_i(x)(\alpha)\right)$$

$$= \left(\vee_{i=1}^{m}(\widehat{A}_{bp}{}^{-})_i\right)(x)(\alpha)$$

Therefore $\left(\vee_{i=1}^{m}\widehat{({A_{bp}}^{-})_i}\right) = \left(\vee_{i=1}^{m}(\widehat{A}_{bp}{}^{-})_i\right)$
implies $\cap_{i=1}^{m}\widehat{(A_{bp})_i} = \cap_{i=1}^{m}(\widehat{A}_{bp})_i \in \hat{\tau}_{\mathfrak{B}}$
Therefore $\cap_{i=1}^{m}(\widehat{A}_{bp})_i \in \hat{\tau}_{\mathfrak{B}}$
Therefore $\hat{\tau}_{\mathfrak{B}}$ defines a SOBPFT (lowen) on X.

Theorem : 3.2

Let $\hat{\tau}_{\mathfrak{B}} = \{(\widehat{A}_{bp})_\lambda / \lambda \in \Lambda\}$ be a SOBPFT on X. Fix $x \in X$. Then the collection $(\hat{\tau}_{\mathfrak{B}})_x$= distinct elements of the collection $\{\left((\widehat{A}_{bp})_\lambda\right)_{(x)} / (\widehat{A}_{bp})_\lambda \in \hat{\tau}_{\mathfrak{B}}\}$ defines a FOBPFT on I(I is the closed unit interval [0,1]), where $\left((\widehat{A}_{bp}{}^{+})_\lambda\right)_{(x)} = (\widehat{A}_{bp}{}^{+})_\lambda(x)$, $\left((\widehat{A}_{bp}{}^{-})_\lambda\right)_{(x)} = (\widehat{A}_{bp}{}^{-})_\lambda(x)$. The correspondence $\hat{\tau}_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_x$ is denoted as $\mathbb{C}_2$.

Proof:

To prove:$(\hat{\tau}_{\mathfrak{B}})_x$ is a FOBPFT on I. By the definition $(\hat{\tau}_{\mathfrak{B}})_x$, there exists $\Lambda_0 \subseteq \Lambda$ such that for $\lambda, \mu \in \Lambda_0$, $\lambda \neq \mu$.

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

$\left((\widehat{A}_{bp})_{\lambda}\right)_{(x)} \neq \left((\widehat{A}_{bp})_{\mu}\right)_{(x)}$ and $(\hat{\tau}_{\mathfrak{B}})_x$ can be written as $(\hat{\tau}_{\mathfrak{B}})_x = \{\left((\widehat{A}_{bp})_{\lambda}\right)_{(x)} / \lambda \in \Lambda_0\}$.

(i) To prove :- $(\hat{0}_{bp})_{(x)}, (\hat{1}_{bp})_{(x)}, (\hat{\alpha}_{bp})_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

Given $\hat{0}_{bp} = (\hat{0}_{bp}{}^{+}, \hat{0}_{bp}{}^{-}) \in \hat{\tau}_{\mathfrak{B}}$,

Let $(\hat{0}_{bp})_{(x)} = \left((\hat{0}_{bp}{}^{+})_{(x)}, (\hat{0}_{bp}{}^{-})_{(x)}\right)$

$\hat{0}_{bp}{}^{+}(x) = (\hat{0}_{bp}{}^{+})_{(x)} = \mathbf{0}, \hat{0}_{bp}{}^{-}(x) = (\hat{0}_{bp}{}^{-})_{(x)} = \mathbf{0}$

Therefore $(\hat{0}_{bp})_{(x)} = 0_{bp} \in (\hat{\tau}_{\mathfrak{B}})_x$

Similarly $\hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $(\hat{1}_{bp})_{(x)}, (\hat{\alpha}_{bp})_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

(ii) To prove : $(\hat{\tau}_{\mathfrak{B}})_x$ is closed with respect to arbitrary union

Given $\left((\widehat{A}_{bp})_{\lambda}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$, for $\lambda \in \Lambda_1 \subseteq \Lambda_0$

To prove: $\cup_{\lambda \in \Lambda_1} \left((\widehat{A}_{bp})_{\lambda}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

$\left((\widehat{A}_{bp})_{\lambda}\right)_{(x)} \in \hat{\tau}_{\mathfrak{B}_x}$, for $\lambda \in \Lambda_1 \subseteq \Lambda_0$ implies $(\widehat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda_1 \subseteq \Lambda_0$

implies $\cup_{\lambda \in \Lambda_1} (\widehat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, where $\cup_{\lambda \in \Lambda_1} (\widehat{A}_{bp})_{\lambda} = \left(\vee_{\lambda \in \Lambda_1} (\widehat{A}_{bp}{}^{+})_{\lambda}, \wedge_{\lambda \in \Lambda_1} (\widehat{A}_{bp}{}^{-})_{\lambda}\right)$

Let $\widehat{B}_{bp} = \cup_{\lambda \in \Lambda_1} (\widehat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$,

where $\widehat{B}_{bp} = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right) = \left(\vee_{\lambda \in \Lambda_1} (\widehat{A}_{bp}{}^{+})_{\lambda}, \wedge_{\lambda \in \Lambda_1} (\widehat{A}_{bp}{}^{-})_{\lambda}\right)$

$\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $(\widehat{B}_{bp})_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

Consider ,

$$\begin{aligned}
(\widehat{B}_{bp}{}^{+})_{(x)} &= \widehat{B}_{bp}{}^{+}(x) \\
&= \left(\vee_{\lambda \in \Lambda_1} (\widehat{A}_{bp}{}^{+})_{\lambda}\right)(x) \\
&= \vee_{\lambda \in \Lambda_1} \left((\widehat{A}_{bp}{}^{+})_{\lambda}(x)\right) \\
&= \vee_{\lambda \in \Lambda_1} \left((\widehat{A}_{bp}{}^{+})_{\lambda}\right)_{(x)}
\end{aligned}$$

implies $(\widehat{B}_{bp}{}^{+})_{(x)} = \vee_{\lambda \in \Lambda_1} \left((\widehat{A}_{bp}{}^{+})_{\lambda}\right)_{(x)}$

$(\widehat{B}_{bp}{}^{-})_{(x)} = \widehat{B}_{bp}{}^{-}(x)$

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

$$= \left(\wedge_{\lambda\in\Lambda_1}(\widehat{A}_{bp}{}^{-})_{\lambda}\right)(x)$$
$$= \wedge_{\lambda\in\Lambda_1}\left((\widehat{A}_{bp}{}^{-})_{\lambda}(x)\right)$$
$$= \wedge_{\lambda\in\Lambda_1}\left((\widehat{A}_{bp}{}^{-})_{\lambda}\right)_{(x)}$$

implies $\left(\widehat{B}_{bp}{}^{-}\right)_{(x)} = \wedge_{\lambda\in\Lambda_1}\left((\widehat{A}_{bp}{}^{-})_{\lambda}\right)_{(x)}$

Therefore $\left(\widehat{B}_{bp}\right)_{(x)} = \left(\vee_{\lambda\in\Lambda_1}\left((\widehat{A}_{bp}{}^{+})_{\lambda}\right)_{(x)}, \wedge_{\lambda\in\Lambda_1}\left((\widehat{A}_{bp}{}^{-})_{\lambda}\right)_{(x)}\right) \in (\hat{\tau}_{\mathfrak{B}})_x$

implies $\cup_{\lambda\in\Lambda_1}\left((\widehat{A}_{bp})_{\lambda}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

(iii) To prove :$(\hat{\tau}_{\mathfrak{B}})_x$ is closed with respect to finite intersection.

Given $\left((\widehat{A}_{bp})_i\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$, for $i = 1,2, \ldots m$

To prove: $\cap_{i=1}^{m}\left((\widehat{A}_{bp})_i\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

$\left((\widehat{A}_{bp})_i\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$, for $i = 1$ to m implies$\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$, for $i = 1,2, \ldots m$

implies $\cap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$

Let $\widehat{B}_{bp} = \cap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_i \in \hat{\tau}_{\mathfrak{B}}$, where

$\widehat{B}_{bp} = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right) = \left(\wedge_{i=1}^{m}\left(\widehat{A}_{bp}{}^{+}\right)_i, \vee_{i=1}^{m}\left(\widehat{A}_{bp}{}^{-}\right)_i\right)$

$\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\widehat{B}_{bp}\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

For $\alpha \in I$

Consider

$\left(\widehat{B}_{bp}{}^{+}\right)_{(x)} = \widehat{B}_{bp}{}^{+}(x)$

$$= \left(\wedge_{i=1}^{m}\left(\widehat{A}_{bp}{}^{+}\right)_i\right)(x)$$
$$= \wedge_{i=1}^{m}\left(\left(\widehat{A}_{bp}{}^{+}\right)_i(x)\right)$$
$$= \wedge_{i=1}^{m}\left(\left(\widehat{A}_{bp}{}^{+}\right)_i\right)_{(x)}$$

implies $\left(\widehat{B}_{bp}{}^{+}\right)_{(x)} = \wedge_{i=1}^{m}\left(\left(\widehat{A}_{bp}{}^{+}\right)_i\right)_{(x)}$ and

$\left(\widehat{B}_{bp}{}^{-}\right)_{(x)} = \widehat{B}_{bp}{}^{-}(x)$

$$= \left(\vee_{i=1}^{m}\left(\widehat{A}_{bp}{}^{-}\right)_i\right)(x)$$

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

$= \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{-}\right)_i(x)\right)$
$= \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{-}\right)_i\right)_{(x)}$

implies $\left(\hat{B}_{bp}{}^{-}\right)_{(x)} = \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{-}\right)_i\right)_{(x)}$

$\left(\hat{B}_{bp}\right)_{(x)} = \left(\left(\hat{B}_{bp}{}^{+}\right)_{(x)}, \left(\hat{B}_{bp}{}^{-}\right)_{(x)}\right) = \left(\wedge_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{+}\right)_i\right)_{(x)}, \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{-}\right)_i\right)_{(x)}\right) \in (\hat{\tau}_{\mathfrak{B}})_x$

implies $\cap_{i=1}^{m}\left(\left(\hat{A}_{bp}\right)_i\right)_{(x)} \in (\hat{\tau}_{\mathfrak{B}})_x$

Hence $(\hat{\tau}_{\mathfrak{B}})_x$ is a FOBPFT on I.

Theorem : 3.3

Let $\hat{\tau}_{\mathfrak{B}} = \{(\hat{A}_{bp})_\lambda / \lambda \in \Lambda\}$ be a SOBPFT (Lowen) on X. Fix $\alpha \in I$, then the collection $(\hat{\tau}_{\mathfrak{B}})_\alpha$ = distinct elements of the collection $\{\left((A_{bp})_\lambda\right)_\alpha / (\hat{A}_{bp})_\lambda \in \hat{\tau}_{\mathfrak{B}}\}$ defines a FOBPFT on X where $\left(\left(\hat{A}_{bp}{}^{+}\right)_\lambda\right)_\alpha : X \to [0,1]$ such that $\left(\left(\hat{A}_{bp}{}^{+}\right)_\lambda\right)_\alpha (x) = \left(\hat{A}_{bp}{}^{+}\right)_\lambda (x)(\alpha)$ for every $x \in X$ and $\left(\left(\hat{A}_{bp}{}^{-}\right)_\lambda\right)_\alpha : X \to [-1,0]$ such that $\left(\left(\hat{A}_{bp}{}^{-}\right)_\lambda\right)_\alpha (x) = \left(\hat{A}_{bp}{}^{-}\right)_\lambda (x)(\alpha)$ for every $x \in X$. The correspondence $\hat{\tau}_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_\alpha$ is denoted as $\mathbb{C}_3$.

Proof:-

To prove $(\hat{\tau}_{\mathfrak{B}})_\alpha$ is a FOBPFT on X.

By the definition of $(\hat{\tau}_{\mathfrak{B}})_\alpha$, there exists $\Lambda_0 \subseteq \Lambda$ such that for $\lambda, \mu \in \Lambda_0$, $\lambda \neq \mu$.

$\left((\hat{A}_{bp})_\lambda\right)_\alpha \neq \left((\hat{A}_{bp})_\mu\right)_\alpha$ and $(\hat{\tau}_{\mathfrak{B}})_\alpha$ can be written as $(\hat{\tau}_{\mathfrak{B}})_\alpha = \left\{\left((\hat{A}_{bp})_\lambda\right)_\alpha / \lambda \in \Lambda_0\right\}$.

(i)To prove : $\left(\hat{0}_{bp}\right)_\alpha, \left(\hat{1}_{bp}\right)_\alpha, \left(\hat{\alpha}_{bp}\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$
Let $\left(\hat{0}_{bp}\right)_\alpha = \left(\left(\hat{0}_{bp}{}^{+}\right)_\alpha, \left(\hat{0}_{bp}{}^{-}\right)_\alpha\right)$

Given $\hat{0}_{bp} = \left(\hat{0}_{bp}{}^{+}, \hat{0}_{bp}{}^{-}\right) \in \hat{\tau}_{\mathfrak{B}}$. Then

$\left(\hat{0}_{bp}{}^{+}\right)_\alpha (x) = \left(\hat{0}_{bp}{}^{+}\right)(x)(\alpha) = 0 = 0_{bp}{}^{+}(x) \Rightarrow \left(\hat{0}_{bp}{}^{+}\right)_\alpha = 0_{bp}{}^{+}$

$\left(\hat{0}_{bp}{}^{-}\right)_\alpha (x) = \left(\hat{0}_{bp}{}^{-}\right)(x)(\alpha) = 0 = 0_{bp}{}^{-}(x) \Rightarrow \left(\hat{0}_{bp}{}^{-}\right)_\alpha = 0_{bp}{}^{-}$

implies $\left(\hat{0}_{bp}\right)_\alpha = 0_{bp} \in (\hat{\tau}_{\mathfrak{B}})_\alpha$

Similarly $\hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\hat{1}_{bp}\right)_\alpha, \left(\hat{\alpha}_{bp}\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$

(ii)To prove:$(\hat{\tau}_{\mathfrak{B}})_\alpha$ is closed with respect to arbitrary union

Given $\left((\hat{A}_{bp})_\lambda\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$ for $\lambda \in \Lambda_0 \subseteq \Lambda$ implies $(\hat{A}_{bp})_\lambda \in \hat{\tau}_{\mathfrak{B}}$ for $\lambda \in \Lambda_0 \subseteq \Lambda$

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

implies $\cup_{\lambda\in\Lambda_0}(\widehat{A}_{bp})_\lambda \in \hat{\tau}_{\mathfrak{B}}$.

Let $\widehat{B}_{bp} = \cup_{\lambda\in\Lambda_0}(\widehat{A}_{bp})_\lambda$,

where $\widehat{B}_{bp} = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right) = \left(\vee_{\lambda\in\Lambda_0}\left(\widehat{A}_{bp}{}^{+}\right)_\lambda, \wedge_{\lambda\in\Lambda_0}\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right) \in \hat{\tau}_{\mathfrak{B}}$

$\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\widehat{B}_{bp}\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$

For $x \in X$,

Consider,

$$\left(\widehat{B}_{bp}{}^{+}\right)_\alpha(x) = \left(\widehat{B}_{bp}{}^{+}\right)(x)(\alpha)$$

$$= \left(\vee_{\lambda\in\Lambda_0}\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)(x)(\alpha)$$
$$= \vee_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda(x)(\alpha)\right)$$
$$= \vee_{\lambda\in\Lambda_0}\left(\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_\alpha(x)\right)$$

$$= \left(\vee_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_\alpha\right)(x)$$

Therefore $\left(\widehat{B}_{bp}{}^{+}\right)_\alpha = \vee_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_\alpha$

$$\left(\widehat{B}_{bp}{}^{-}\right)_\alpha(x) = \left(\widehat{B}_{bp}{}^{-}\right)(x)(\alpha)$$

$$= \left(\wedge_{\lambda\in\Lambda_0}\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)(x)(\alpha)$$

$$= \wedge_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda(x)(\alpha)\right)$$

$$= \wedge_{\lambda\in\Lambda_0}\left(\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_\alpha(x)\right)$$

$$= \left(\wedge_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_\alpha\right)(x)$$

implies $\left(\widehat{B}_{bp}{}^{-}\right)_\alpha = \wedge_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_\alpha$

Therefore $\left(\widehat{B}_{bp}\right)_\alpha = \left(\left(\widehat{B}_{bp}{}^{+}\right)_\alpha, \left(\widehat{B}_{bp}{}^{-}\right)_\alpha\right)$

$$= \left(\vee_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_\alpha, \wedge_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_\alpha\right)$$

$$= \cup_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$$

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

Therefore $\cup_{\lambda\in\Lambda_0}\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$

The proof for the finite intersection can also be proved in similar manner.
Therefore $(\hat{\tau}_{\mathfrak{B}})_\alpha$ is a FOBPFT (Lowen) on X.

Theorem : 3.4

Let $\tau_{\mathfrak{B}} = \left\{\left(A_{bp}\right)_\lambda / \lambda \in \Lambda\right\}$ be a FOBPFT on I. Then the collection $(\hat{\tau}_{\mathfrak{B}})_I = \{\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I / \left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}\}$ defines a SOBPFT (Lowen) on a non-empty set X where $\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I = \left(\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I, \left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I\right)$ such that $\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I(x) = \left(A_{bp}{}^{+}\right)_\lambda$, $\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I(x) = \left(A_{bp}{}^{-}\right)_\lambda$ for every $x \in X$. The correspondence $\tau_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_I$ is denoted as $\mathbb{C}_4$.

Proof:

To prove $(\hat{\tau}_{\mathfrak{B}})_I$ is a SOBPFT (Lowen) over X.

By the definition of $(\hat{\tau}_{\mathfrak{B}})_I$, there exists $\Lambda_0 \subseteq \Lambda$ such that for $\lambda, \mu \in \Lambda_0, \lambda \neq \mu$,

$\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I \neq \left(\left(\widehat{A}_{bp}\right)_\mu\right)_I$ and $(\hat{\tau}_{\mathfrak{B}})_I$ can be written as $(\hat{\tau}_{\mathfrak{B}})_I = \{(\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I / \lambda \in \Lambda_0\}$.

(i) To prove : $(\hat{0}_{bp})_I, (\hat{1}_{bp})_I, (\hat{\alpha}_{bp})_I \in (\hat{\tau}_{\mathfrak{B}})_I$

Given $0_{bp} = (0_{bp}{}^{+}, 0_{bp}{}^{-}) \in \tau_{\mathfrak{B}}$

Let $(\hat{0}_{bp})_I = \left((\hat{0}_{bp}{}^{+})_I, (\hat{0}_{bp}{}^{-})_I\right)$, then

$$\left(\hat{0}_{bp}{}^{+}\right)_I(x) = 0_{bp}{}^{+} = \mathbf{0} = \hat{0}_{bp}{}^{+}(x) \Rightarrow \left(\hat{0}_{bp}{}^{+}\right)_I = \hat{0}_{bp}{}^{+}$$

$$\left(\hat{0}_{bp}{}^{-}\right)_I(x) = 0_{bp}{}^{-} = \mathbf{0} = \hat{0}_{bp}{}^{-}(x) \Rightarrow \left(\hat{0}_{bp}{}^{-}\right)_I = \hat{0}_{bp}{}^{-}$$

implies $(\hat{0}_{bp})_I = \hat{0}_{bp} \in (\hat{\tau}_{\mathfrak{B}})_I$

Similarly $\hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(\hat{1}_{bp}\right)_I, \left(\hat{\alpha}_{bp}\right)_I \in (\hat{\tau}_{\mathfrak{B}})_I$

(ii) To prove: $(\hat{\tau}_{\mathfrak{B}})_I$ is closed with respect to arbitrary union.

Consider $\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I \in (\hat{\tau}_{\mathfrak{B}})_I$ for $\lambda \in \Lambda_1 \subseteq \Lambda_0$

implies $\left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}$, for $\lambda \in \Lambda_1 \subseteq \Lambda_0$ implies $\cup_{\lambda\in\Lambda_1}\left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}$

Let $B_{bp} = \cup_{\lambda\in\Lambda_1}\left(A_{bp}\right)_\lambda \in \tau_{\mathfrak{B}}$

where $B_{bp} = \left(B_{bp}{}^{+}, B_{bp}{}^{-}\right) = \left(\vee_{\lambda\in\Lambda_1}\left(A_{bp}{}^{+}\right)_\lambda, \wedge_{\lambda\in\Lambda_1}\left(A_{bp}{}^{-}\right)_\lambda\right)$.

$B_{bp} \in \tau_{\mathfrak{B}}$ implies $\left(B_{bp}\right)_I \in (\hat{\tau}_{\mathfrak{B}})_I$

For $x \in X$,

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

Consider,

$$\left(\widehat{B}_{bp}{}^{+}\right)_I(x) = B_{bp}{}^{+}$$

$$= \vee_{\lambda\in\Lambda_1}\left(A_{bp}{}^{+}\right)_\lambda$$
$$= \vee_{\lambda\in\Lambda_1}\left(\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I(x)\right)$$
$$= \left(\vee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I\right)(x)$$

implies $\left(\widehat{B}_{bp}{}^{+}\right)_I = \vee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I$

$$\left(\widehat{B}_{bp}{}^{-}\right)_I(x) = B_{bp}{}^{-}$$

$$= \wedge_{\lambda\in\Lambda_1}\left(A_{bp}{}^{-}\right)_\lambda$$

$$= \wedge_{\lambda\in\Lambda_1}\left(\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I(x)\right)$$
$$= \left(\wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I\right)(x)$$

implies$\left(\widehat{B}_{bp}{}^{-}\right)_I = \wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I$

Therefore $(\widehat{B}_{bp})_I = \left(\left(\widehat{B}_{bp}{}^{+}\right)_I, \left(\widehat{B}_{bp}{}^{-}\right)_I\right) = \left(\vee_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I, \wedge_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I\right)$

Therefore$(\widehat{B}_{bp})_I = \cup_{\lambda\in\Lambda_1}\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I \in (\hat{\tau}_{\mathfrak{B}})_I$.
The proof for the finite intersection can also be proved in similar manner.

Theorem : 3.5

Let $\hat{\tau}_{\mathfrak{B}} = \{(\widehat{A}_{bp})_\lambda / \lambda \in \Lambda\}$ be a SOBPFT (lowen) on X. Then the collection $(\hat{\tau}_{\mathfrak{B}})_c = \left\{\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_c / \left(\widehat{A}_{bp}\right)_\lambda \in \hat{\tau}_{\mathfrak{B}}\right\}$ is also a SOBPFT (Lowen) on X where

$\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_c(x)(\alpha) = \left(\widehat{A}_{bp}{}^{+}\right)_\lambda(x)(1-\alpha)$ and

$\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_c(x)(\alpha) = \left(\widehat{A}_{bp}{}^{-}\right)_\lambda(x)(1-\alpha)$. The correspondence $\hat{\tau}_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_c$ is denoted as $\mathbb{C}_5$.

Proof:

The proof of this theorem is obvious.

4. Connections between crisp topological spaces and second order bipolar fuzzy topological spaces

Definition : 4.1

Let $\widehat{A}_{bp} = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right) \in$ SOBPFS

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

For $\varepsilon \in (0,1)$, define

$$\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} = \left\{x \in X : \left(\widehat{A}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{A_{bp}}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I\right\}.$$

Definition : 4.2

Let $\widehat{A}_{bp} = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right) \in$ SOBPFS. Define

$$\left(L_{\widehat{A}_{bp}}\right) = \left\{x \in X : \left(\widehat{A}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{A}_{bp}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I, \text{for some } \varepsilon \in (0,1)\right\}$$

Remark : 4.3

1) $\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} \subseteq \left(L_{\widehat{A}_{bp}}\right)$, for every $\varepsilon \in (0,1)$
2) $\left(L_{\widehat{A}_{bp}}\right) = \cup_{\varepsilon \in (0,1)} \left(L_{\widehat{A}_{bp}}\right)_{\varepsilon}$

Proposition : 4.4

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a SOBPFTS. Then the collection $\left\{\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} / \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$ is closed with respect to finite intersection.

Proof:

Consider $\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} \cap \left(L_{\widehat{B}_{bp}}\right)_{\varepsilon}$

$$= \{x \in X : \left(\widehat{A}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{A}_{bp}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I \text{ and}$$

$$\left(\widehat{B}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{B}_{bp}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I, \text{ for every } \varepsilon \in (0,1)\}$$

$$= \{x \in X : \widehat{A}_{bp}{}^{+}(x)(\alpha) > \varepsilon, \widehat{A}_{bp}{}^{-}(x)(\alpha) < -\varepsilon \text{ and } \widehat{B}_{bp}{}^{+}(x)(\alpha) > \varepsilon, \widehat{B}_{bp}{}^{-}(x)(\alpha) < -\varepsilon$$

$$\text{for every } \varepsilon \in (0,1) \text{ and for every } \alpha \in I\}$$

$$= \{x \in X : \left(\widehat{A}_{bp}{}^{+}(x)(\alpha) \wedge \widehat{B}_{bp}{}^{+}(x)(\alpha)\right) > \varepsilon, \left(\widehat{A}_{bp}{}^{-}(x)(\alpha) \vee \widehat{B}_{bp}{}^{-}(x)(\alpha)\right) < -\varepsilon,$$

$$\text{for every } \varepsilon \in (0,1) \text{ and for every } \alpha \in I\}$$

$$= \{x \in X : \left(\widehat{A}_{bp}{}^{+} \wedge \widehat{B}_{bp}{}^{+}\right)(x)(\alpha) > \varepsilon, \left(\widehat{A}_{bp}{}^{-} \vee \widehat{B}_{bp}{}^{-}\right)(x)(\alpha) < -\varepsilon, \text{ for every } \varepsilon \in (0,1)$$

$$\text{and for every } \alpha \in I\}$$

$$= \{x \in X : \left(\left(\widehat{A}_{bp}{}^{+} \wedge \widehat{B}_{bp}{}^{+}\right)(x)\right)^{-1}(\varepsilon, 1] = I, \left(\left(\widehat{A}_{bp}{}^{-} \vee \widehat{B}_{bp}{}^{-}\right)(x)\right)^{-1}[-1, -\varepsilon) = I\}$$

$= \left(L_{(\widehat{A}_{bp} \cap \widehat{B}_{bp})}\right)_{\varepsilon}$ (Therefore the given collection is closed with respect to finite intersection)

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

Proposition : 4.5

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a SOBPFTS Then the collection $\left\{\left(L_{\hat{A}_{bp}}\right) / \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$ is closed with respect to finite intersection.

Proof:

The proof of this proposition is similar as above.

Definition : 4.6

Let (X, τ) be a topological space

(1) For $\varepsilon \in (0,1)$, define $\left(\hat{K}_{bp}\right)_{\varepsilon} = \{\hat{A}_{bp} \in SBPF(X) / \left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau\}$

(2) Define $\hat{K}_{bp} = \{\hat{A}_{bp} \in SBPF(X) / \left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau$, for every $\varepsilon \in (0,1)\}$.

(3) Define $\left(\hat{K}_{bp}\right)_{*} = \{\hat{A}_{bp} \in SBPF(X) / \left(L_{\hat{A}_{bp}}\right) \in \tau\}$

Proposition : 4.7

Each of the above three sets $\left(\hat{K}_{bp}\right)_{\varepsilon}, \hat{K}_{bp}, \left(\hat{K}_{bp}\right)_{*}$ is closed with respect to finite intersection.

The proof is immediate from the preposition 2.3.4 and preposition 2.3.5.

Definition : 4.8

Let (X, τ) be a topological space. Then define

(i) $\widehat{\omega_{\varepsilon}(\tau)}$ to be a SOBPFT generated by $\left(\hat{K}_{bp}\right)_{\varepsilon}$

(ii) $\widehat{\omega(\tau)}$ to be a SOBPFT generated by $\hat{K}_{bp}$

(iii) $\widehat{\omega_{*}(\tau)}$ to be a SOBPFT generated by $\left(\hat{K}_{bp}\right)_{*}$

Theorem : 4.9

Let (X, τ) be a topological space. Then

(i) $\widehat{\omega(\tau)} \subseteq \widehat{\omega_{\varepsilon}(\tau)}$, for every $\varepsilon \in (0,1)$.

(ii) $\widehat{\omega(\tau)} \subseteq \widehat{\omega_{*}(\tau)}$.

Proof:

(i) Consider $\hat{A}_{bp} \in \hat{K}_{bp}$

Then by the definition $\left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau$, for every $\varepsilon \in (0,1)$

implies $\hat{A}_{bp} \in \left(\hat{K}_{bp}\right)_{\varepsilon}$ for every $\varepsilon \in (0,1)$

Therefore $\widehat{\omega(\tau)} \subseteq \widehat{\omega_{\varepsilon}(\tau)}$, for every $\varepsilon \in (0,1)$.

(ii) Consider $\hat{A}_{bp} \in \hat{K}_{bp}$

implies $\left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau$, for every $\varepsilon \in (0,1)$

implies $\bigcup_{\varepsilon \in (0,1)} \left(L_{\hat{A}_{bp}}\right)_{\varepsilon} \in \tau$

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

implies $\left(L_{\widehat{A}_{bp}}\right) \in \tau$ (from remark (2.3.3))

implies $\widehat{A}_{bp} \in \left(\widehat{K}_{bp}\right)_*$

Therefore $\widehat{\omega(\tau)} \subseteq \widehat{\omega_*(\tau)}$.

Proposition : 4.10

Let τ_1, τ_2 be two topologies on X such that $\tau_1 \subseteq \tau_2$. Then

(i) $\widehat{\omega(\tau_1)} \subseteq \widehat{\omega(\tau_2)}$
(ii) $\widehat{\omega_\varepsilon(\tau_1)} \subseteq \widehat{\omega_\varepsilon(\tau_2)}$
(iii) $\widehat{\omega_*(\tau_1)} \subseteq \widehat{\omega_*(\tau_2)}$

Proof:

(i) Let $\widehat{K}_{bp} \in \widehat{\omega(\tau_1)}$.

implies $\left(L_{\widehat{A}_{bp}}\right)_\varepsilon \in \tau_1$, for any $\varepsilon \in (0,1)$

implies $\left(L_{\widehat{A}_{bp}}\right)_\varepsilon \in \tau_2$, for any $\varepsilon \in (0,1)$ (since $\tau_1 \subseteq \tau_2$)

implies $\widehat{K}_{bp} \in \widehat{\omega(\tau_2)}$

Therefore $\widehat{\omega(\tau_1)} \subseteq \widehat{\omega(\tau_2)}$

The proofs of the other are similar.

Definition: 4.11

Let $(X, \hat{\tau}_{\mathfrak{B}})$ SOBPFT. Define

(i) $i_\varepsilon(\hat{\tau}_{\mathfrak{B}})$ to be the topology generated by the collection $\left\{\left(L_{\widehat{A}_{bp}}\right)_\varepsilon \;/\; \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$
(ii) $i^*(\hat{\tau}_{\mathfrak{B}})$ to be the topology generated by the collection $\left\{\left(L_{\widehat{A}_{bp}}\right) \;/\; \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$
(iii) $i(\hat{\tau}_{\mathfrak{B}})$ to be the topology generated by the collection $\left\{\left(L_{\widehat{A}_{bp}}\right)_\varepsilon \;/\; \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}, \varepsilon \in (0,1)\right\}$ as a subbasis.

Theorem : 4.12

Let (X, τ) be a topological space. Then

(i) $\tau \subseteq i^*\left(\widehat{\omega_*(\tau)}\right)$
(ii) For $\varepsilon \in (0,1)$, $\tau \subseteq i_\varepsilon\left(\widehat{\omega_\varepsilon(\tau)}\right)$
(iii) $\tau \subseteq i\left(\widehat{\omega(\tau)}\right)$

Proof:

(i) Consider $M \in \tau$

Consider the SOBPF characteristic function

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

$\left(\hat{\chi}_{bp}\right)_M = \left(\left(\hat{\chi}_{bp}{}^{+}\right)_M, \left(\hat{\chi}_{bp}{}^{-}\right)_M\right)$ on X as follows:

$\left(\hat{\chi}_{bp}{}^{+}\right)_M(x) = \mathbf{1}$, if $x \in M$

$= \mathbf{0}$, if $x \notin M$

$\left(\hat{\chi}_{bp}{}^{-}\right)_M(x) = -\mathbf{1}$, if $x \in M$

$= \mathbf{0}$, if $x \notin M$

Consider,

$\left(L_{\left(\hat{\chi}_{bp}\right)_M}\right)$

$= \left\{x \in X \,/\, \left(\left(\hat{\chi}_{bp}{}^{+}\right)_M(x)\right)^{-1}(\varepsilon, 1] = I, \left(\left(\hat{\chi}_{bp}{}^{-}\right)_M(x)\right)^{-1}[-1, -\varepsilon) = I \text{ for some } \varepsilon \in (0,1)\right\}$

$= \left\{x \in X \,/\, \left(\hat{\chi}_{bp}{}^{+}\right)_M(x)(\alpha) > \varepsilon, \left(\hat{\chi}_{bp}{}^{-}\right)_M(x)(\alpha) < -\varepsilon, \text{for some } \varepsilon \in (0,1) \text{ and for every } \alpha \in I\right\}$

$= \left\{x \in X \,/\, \left(\hat{\chi}_{bp}{}^{+}\right)_M(x)(\alpha) \neq 0, \left(\hat{\chi}_{bp}{}^{-}\right)_M(x)(\alpha) \neq 0, \text{for every } \alpha \in I\right\}$

$= \left\{x \in X \,/\, \left(\hat{\chi}_{bp}{}^{+}\right)_M(x) \neq \mathbf{0}, \left(\hat{\chi}_{bp}{}^{-}\right)_M(x) \neq \mathbf{0}\right\}$

$= \left\{x \in X \,/\, \left(\hat{\chi}_{bp}{}^{+}\right)_M(x) = \mathbf{1}, \left(\hat{\chi}_{bp}{}^{-}\right)_M(x) = -\mathbf{1}\right\}$

$= M$

$M \in \tau$ implies $\left(L_{\left(\hat{\chi}_{bp}\right)_M}\right) \in \tau$

implies $\left(\hat{\chi}_{bp}\right)_M \in \left(\widehat{K}_{bp}\right)_*$

implies $\left(\hat{\chi}_{bp}\right)_M$ is a basis element of $\widehat{\omega_*(\tau)}$

implies $\left(L_{\left(\hat{\chi}_{bp}\right)_M}\right)$ is a basis element of $i^*\left(\widehat{\omega_*(\tau)}\right)$

implies M is a basis element of $i^*\left(\widehat{\omega_*(\tau)}\right)$

implies $\tau \subseteq i^*\left(\widehat{\omega_*(\tau)}\right)$

Proofs of (ii) and (iii) are similar.

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

Example : 4.13

Let $\hat{\tau}_{\mathfrak{B}} = \{\hat{0}_{bp}\} \cup \{\hat{A}_{bp} \in SBPF(X) \,/ supp\left(\mathbf{1} - \hat{A}_{bp}{}^{+}(x)\right), supp\left(-\mathbf{1} - \hat{A}_{bp}{}^{-}(x)\right)$

is finite for every $x \in X\}$.

Then $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Proof:

(i) Since $supp\left(\mathbf{1} - \hat{1}_{bp}{}^{+}(x)\right) = \emptyset$, $supp\left(-\mathbf{1} - \hat{1}_{bp}{}^{-}(x)\right) = \emptyset$, $\hat{1}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) $\left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda$

implies $supp\left(\mathbf{1} - \left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)\right)$, $supp\left(-\mathbf{1} - \left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)\right)$ are finite

Consider $supp\left(\mathbf{1} - \left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)\right) = supp\left(\mathbf{1} - \vee_{\lambda\in\Lambda}\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)\right)\right)$

$\subseteq supp\left(\mathbf{1} - \left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)\right)$

implies $supp\left(\mathbf{1} - \left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)\right)$ is finite

Similarly $supp\left(-\mathbf{1} - \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)\right) = supp\left(-\mathbf{1} - \wedge_{\lambda\in\Lambda}\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)\right)\right)$

$\supset supp\left(-\mathbf{1} - \left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)\right)$

implies $supp\left(-\mathbf{1} - \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)\right)$ is finite

Therefore $\cup_{\lambda\in\Lambda}\left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$

(iii) $\left(\hat{A}_{bp}\right)_{i} \in \hat{\tau}_{\mathfrak{B}}$, for $i = 1,2,\ldots m$

implies $supp\left(\mathbf{1} - \left(\hat{A}_{bp}{}^{+}\right)_{i}(x)\right)$, $supp\left(-\mathbf{1} - \left(\hat{A}_{bp}{}^{-}\right)_{i}(x)\right)$ are finite

Consider $supp\left(\mathbf{1} - \left(\wedge_{i=1}^{m}\left(\hat{A}_{bp}{}^{+}\right)_{i}\right)(x)\right) = supp\left(\mathbf{1} - \wedge_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{+}\right)_{i}(x)\right)\right)$

There exists k such that

$supp\left(\mathbf{1} - \wedge_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{+}\right)_{k}(x)\right)\right) \subseteq supp\left(\mathbf{1} - \left(\hat{A}_{bp}{}^{+}\right)_{K}(x)\right)$ which is finite

Similarly $supp\left(-\mathbf{1} - \left(\vee_{i=1}^{m}\left(\hat{A}_{bp}{}^{-}\right)_{i}\right)(x)\right) = supp\left(-\mathbf{1} - \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{-}\right)_{i}(x)\right)\right)$

There exists k such that

$supp\left(-\mathbf{1} - \vee_{i=1}^{m}\left(\left(\hat{A}_{bp}{}^{-}\right)_{k}(x)\right)\right) \supset supp\left(-\mathbf{1} - \left(\hat{A}_{bp}{}^{-}\right)_{K}(x)\right)$ which is also finite

Therefore $\cap_{i=1}^{m}\left(\hat{A}_{bp}\right)_{i} \in \hat{\tau}_{\mathfrak{B}}$

Therefore $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Example : 4.14

Let $\hat{\tau}_{\mathfrak{B}} = \{\hat{1}_{bp}\} \cup \{\hat{A}_{bp} \in SBPF(X) / \hat{A}_{bp}{}^{+}(x)(\alpha) = 0, \hat{A}_{bp}{}^{-}(x)(\alpha) = 0$, for every $x \in X$ and

for $\alpha \neq \frac{r}{n}$, $r = 0,1,2,\ldots n\}$

Then $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Proof:

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

(i) Since $\hat{0}_{bp}{}^{+}(x)(\alpha) = 0, \hat{0}_{bp}{}^{-}(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$.
implies $\hat{0}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) $(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda$
$(\hat{A}_{bp}{}^{+})_{\lambda}(x)(\alpha) = 0, (\hat{A}_{bp}{}^{-})_{\lambda}(x)(\alpha) = 0$, for every $x \in X$, for $\alpha \neq \frac{r}{n}$,
$r = 0,1,2,\ldots n$
implies $\vee_{\lambda\in\Lambda}\left((\hat{A}_{bp}{}^{+})_{\lambda}(x)(\alpha)\right) = 0$, $\wedge_{\lambda\in\Lambda}\left((\hat{A}_{bp}{}^{-})_{\lambda}(x)(\alpha)\right) = 0$
for every $x \in X$ and for $\alpha \neq \frac{r}{n}$, $r = 0,1,2,\ldots n$
implies $\left(\vee_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{+})_{\lambda}\right)(x)(\alpha) = 0, \left(\wedge_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{-})_{\lambda}\right)(x)(\alpha) = 0$,
for every $x \in X$ and for $\alpha \neq \frac{r}{n}$, $r = 0,1,2,\ldots n$
Therefore $\cup_{\lambda\in\Lambda}(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$

(iii) Similarly we can prove for finite intersection.
Therefore $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Example : 4.15

Le $\hat{\tau}_{\mathfrak{B}} = \{\hat{A}_{bp} \in SBPF(X) /$ for every $x \in X$, either $\hat{A}_{bp}{}^{+}(x) = \mathbf{0}, \hat{A}_{bp}{}^{-}(x) = \mathbf{0}$ (or)

$\hat{A}_{bp}{}^{+}(x)(\alpha) > 0,\ \hat{A}_{bp}{}^{-}(x)(\alpha) < 0$, for every $\alpha \in I\}$

Then $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Proof:

(i) Obviously $\hat{0}_{bp}, \hat{1}_{bp}, \hat{\alpha}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) Consider $(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$, for $\lambda \in \Lambda$
Suppose $\left(\vee_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{+})_{\lambda}\right)(x) = \mathbf{0}$, $\left(\wedge_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{-})_{\lambda}\right)(x) = \mathbf{0}$, for every $x \in X$
Then $\cup_{\lambda\in\Lambda}(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$
Suppose $\left(\vee_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{+})_{\lambda}\right)(x) \neq \mathbf{0}, \left(\wedge_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{-})_{\lambda}\right)(x) \neq \mathbf{0}$,
Then $\left(\vee_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{+})_{\lambda}\right)(x)(\alpha) > 0, \left(\wedge_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{-})_{\lambda}\right)(x)(\alpha) < 0$, for some $\alpha \in I$
there exists a $\lambda \in \Lambda$ such that
$(\hat{A}_{bp}{}^{+})_{\lambda}(x)(\alpha) > 0, (\hat{A}_{bp}{}^{-})_{\lambda}(x)(\alpha) < 0$ for some $\alpha \in I$
for that $\lambda \in \Lambda, (\hat{A}_{bp}{}^{+})_{\lambda}(x)(\alpha) > 0, (\hat{A}_{bp}{}^{-})_{\lambda}(x)(\alpha) < 0$, for every $\alpha \in I$
implies $\left(\vee_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{+})_{\lambda}\right)(x)(\alpha) > 0, \left(\wedge_{\lambda\in\Lambda}(\hat{A}_{bp}{}^{-})_{\lambda}\right)(x)(\alpha) < 0$, for every $\alpha \in I$
Therefore $\cup_{\lambda\in\Lambda}(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$

(iii) Consider $(\hat{A}_{bp})_{i} \in \hat{\tau}_{\mathfrak{B}}$, for i = 1 to m
Suppose $\left(\wedge_{i=1}^{m}(\hat{A}_{bp}{}^{+})_{i}\right)(x) = \mathbf{0}$, $\left(\vee_{i=1}^{m}(\hat{A}_{bp}{}^{-})_{i}\right)(x) = \mathbf{0}$
Then $\cap_{i=1}^{m}(\hat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$
Suppose $\left(\wedge_{i=1}^{m}(\hat{A}_{bp}{}^{+})_{i}\right)(x) \neq \mathbf{0}$, $\left(\vee_{i=1}^{m}(\hat{A}_{bp}{}^{-})_{i}\right)(x) \neq \mathbf{0}$
Then $\left(\wedge_{i=1}^{m}(\hat{A}_{bp}{}^{+})_{i}\right)(x)(\alpha) > 0$, $\left(\vee_{i=1}^{m}(\hat{A}_{bp}{}^{-})_{i}\right)(x)(\alpha) < 0$, for some $\alpha \in I$
Therefore $(\hat{A}_{bp}{}^{+})_{i}(x)(\alpha) > 0, (\hat{A}_{bp}{}^{-})_{i}(x)(\alpha) < 0$, for some $\alpha \in I$ & for $i = 1,2, \ldots m$
Therefore $(\hat{A}_{bp}{}^{+})_{i}(x)(\alpha) > 0, (\hat{A}_{bp}{}^{-})_{i}(x)(\alpha) < 0$, for every $\alpha \in I$ & for $i = 1,2, \ldots m$
Therefore $\left(\wedge_{i=1}^{m}(\hat{A}_{bp}{}^{+})_{i}\right)(x)(\alpha) > 0$, $\left(\vee_{i=1}^{m}(\hat{A}_{bp}{}^{-})_{i}\right)(x)(\alpha) < 0$, for every $\alpha \in I$

DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

Therefore $\bigcap_{i=1}^{m}(\widehat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}$
Therefore $\hat{\tau}_{\mathfrak{B}}$ is a SOBPFT on X.

Example : 4.16

Consider the closed unit interval I. For $\alpha \in I$, define $(\widehat{A}_{bp})_{\alpha} : I \to I^{I}$ such that

$(\widehat{A}_{bp}{}^{+})_{\alpha}(\beta)(\gamma) = \alpha\beta\gamma$, $(\widehat{A}_{bp}{}^{-})_{\alpha}(\beta)(\gamma) = -\alpha\beta\gamma$, for every $\beta, \gamma \in I$.

Then the collection $\hat{\tau}_{\mathfrak{B}} = \{(\widehat{A}_{bp})_{\alpha} / \alpha \in I\} \cup \{\hat{1}_{bp}\}$ is a SOBPFT on I.

Proof:

(i) Since $\hat{0}_{bp}{}^{+} = (\widehat{A}_{bp}{}^{+})_{0} = \hat{0}$, $\hat{0}_{bp}{}^{-} = (\widehat{A}_{bp}{}^{-})_{0} = \hat{0}$
implies $\hat{0}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

(ii) Let $(\widehat{A}_{bp})_{\alpha} \in \hat{\tau}_{\mathfrak{B}}$ for α belongs to an arbitrary set $S \subseteq I$
For $\beta, \gamma \in I$, Consider
Now $\bigcup_{\alpha\in S}(\widehat{A}_{bp})_{\alpha} = \left(\left(\bigvee_{\alpha\in S}(\widehat{A}_{bp}{}^{+})_{\alpha}\right), \left(\bigwedge_{\alpha\in S}(\widehat{A}_{bp}{}^{-})_{\alpha}\right)\right)$

$$\left(\bigvee_{\alpha\in S}(\widehat{A}_{bp}{}^{+})_{\alpha}\right)(\beta)(\gamma) = \bigvee_{\alpha\in S}\left((\widehat{A}_{bp}{}^{+})_{\alpha}(\beta)(\gamma)\right)$$
$$= \bigvee_{\alpha\in S}(\alpha\beta\gamma)$$
$$= (\bigvee_{\alpha\in S}\alpha)\beta\gamma$$
$$= \alpha'\beta\gamma$$
$$= (\widehat{A}_{bp}{}^{+})_{\alpha'}$$

$$\left(\bigwedge_{\alpha\in S}(\widehat{A}_{bp}{}^{-})_{\alpha}\right)(\beta)(\gamma) = \bigwedge_{\alpha\in S}(\widehat{A}_{bp}{}^{-})_{\alpha}(\beta)(\gamma)$$
$$= \bigwedge_{\alpha\in S}(-\alpha\beta\gamma)$$
$$= (\bigwedge_{\alpha\in S}(-\alpha))\beta\gamma$$
$$= (-\alpha')\beta\gamma$$
$$= (\widehat{A}_{bp}{}^{-})_{\alpha'}$$

Therefore $\bigcup_{\alpha\in S}(\widehat{A}_{bp})_{\alpha} = (\widehat{A}_{bp})_{\alpha'} \in \hat{\tau}_{\mathfrak{B}}$

(iii) The proof for finite intersection is similar
Therefore $\hat{\tau}_{\mathfrak{B}}$ is a s SOBPFT on I.

5. Conclusion:

In this paper, the detailed study about SOBPFT relating to FOBPFT and crisp topology are established and some examples for SOBPFT are provided. The connections between FOBPFT and SOBPFT under five different cases are discussed. And from a crisp topology τ on X there exists three different SOBPFT denoted by $\widehat{\omega(\tau)}$, $\widehat{\omega_{*}(\tau)}$ and $\widehat{\omega_{\varepsilon}(\tau)}$ and from a SOBPFT on X there exists three crisp topologies denoted by $i(\hat{\tau}_{\mathfrak{B}})$, $i^{*}(\hat{\tau}_{\mathfrak{B}})$ and $i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})$.

Funding: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest.

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Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024

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DOI: https://doi.org/10.54216/IJNS.240410
Received: November 03, 2023 Revised: February 01, 2024 Accepted: June 01, 2024