The Impact of Anti-Predator Behavior and Toxicant on An Ecological Neutrosophic Model

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Abstract

This paper discusses a neutrosophic mathematical model consisting of three nonlinear ordinary differential equations describing the interaction between two prey and a predator with the use of function response Holling's type IV and Lotka Volttra. It appears that the first prey has a way to defend itself by using the toxic substance directly to the predator, as well as the effect of the predator on the toxic substance. The conditions for the existence of the solution and the uniqueness of the boundaries were discussed, and then the different equilibrium points and the stability of the system around the equilibrium points were analyzed. The Lyapunov function was used to study the global dynamics of this proposed model. Finally, numerical simulations were performed to show the analytical results.

Keywords: Prey- Predator; Holling's type IV Function Response; Toxicity; Anti-Predator; neutrosophic model

1. Introduction

In population dynamics, mathematical model formulations are created based on experiments and observations that are used to understand specific predation events, interactions, and are represented mathematically by interactions between predators and animals living in the same environment [1–3].

In mathematical ecology, the predator-prey model is an essential tool for understanding that ecosystem, the current food chain, the habits of existing species, and their behavior toward other species. Researchers have studied dynamic behavior in prey-predator interactions with different functional responses [4–8].

The anti-predator behavior characteristic of the first prey population is presented as shown in the proposed mathematical model. Anti-predator behavior always affects more than one predator and makes competition more complex.

A predator may use speed or stealth, the ability to approach unnoticed by being quiet and deliberate in its movements or by approaching upwind, camouflage, sense of smell, sight, hearing, and other means to help it prey on prey. Animals use many means to defend themselves. These means are represented by the weapons that animals possess, whether in one of their body parts or their ability to hide and escape. Each animal also develops its own methods to defend itself. For example, a cobra snake secretes venom to help it avoid being killed by a predator [3,9].

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Propose (ARDITI) and (GINBURG) in (1989) the response function depends on the ratio, which is type specific in predator dependence. Here, Holling's type IV function response was used, the basis of which is prey and predator dependent [10,11].

Toxicity or toxic effect is the harm that a substance can cause to an organism. Toxicity may reflect an effect on the entire organism. This technique has been used in a variety of biological systems. The global increase in the prevalence of venomous prey over the past three decades has sparked interest in studying the dynamics and regulation of toxin-producing prey [12]. Toxicology focuses on the fact that the effects caused by toxins depend on the dose at which they are taken. Poisoned animals can inject poison when attacked by predators. In the method proposed by (M.BANERJEE) and (E.VENTURINO), the reproduction of the predator after preying on the prey is immediate, and the practical form of the predator is taken, which illustrates the behavior of the predator in the presence of huge quantities of toxic prey [5,13,14].

Neutrosophic logic was presented by Smarandache [16], and then it was applied to statistics and probability theory by many different authors around the world [17-19].

These results indicate that toxic substances as well as toxic prey play an important role in predator population development and have a significant impact on prey-predator encounters. A mathematical model of three differential equations was proposed that describes anti-predator behavior, as well as the effect of the predator on the toxic substance secreted by the prey to defend itself from the predator during hunting.

2. Classical Model formulation

In this research, a mathematical model of three differential equations for living organisms consisting of a first prey S_1, a second prey S_1 which represents the population size at the time and a predator S_3 which represents the population size at the time. It appears that the first prey secretes the toxic substance to defend itself when hunting, knowing that the predator feeds on it the second prey does not secrete the toxic substance, but rather feeds on it only the third predator feeds on the first and second prey and is negatively affected by the toxic substance secreted by the first prey during hunting. Accordingly, the dynamics of this model can be interpreted as shown.

$$\frac{dS_1}{dt} = S_1\left(\alpha_1\left(1-\frac{S_1}{h_1}\right)-\frac{K_1S_3}{n+s_1^2}-C_1S_3S_1\right),$$
$$\frac{dS_2}{dt} = S_2\left(\alpha_2\left(1-\frac{S_2}{h_2}\right)-K_2S_3-d_1\right), \quad (1)$$
$$\frac{dS_3}{dt} = S_3\left(\frac{R_1S_1}{n+S_1^2}+R_2S_2-eS_1-d_2\right).$$

The parameters and variables system (1) can be described in the following table.

Table 1: The parameters of the system (1):

Parameters	Described
$\alpha_i > 0$, $i = 1,2$	The logistic growth rate of first and second prey, respectively.
$h_i > 0$, $i = 1.2$	The carrying capacity of the first and second prey, respectively.
$K_i > 0$, $i = 1,2$	The rate of predator attack on the first and second prey, respectively.
$n > 0$	The rate is half the saturation limit.
$C_1 > 0$	The toxicity rate released by the first prey on the predator

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$d_i > 0, i = 1,2$	The mortality rates of the second prey and a predator, respectively.
$R_i > 0, i = 1,2$	The conversion rates of food to a predator, respectively.
$e > 0$	The rate of anti-predator behavior of the first prey.

Theorem 1: Every solution of system (1) which initiate in the R_+^3 are uniformly bounded.

Proof: Let $(S_1(t), S_2(t), S_3(t))$ be any solution of the system (1) with a non-negative initial Condition $(S_1(0), S_2(0), S_3(0))$ Now consider a function: $E(t) = S_1(t) + S_2(t) + S_3(t)$,we get:

$$\frac{dE}{dt} < \alpha_1 S_1\left(1 - \frac{S_1}{h_1}\right) + \alpha_2 S_2\left(1 - \frac{S_2}{h_2}\right) - d_1 S_2 - d_2 S_3 - S_1 S_3(C_1 S_3 - e) - (K_2 - R_2) S_2 S_3 - (K_1 - R_1)\frac{S_1 S_3}{n + S_1^2} .$$

So according to the biological facts always $K_i > R_i$, $i = 1,2$,we get:

$$\frac{dE}{dt} < 2\alpha_1 S_1 + \alpha_2 S_2\left(1 - \frac{S_2}{h_2}\right) - (\alpha_1 S_1 + d_1 S_2 + d_2 S_3) .$$

By comparison theorem [15] for solving this differential inequality, we get:

$$\frac{dE}{dt} \le 2\alpha_1 h_1 + \frac{\alpha_2 h_2}{4} - (\alpha_1 + d_1 + d_2)\, E.$$

$$\frac{dE}{dt} + LE \le 2\alpha_1 h_1 + \frac{\alpha_2 h_2}{4}, \quad Where \quad L = min\ \{\alpha_1, d_1, d_2\}.$$

$$\frac{dE}{dt} + LE \le N, \ where \quad N = \left(2\alpha_1 h_1 + \frac{\alpha_2 h_2}{4}\right).$$

Again, by comparison theorem to solving this differential inequality for the initial value $E(0) = E_0$, we get:

$$E(t) \le \frac{N}{L} + \left(E_0 - \frac{N}{L}\right) e^{-Lt},$$

$$Then, \quad \lim_{t\to\infty} E(t) \le \frac{N}{L}.$$

So, $0 \le E(t) \le \frac{N}{L}$,$\forall\, t > 0$.

Hence the theorem is proved.

3. Existence of equilibrium points

In this section, all possible equilibrium points of the system (1) whose conditions of existence can be achieved are discussed, as shown below:

1) The trivial equilibrium point $m_0 = (0, 0, 0)$, always exists.
2) The equilibrium point $m_1 = (h_1, 0, 0)$ always exists.
3) The equilibrium point $m_2 = \left(0, \frac{h_2(\alpha_2 - d_1)}{\alpha_2}, 0\right)$, exists provided that:

$$\alpha_2 > d_1 \qquad (1.a)$$

4) The equilibrium point $m_3 = (\hat{S}_1, \hat{S}_2, 0)$ exists uniquely the following set of equations:

$$\alpha_1 - \frac{\alpha_1 S_1}{h_1} - \frac{K_1 S_3}{(n + S_1^2)} - C_1 S_2 S_3 = 0, \qquad (1.b)$$

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$$\alpha_2 - \frac{\alpha_2 S_2}{h_2} - K_2 S_3 - d_1 = 0, \quad (1.c)$$

From equation (1.b) we have,

$$S_1 = h_1 \quad (1.d)$$

From equation (1.c) we have,

$$S_2 = \frac{h_2(\alpha_2 - d_1)}{\alpha_2} \quad (1.e)$$

5) The equilibrium point $m_4 = (\tilde{S}_1, 0, \tilde{S}_3)$ exists uniquely the following set of equations:

$$\alpha_1 - \frac{\alpha_1 S_1}{h_1} - \frac{K_1 S_3}{(n+S_1^2)} - C_1 S_1 S_3 = 0, \quad (1.f)$$

$$\frac{R_1 S_1}{(n+S_1^2)} + R_2 S_2 - e S_1 - d_2, \quad (1.g)$$

From equation (1.f) we have,

$$S_3 = \frac{\alpha_1(h_1 - S_1)(n+S_1^2)}{h_1(K_1 + C_1 S_1(n+S_1^2))}, \quad (1.h)$$

From equation (1.g) we have,

$$S_1 = \frac{d_2(n+S_1^2)}{R_1 - e(n+S_1^2)}, \quad (1.i)$$

So, $m_4 = (\tilde{S}_1, 0, \tilde{S}_3)$ exists,provided that depending on the following conditions:

$$h_1 > S_1, \quad (1.j)$$

$$R_1 > e(n + S_1^2), \quad (1.k)$$

6) The equilibrium point $m_5 = (0, \dot{S}_2, \dot{S}_3)$ exists uniquely the following set of equations:

$$\alpha_2 - \frac{\alpha_2 S_2}{h_2} - K_2 S_3 - d_1 = 0, \quad (1.L)$$

$$\frac{R_1 S_1}{(n+S_1^2)} + R_2 S_2 - e S_1 - d_2, \quad (1.m)$$

From equation (1.m) we have,

$$S_2 = \frac{d_2}{R_2}, \quad (1.n)$$

Now by substituting (1.n) in (1.L) we get:

$$S_3 = \frac{h_2(\alpha_2 - d_1)R_2 - \alpha_2 d_2}{h_2 K_2 R_2},$$

So, $m_5 = \left(0, \dot{S}_2, \dot{S}_3\right)$exists if, in addition to condition (1.a) the following conditions hold.

$$h_2(\alpha_2 - d_1)R_2 > \alpha_2 d_2, \quad (1.o)$$

7) Finally the positive equilibrium point $m_6 = \left(\bar{\bar{S}}_1, \bar{\bar{S}}_2, \bar{\bar{S}}_3\right)$ exists in the Int. R_+^3 if and only if there is appositive solution of the following set of equations :

$$\alpha_1 - \frac{\alpha_1 S_1}{h_1} - \frac{K_1 S_3}{(n + S_1^2)} - C_1 S_1 S_3 = 0, \quad (1.p)$$

$$\alpha_2 - \frac{\alpha_2 S_2}{h_2} - K_2 S_3 - d_1 = 0, \quad (1.q)$$

$$\frac{R_1 S_1}{(n+S_1^2)} + R_2 S_2 - e S_1 - d_2 = 0, \quad (1.r)$$

From equation (1.p) we have,

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$$S_3 = \frac{\alpha_1(h_1 - S_1)(n + S_1^2)}{h_1(K_1 + C_1 S_1(n + S_1^2))}, \qquad (1.s)$$

Now by substituting (1.s) in (1.q) we get:

$$S_2 = \frac{h_2[h_1(K_1 + C_1 S_1(n + S_1^2))(\alpha_2 - d_1) - K_2(\alpha_1(h_1 - S_1)(n + S_1^2))]}{\alpha_2 h_1(K_1 + C_1 S_1(n + S_1^2))}, \qquad (1.t)$$

Now by substituting (1.t) in (1.r) we get:

$$f(S_1) = \gamma_1 S_1^6 + \gamma_2 S_1^5 + \gamma_3 S_1^4 + \gamma_4 S_1^3 + \gamma_5 S_1^2 + \gamma_6 S_1 + \gamma_7 = 0, \qquad (1.u)$$

Where:

$\gamma_1 = e\,\alpha_2 h_1 C_1 < 0,$

$\gamma_2 = \alpha_1 h_2 R_2 K_2 + C_1 h_1(h_2 R_2(\alpha_2 - d_1) - d_2\alpha_2),$

$\gamma_3 = h_1(\alpha_2 c_1(R_1 - en) - \alpha_1 h_2 R_2 K_2),$

$\gamma_4 = 2h_2 R_2 n(h_1 C_1(\alpha_2 - d_1) + K_2\alpha_2) - \alpha_2 h_1(k_1 e + 2nd_2 C_1)),$

$\gamma_5 = h_1\left(h_2 R_2 K_2((\alpha_2 - d_1) - 2n\alpha_1) + \alpha_2 h_1(C_1 n(R_1 - en) - d_2 K_1)\right),$

$\gamma_6 = h_2 R_2 n^2(h_1 C_1(\alpha_2 - d_1) + K_2\alpha_2) + \alpha_2 h_1(k_1(R_1 - en) - d_2 C_1 n^2),$

$\gamma_7 = h_1 n(R_2 h_2(k_1\alpha_2 - d_1) - K_2\alpha_1 n) - \alpha_2 K_1 d_2),$

So, $m_6 = (\bar{\bar{S}}_1, \bar{\bar{S}}_2, \bar{\bar{S}}_3)$ exists if, in addition to conditions (1.a) and (1.j) the following conditions hold.

$$h_1 > \frac{K_2(\alpha_1(h_1 - S_1)(n + S_1^2))}{(K_1 + C_1 S_1(n + S_1^2))(\alpha_2 - d_1)}, \qquad (1.v)$$

$$R_1 > en, \qquad (1.w)$$

$$(\alpha_2 - d_1) > \max\left\{\frac{d_2\alpha_2}{R_2 h_2}, 2n\alpha_1, \frac{K_2\alpha_1 n}{K_1}\right\}, \qquad (1.x)$$

$$(R_1 - en) > \max\left\{\frac{d_2 C_1 n^2}{K_1}, \frac{h_2 R_2 K_2 \alpha_1}{C_1\alpha_2}, \frac{K_1 d_2}{C_1 n}\right\}, \qquad (1.y)$$

$$h_2 R_2 > \max\left\{\frac{d_2\alpha_2 K_1}{k_1(\alpha_2 - d_1) - K_2\alpha_1 n}, \frac{\alpha_2 h_1(k_1 e + 2nd_2 C_1)}{2nh_1 C_1(\alpha_2 - d_1) + K_2\alpha_2}\right\}, \qquad (1.z)$$

4. Local stability analysis

In this section, the local stability analysis of system (1) around each of the previous equilibrium points is discussed by calculating the Jacobean matrix $J(S_1, S_2, S_3)$ of the system (1) as follows:

$$J = [a_{ij}]_{3\times 3}, \qquad (2.a)$$

Where:

$$a_{11} = \frac{h_1\alpha_1 - 2S_1(\alpha_1 + h_1 C_1 S_3)}{h_1} - \frac{K_1 S_3(n - S_1^2)}{(n + S_1^2)^2}, a_{12} = 0, a_{13} = -\frac{K_1 S_1}{(n + S_1^2)} - C_1 S_1^2,$$

$$a_{21} = \alpha_2 - \frac{2\alpha_2 S_2}{h_2} - K_2 S_3 - d_1, a_{22} = 0, a_{23} = -K_2 S_2,$$

$$a_{31} = \frac{K_1 S_3(n - S_1^2)}{(n + S_1^2)^2} - eS_3, a_{32} = R_2 S_3, a_{33} = \frac{R_1 S_1}{n + S_1^2} + R_2 S_2 - eS_1 - d_2,$$

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A. Local stability of m_0

The Jacobean matrix at $m_0 = (0,0,0)$ is given by:

$$J_0 = J(m_0) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 - d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}, \tag{2.b}$$

Then the characteristic equation of J_0 is given by:

$(\alpha_1 - \lambda_0)(\alpha_2 - d_1 - \lambda_0)(-d_2 - \lambda_0) = 0,$

which gives:

$\lambda_{0S_1} = \alpha_1 > 0,\ \ \lambda_{0S_2} = (\alpha_2 - d_1),\ \lambda_{0S_3} = -d_2 < 0$,

Hence m_0 is saddle point, and it is (unstable).

B. Local stability of m1

The Jacobean matrix at $m_1 = (h_1, 0,0)$ is given by:

$$J_1 = J(m_1) = \begin{bmatrix} -\alpha_1 & 0 & -\dfrac{K_1 h_1}{(n + h_1^2)} - C_1 h_1^2 \\ 0 & \alpha_2 - d_1 & 0 \\ 0 & 0 & \dfrac{R_1 h_1}{n + h_1^2} - e h_1 - d_2 \end{bmatrix}, \tag{2.c}$$

Then the eigenvalues of $J_1$are:

$(-\alpha_1 - \lambda_1)(\alpha_2 - d_1 - \lambda_1)(\dfrac{R_1 h_1}{n + h_1^2} - e h_1 - d_2 - \lambda_1) = 0,$

which gives:

$\lambda_{1S_1} = -\alpha_1 < 0,\ \ \lambda_{1S_2} = (\alpha_2 - d_1),\ \lambda_{1S_3} = (\frac{R_1}{n + h_1^2} - e)h_1 - d_2,$,

Hence m_1 is locally asymptotically stable provided that the following conditions hold.

$$\alpha_2 < d_1, \tag{2.d}$$

$$\frac{R_1}{n + h_1^2} < e, \tag{2.e}$$

Otherwise it is unstable.

C. Local stability of m_2

The Jacobean matrix of $m_2 = \left(0, \frac{h_2(\alpha_2 - d_1)}{\alpha_2}, 0\right)$ is given by:

$$J_2 = J(m_2) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \dfrac{\alpha_2 h_2 - 2\alpha_2 S_2 - d_1 h_2}{h_2} & 0 \\ 0 & 0 & R_2 S_2 - d_2 \end{bmatrix}, \tag{2.f}$$

Then the eigenvalues of J_2 are:

$(\alpha_1 - \lambda_2)(\dfrac{\alpha_2 h_2 - 2\alpha_2 S_2 - d_1 h_2}{h_2} - \lambda_2)(R_2 S_2 - d_2 - \lambda_2) = 0,$

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which gives:

$\lambda_{2S_1} = \alpha_1 > 0,\ \ \lambda_{2S_2} = -\alpha_2 + d_1,\ \lambda_{2S_3} = \frac{R_1 h_2(\alpha_2 - d_1) - d_2\alpha_2}{\alpha_2},$

Hence m_2 is saddle point, and it is (unstable).

D. Local stability of m_3

The Jacobean matrix at $m_3 = (h_1, \frac{h_2(\alpha_2 - d_1)}{\alpha_2}, 0)$ is given by:

$$J_3 = J(m_3) = \begin{bmatrix} \frac{h_1\alpha_1 - 2\alpha_1 S_1}{h_1} & 0 & -\frac{K_1 S_1}{(n+S_1^2)} - C_1 S_1^2 \\ 0 & \frac{\alpha_2 h_2 - 2\alpha_2 S_2 - d_1 h_2}{h_2} & -K_2 S_2 \\ 0 & 0 & \frac{R_1 S_1}{n+S_1^2} + R_2 S_2 - eS_1 - d_2 \end{bmatrix}, \qquad (2.g)$$

Then the eigenvalues of $J_3$are:

$$\left(\frac{h_1\alpha_1 - 2\alpha_1 S_1}{h_1} - \lambda_3\right)\left(\frac{\alpha_2 h_2 - 2\alpha_2 S_2 - d_1 h_2}{h_2} - \lambda_3\right)\left(\frac{R_1 S_1}{n + S_1^2} + R_2 S_2 - eS_1 - d_2 - \lambda_3\right) = 0,$$

Which gives:

$\lambda_{3S_1} = -\alpha_1 < 0,\ \ \lambda_{3S_2} = -\alpha_2 + d_1,\ \lambda_{3S_3} = \left(\frac{R_1}{n+h_1^2} - e\right) h_1 - d_2 + \frac{R_2 h_2(\alpha_2 - d_1)}{\alpha_2},$

Hence m_3 is locally asymptotically stable provided that conditions (1.a) and (2.d-2.e).

E. Local stability of m_4

The Jacobean matrix at $m_4 = (\tilde{S}_1, 0, \tilde{S}_3)$ is given by:

$$J_4 = J(m_4) = [u_{ij}]_{3\times 3}, \qquad (2.h)$$

Where:

$$u_{11} = \frac{h_1\alpha_1 - 2\tilde{S}_1(\alpha_1 + h_1 C_1 \tilde{S}_3)}{h_1} - \frac{K_1\tilde{S}_3(n - \tilde{S}_1^2)}{(n+\tilde{S}_1^2)^2}, u_{12} = 0, u_{13} = -\frac{K_1\tilde{S}_1}{(n+\tilde{S}_1^2)} - C_1\tilde{S}_1^2,$$

$$u_{21} = 0, u_{22} = \alpha_2 - K_2\tilde{S}_3 - d_1, u_{23} = 0, u_{13} = \frac{K_1\tilde{S}_3(n - \tilde{S}_1^2)}{(n + \tilde{S}_1^2)^2} - e\tilde{S}_3,$$

$$a_{23} = R_2\tilde{S}_3, a_{33} = \frac{R_1\tilde{S}_1}{n + \tilde{S}_1^2} - e\tilde{S}_1 - d_2.$$

Then the characteristic equation of J_4 is given by:

$$(u_{22} - \lambda)[\lambda^2 + (u_{11} + u_{33})\lambda + (u_{11})(u_{33}) - (u_{13})(u_{31})] = 0,$$

So, either $(u_{22} - \lambda) = 0,$

which gives:

$\lambda_{4S_2} = \alpha_2 - K_2\tilde{S}_3 - d_1,$

Or,$[\lambda^2 + (u_{11} + u_{33})\lambda + (u_{11})(u_{33}) - (u_{13})(u_{31})] = 0,$

which gives:

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$$\lambda_{4S_1}+\lambda_{4S_3}=\frac{h_1\alpha_1-2\tilde{S}_1(\alpha_1+h_1C_1\tilde{S}_3)}{h_1}-\frac{K_1\tilde{S}_3(n-\tilde{S}_1^2)}{(n+\tilde{S}_1^2)^2}+(\frac{R_1}{n+\tilde{S}_1^2}-e)\tilde{S}_1-d_2.$$

$$\lambda_{5E_2}\cdot\lambda_{4E_3}=\left(\frac{h_1\alpha_1-2\tilde{S}_1(\alpha_1+h_1C_1\tilde{S}_3)}{h_1}-\frac{K_1\tilde{S}_3(n-\tilde{S}_1^2)}{(n+\tilde{S}_1^2)^2}\right)\left(\left(\frac{R_1}{(n+\tilde{S}_1^2)}-e\right)\tilde{S}_1-d_2.\right)$$
$$+\left(\tilde{S}_1\left(\frac{K_1}{(n+\tilde{S}_1^2)}+C_1\tilde{S}_1\right)\right)\left(\left(\frac{K_1(n-\tilde{S}_1^2)}{(n+\tilde{S}_1^2)^2}-e\right)\tilde{S}_3\right).$$

Hence m_4 is locally asymptotically stable provided that the following conditions hold.

$$\alpha_2<K_2\tilde{S}_3+d_1\,, \tag{2.i}$$

$$n>\tilde{S}_1^2\,, \tag{2.j}$$

$$\frac{R_1}{(n+\tilde{S}_1^2)}<e, \tag{2.K}$$

$$h_1\alpha_1<2\tilde{S}_1(\alpha_1+h_1C_1\tilde{S}_3), \tag{2.l}$$

$$\frac{K_1(n-\tilde{S}_1^2)}{(n+\tilde{S}_1^2)^2}>e. \tag{2.m}$$

Otherwise it is saddle point.

F. Local stability of m_5

The Jacobean matrix of $m_5=(0,\dot{S}_2,\dot{S}_3)$ is given by:

$$J_5=J(m_5)=[x_{ij}]_{3\times3}\,, \tag{2.n}$$

where:

$$x_{11}=\alpha_1-\frac{K_1\dot{S}_3}{n},x_{12}=0,x_{13}=0,x_{21}=0,x_{22}=\alpha_2-\frac{2\alpha_2\dot{S}_2}{h_2}-K_2\dot{S}_3-d_1,$$

$$x_{23}=-K_2\dot{S}_2,x_{31}=\frac{K_1\dot{S}_3}{n}-e\dot{S}_3,x_{32}=R_2\dot{S}_3,x_{33}=R_2\dot{S}_2-d_2.$$

Then the characteristic equation of J_5 is given by:

$(x_{11}-\lambda)[\lambda^2+(x_{22}+x_{33})\lambda+(x_{22})(x_{33})-(x_{23})(x_{32})]=0,$

So, either $(x_{11}-\lambda)=0,$

which gives:

$$\lambda_{5S_1}=\alpha_1-\frac{K_1\dot{S}_3}{n},$$

Or, $[\lambda^2+(x_{22}+x_{33})\lambda+(x_{22})(x_{33})-(x_{23})(x_{32})]=0,$

$$\lambda_{5S_2}+\lambda_{5S_3}=\frac{\alpha_2(h_2-2\dot{S}_2)}{h_2}-K_2\dot{S}_3-d_1+(R_2\dot{S}_2-d_2).$$

$$\lambda_{5S_2}\cdot\lambda_{5S_3}=\left(\frac{\alpha_2(h_2-2\dot{S}_2)}{h_2}-K_2\dot{S}_3-d_1\right)(R_2\dot{S}_2-d_2)+K_2R_2\dot{S}_3\dot{S}_2.$$

Hence m_5 is locally asymptotically stable provided that the following conditions hold.

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$$\alpha_1 < \frac{K_1\dot{S}_3}{n}, \quad (2.o)$$

$$R_2\dot{S}_2 < d_2, \quad (2.p)$$

$$h_2 < 2\dot{S}_2. \quad (2.q)$$

Otherwise it is saddle point.

G. Local stability of m_6

The Jacobean matrix at $m_6 = (\bar{\bar{S}}_1, \bar{\bar{S}}_2, \bar{\bar{S}}_3)$ is given by:

$$J_6 = J(m_6) = [q_{ij}]_{3\times 3}, \quad (2.r)$$

$$q_{11} = \frac{h_1\alpha_1 - 2\bar{\bar{S}}_1(\alpha_1 + h_1C_1\bar{\bar{S}}_3)}{h_1} - \frac{K_1\bar{\bar{S}}_3(n - \bar{\bar{S}}_1^2)}{(n + \bar{\bar{S}}_1^2)^2}, q_{12} = 0, q_{13} = -\frac{K_1\bar{\bar{S}}_1}{(n + \bar{\bar{S}}_1^2)} - C_1\bar{\bar{S}}_1^2,$$

$$q_{21} = \alpha_2 - \frac{2\alpha_2\bar{\bar{S}}_2}{h_2} - K_2\bar{\bar{S}}_3 - d_1, q_{22} = 0, q_{23} = -K_2\bar{\bar{S}}_2,$$

$$q_{31} = \left(\frac{K_1(n - \bar{\bar{S}}_1^2)}{(n + \bar{\bar{S}}_1^2)^2} - e\right)\bar{\bar{S}}_3, q_{32} = R_2\bar{\bar{S}}_3, q_{33} = \left(\frac{R_1}{n + \bar{\bar{S}}_1^2} - e\right)\bar{\bar{S}}_1 + R_2\bar{\bar{S}}_2 - d_2,$$

Then the characteristic equation of J_6 is given by:

$$[\lambda^3 + \bar{\bar{\mu}}_1\lambda^2 + \bar{\bar{\mu}}_2\lambda^1 + \bar{\bar{\mu}}_3] = 0, \quad (2.s)$$

where:

$$\bar{\bar{\mu}}_1 = -(q_{11} + q_{22} + q_{33}) > 0,$$

$$\bar{\bar{\mu}}_2 = (q_{22})(q_{11}) + (q_{11})(q_{33}) + (q_{22})(q_{33}) - (q_{23})(q_{32}) + (q_{13})(q_{31}) > 0,$$

$$\bar{\bar{\mu}}_3 = (q_{22})(q_{31})(q_{13}) + (q_{11})(q_{32})(q_{23}) - (q_{11})(q_{22})(q_{33}) > 0,$$

By the Routh-Hawirtiz criterion, equation (2.s) has real negative parts, if $\bar{\bar{\mu}}_i > 0$, i = 1, 3 and $\Delta = (\bar{\bar{\mu}}_1\bar{\bar{\mu}}_2 - \bar{\bar{\mu}}_3)\bar{\bar{\mu}}_3 > 0$.

Clearly, $\bar{\bar{\mu}}_i > 0$ if the following conditions hold:

$$n > \bar{\bar{S}}_1^2, \quad (2.t)$$

$$h_1\alpha_1 < 2\bar{\bar{S}}_1(\alpha_1 + h_1C_1\bar{\bar{S}}_3), \quad (2.u)$$

$$h_2 < 2\bar{\bar{S}}_2, \quad (2.v)$$

$$\frac{R_1}{n + \bar{\bar{S}}_1^2} < e, \quad (2.w)$$

$$\left(\frac{R_1}{n + \bar{\bar{S}}_1^2} - e\right)\bar{\bar{S}}_1 + d_2 > R_2\bar{\bar{S}}_2, \quad (2.x)$$

$$\frac{K_1(n - \bar{\bar{S}}_1^2)}{(n + \bar{\bar{S}}_1^2)^2} > e. \quad (2.y)$$

Straightforward computation shows that: $\bar{\bar{\Delta}} = E_1 - E_2$, where:

$$E_1 = -(q_{11} + q_{22} + q_{33})[q_{22}(q_{31}q_{11}q_{13}(q_{22} + 2q_{33}) - q_{11}^2q_{33}(q_{22} + q_{33}) + q_{31}q_{31}) + (q_{11})(q_{23})(q_{32})(q_{11}(q_{22} + q_{33}) - q_{32}q_{23} + q_{33}q_{22} - q_{31}q_{13})],$$

and

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$$E_2 = (q_{22})(q_{11})\big(q_{23}q_{32}(2q_{31}q_{13} - 2q_{11}q_{22}) + q_{33}q_{22}(q_{33}q_{11} - 2q_{31}q_{13})\big) + q_{13}^2 q_{31}^2 q_{22}^2 + q_{11}^2 q_{23}^2 q_{32}^2,$$

Hence, Δ will be positive if in addition of conditions and [2.t-2.y]
the following condition holds:

$$E_1 > E_2\,, \tag{2.z}$$

So, m_6 is locally asymptotically stable. It's unstable otherwise.

5. Global Stability Analysis

In this section, the global stability of the equilibrium points of system (1) is analyzed using the Lyapunov function as shown.

Theorem 2: The (EP) m_1 is a globally asymptotically stable in the sub region $\Omega_1 \subset \mathrm{R}_+^3$ that satisfies the next condition:

$$\frac{K_1S_3S_1}{(n+S_1^2)} + \frac{\alpha_2h_2}{4} < \left[\left(\frac{\alpha_1}{h_1} + C_1S_3\right)(S_1-h_1)^2 + d_1S_2 + d_2S_3\right], \tag{3.a}$$

Proof: Consider the following function*:*

$$A_1(S_1,S_2,S_3) = \left(S_1 - h_1 - h_1\,ln\frac{S_1}{h_1}\right) + S_2 + S_3.$$

Clearly A_1: $\mathrm{R}_+^3 \to \mathrm{R}$ is a $A_1 \in \mathrm{C}^1$ positive definite function. Now, by differentiating A_1 with respect to time t and use the equations in the system we get:

$$\frac{dA_1}{dt} = -\left(\frac{\alpha_1}{h_1} + C_1S_3\right)(S_1-h_1)^2 + \alpha_2S_2\left(1-\frac{S_2}{h_2}\right) + \frac{K_1S_3h_1}{(n+S_1^2)} - d_1S_2 - d_2S_3 - S_2S_3(K_2-R_2) - \frac{S_1S_3}{(n+S_1^2)}(K_1-R_1).$$

Now since the function $f(S_2) = \alpha_2S_2\left(1-\frac{S_2}{h_2}\right)$ in the second term represents a logistic function with respect to S_2 and hence it is bounded above by the constant $\frac{\alpha_2 h_2}{4}$, then according to the biological facts,$K_i > R_i$, $i = 1{,}2$. Hence,

$$\frac{dA_1}{dt} < \frac{K_1S_3h_1}{(n+S_1^2)} + \frac{\alpha_2h_2}{4} - \left[\left(\frac{\alpha_1}{h_1} + C_1S_3\right)(S_1-h_1)^2 + d_1S_2 + d_2S_3\right].$$

So, $\frac{dA_1}{dt} < 0$ according to condition (3.a). Thus m_1 is a globally asymptotically stable.

Theorem 3: The (EP) m_3 is a globally asymptotically stable in the sub region $\Omega_2 \subset \mathrm{R}_+^3$ that satisfies the next condition:

$$\frac{K_1S_3h_1}{(n+S_1^2)} + \frac{K_2h_2(\alpha_2-d_1)}{\alpha_2}S_3 < \left[\left(\frac{\alpha_1}{h_1} + C_1S_3\right)(S_1-h_1)^2 + \frac{\alpha_1}{h_1}\left(S_2 - \frac{h_2(\alpha_2-d_1)}{\alpha_2}\right)^2\right], \tag{3.b}$$

Proof: Consider the following function*:*

$$A_2(S_1,S_2,S_3) = \left(S_1 - h_1 - h_1\,ln\frac{S_1}{h_1}\right) + \left(S_2 - \frac{h_2(\alpha_2-d_1)}{\alpha_2} - \frac{h_2(\alpha_2-d_1)}{\alpha_2}\,ln\frac{S_2}{\frac{h_2(\alpha_2-d_1)}{\alpha_2}}\right) + S_3.$$

Clearly A_2: $\mathrm{R}_+^3 \to \mathrm{R}$ is a $A_2 \in \mathrm{C}^1$ positive definite function. Now, by differentiating A_2 with respect to time t and use the equations in the system we get:

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$$\frac{dA_2}{dt} = -\left(\frac{\alpha_1}{h_1} + C_1S_3\right)(S_1 - h_1)^2 - \frac{\alpha_1}{h_1}\left(S_2 - \frac{h_2(\alpha_2 - d_1)}{\alpha_2}\right)^2 + \frac{K_1S_3h_1}{(n + S_1^2)} - \frac{K_2h_2(\alpha_2 - d_1)}{\alpha_2}S_3 - S_2S_3(K_2 - R_2) - \frac{S_1S_3}{(n + S_1^2)}(K_1 - R_1).$$

So, according to the biological facts in theorem (1), always $K_1 > R_i, i = 1,2$.

$$\frac{dA_2}{dt} = \frac{K_1S_3h_1}{(n + S_1^2)} + \frac{K_2h_2(\alpha_2 - d_1)}{\alpha_2}S_3 - \left[\left(\frac{\alpha_1}{h_1} + C_1S_3\right)(S_1 - h_1)^2 + \frac{\alpha_1}{h_1}\left(S_2 - \frac{h_2(\alpha_2 - d_1)}{\alpha_2}\right)^2\right].$$

So, $\frac{dA_2}{dt} < 0$ according to conditions (3.b) and (1.a). Thus m_3 is a globally asymptotically stable.

Theorem 4: The (EP) m_4 is a globally asymptotically stable in the sub region $\Omega_3 \subset R_+^3$ that satisfies the next conditions:

$$\widetilde{w}_2 < \widetilde{w}_1, \qquad (3.c)$$

$$\tilde{S}_1\tilde{S}_3 < S_1S_3, \qquad (3.d)$$

$$\tilde{S}_i < S_i, \qquad (3.e)$$

$i = 1,3.$

Where:

$$\widetilde{w}_1 = \frac{\alpha_1}{h_1}\left(S_1 - \tilde{S}_1\right)^2 + \frac{R_1(\tilde{S}_1\tilde{S}_3 + S_1S_3)}{(n+S_1^2)(n+\tilde{S}_1^2)} + e\left(S_1 - \tilde{S}_1\right)\left(S_3 - \tilde{S}_3\right) + C_1\left(S_1S_3 - \tilde{S}_1\tilde{S}_3\right)\left(S_1 - \tilde{S}_1\right),$$

$$\widetilde{w}_2 = \frac{\alpha_2 h_2}{4} + \frac{K_1(\tilde{S}_1\tilde{S}_3 + S_1S_3)}{(n+S_1^2)(n+\tilde{S}_1^2)}.$$

Proof: Consider the following function:

$$A_3(S_1, S_2, S_3) = \left(S_1 - \tilde{S}_1 - \tilde{S}_1 \ln\frac{S_1}{\tilde{S}_1}\right) + S_2 + \left(S_3 - \tilde{S}_3 - \tilde{S}_3 \ln\frac{S_3}{\tilde{S}_3}\right).$$

Clearly $A_3\colon R_+^3 \to R$ is a $A_3 \in C^1$ positive definite function. Now, by differentiating A_3 with respect to time t and use the equations in the system we get:

$$\frac{dA_3}{dt} = -\frac{\alpha_1}{h_1}\left(S_1 - \tilde{S}_1\right)^2 + \alpha_2S_2\left(1 - \frac{S_2}{h_2}\right) - C_1\left(S_1S_3 - \tilde{S}_1\tilde{S}_3\right)\left(S_1 - \tilde{S}_1\right) - e\left(S_1 - \tilde{S}_1\right)\left(S_3 - \tilde{S}_3\right) + \frac{K_1S_3\tilde{S}_1}{(n+S_1^2)} + \frac{K_1\tilde{S}_3\tilde{S}_1}{(n+\tilde{S}_1^2)} - \frac{R_1S_1\tilde{S}_3}{(n+S_1^2)} - \frac{R_1S_3\tilde{S}_1}{(n+\tilde{S}_1^2)} - S_2S_3(K_2 - R_2) - \left(\frac{S_1S_3}{(n+S_1^2)} + \frac{\tilde{S}_3\tilde{S}_1}{(n+\tilde{S}_1^2)}\right)(K_1 - R_1).$$

Now since the function $f(S_2) = \alpha_2S_2\left(1 - \frac{S_2}{h_2}\right)$ in the second term represents a logistic function with respect to S_2 and hence it is bounded above by the constant $\frac{\alpha_2 h_2}{4}$, then according to the biological facts, $K_i > R_i$, $i = 1,2$. Hence,

$$\frac{dA_3}{dt} <= -\frac{\alpha_1}{h_1}\left(S_1 - \tilde{S}_1\right)^2 - C_1\left(S_1S_3 - \tilde{S}_1\tilde{S}_3\right)\left(S_1 - \tilde{S}_1\right) - + \frac{R_1(\tilde{S}_1\tilde{S}_3 + S_1S_3)}{(n + S_1^2)(n + \tilde{S}_1^2)} - e\left(S_1 - \tilde{S}_1\right)\left(S_3 - \tilde{S}_3\right) + \frac{\alpha_2 h_2}{4} + \frac{K_1(\tilde{S}_1\tilde{S}_3 + S_1S_3)}{(n + S_1^2)(n + \tilde{S}_1^2)},$$

Then $\frac{dA_3}{dt} = -\widetilde{w}_1 + \widetilde{w}_2$.

So, $\frac{dA_3}{dt} < 0$ according to conditions (3.c-3.e) .Thus m_4 is a globally asymptotically stable.

Theorem 5: The (EP) m_5 is a globally asymptotically stable in the sub region $\Omega_4 \subset R_+^3$ that satisfies the next condition:

$$\frac{\alpha_1 h_1}{4} + K_1\left(S_2\dot{S}_3 + S_3\dot{S}_2\right) < \left[\frac{\alpha_2}{h_2}\left(S_2 - \dot{S}_2\right)^2 + \frac{R_1S_1\dot{S}_3}{(n+S_1^2)}\right], \qquad (3.f)$$

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Proof: Consider the following function:

$$A_4(S_1,S_2,S_3) = S_1 + \left(S_2 - \dot{S}_2 - \dot{S}_2 \ln\frac{S_2}{\dot{S}_2}\right) + \left(S_3 - \dot{S}_3 - \dot{S}_3 \ln\frac{S_3}{\dot{S}_3}\right).$$

Clearly $A_4: R_+^3 \to R$ is a $A_4 \in C^1$ positive definite function. Now, by differentiating A_4 with respect to time t and use the equations in the system we get:

$$\frac{dA_4}{dt} = -\frac{\alpha_2}{h_2}(S_2-\dot{S}_2)^2 + \alpha_1 S_1\left(1-\frac{S_1}{h_1}\right) + K_1(S_2\dot{S}_3 + S_3\dot{S}_2) + \frac{R_1S_1\dot{S}_3}{(n+S_1^2)} - \frac{S_1S_3}{(n+S_1^2)}(K_1-R_1) - (S_2S_3 + \dot{S}_2\dot{S}_3)(K_2-R_2),$$

Now since the function $f(S_1) = \alpha_1 S_1\left(1-\frac{S_1}{h_1}\right)$ in the second term represents a logistic function with respect to S_1 and hence it is bounded above by the constant $\frac{\alpha_1 h_1}{4}$, then according to the biological facts,$K_i > R_i$, $i = 1,2$. Hence,

$$\frac{dA_4}{dt} < -\frac{\alpha_2}{h_2}(S_2-\dot{S}_2)^2 + \frac{\alpha_1 h_1}{4} + K_1(S_2\dot{S}_3 + S_3\dot{S}_2) - \frac{R_1S_1\dot{S}_3}{(n+S_1^2)}.$$

So, $\frac{dA_4}{dt} < 0$ according to condition (3.f) .Thus m_5 is a globally asymptotically stable.

Theorem 6: The (EP) m_6 is a globally asymptotically stable in the sub region $\Omega_5 \subset R_+^3$ that satisfies the next condition:

$$\bar{\bar{N}}_2 < \bar{\bar{N}}_1\,, \qquad (3.g)$$

$$\bar{\bar{S}}_1\bar{\bar{S}}_3 < S_1S_3, \qquad (3.h)$$

$$\bar{\bar{S}}_i < S_i\,, \qquad (3.i)$$

$i = 1,3.$

Where:

$$\bar{\bar{N}}_1 = \frac{\alpha_1}{h_1}(S_1-\bar{\bar{S}}_1)^2 + \frac{\alpha_2}{h_2}(S_2-\bar{\bar{S}}_2)^2 + e(S_1-\bar{\bar{S}}_1)(S_3-\bar{\bar{S}}_3) + C_1(S_1S_3 - \bar{\bar{S}}_1\bar{\bar{S}}_3)(S_1-\bar{\bar{S}}_1),$$

$$\bar{\bar{N}}_2 = K_1\left(\frac{S_3\bar{\bar{S}}_1}{(n+S_1^2)} + \frac{\bar{\bar{S}}_3S_1}{(n+\bar{\bar{S}}_1^2)}\right) + K_2(S_2\bar{\bar{S}}_3 + \bar{\bar{S}}_2S_3).$$

Proof: Consider the following function:

$$A_5(S_1,S_2,S_3) = \left(S_1 - \bar{\bar{S}}_1 - \bar{\bar{S}}_1 \ln\frac{S_1}{\bar{\bar{S}}_1}\right) + \left(S_2 - \bar{\bar{S}}_2 - \bar{\bar{S}}_2 \ln\frac{S_2}{\bar{\bar{S}}_2}\right) + \left(S_3 - \bar{\bar{S}}_3 - \bar{\bar{S}}_3 \ln\frac{S_3}{\bar{\bar{S}}_3}\right).$$

Clearly $A_5: R_+^3 \to R$ is a $A_5 \in C^1$ positive definite function. Now, by differentiating A_5 with respect to time t and use the equations in the system we get:

$$\frac{dA_5}{dt} = -\frac{\alpha_1}{h_1}(S_1-\bar{\bar{S}}_1)^2 - \frac{\alpha_2}{h_2}(S_2-\bar{\bar{S}}_2)^2 - e(S_1-\bar{\bar{S}}_1)(S_3-\bar{\bar{S}}_3) - C_1(S_1S_3 - \bar{\bar{S}}_1\bar{\bar{S}}_3)(S_1-\bar{\bar{S}}_1) - \frac{S_1S_3}{(n+S_1^2)}(K_1-R_1) - (S_2S_3 + \dot{S}_2\dot{S}_3)(K_2-R_2) + \frac{S_3\bar{\bar{S}}_1K_1}{(n+S_1^2)} + \frac{\bar{\bar{S}}_3S_1K_1}{(n+\bar{\bar{S}}_1^2)} + K_2(S_2\bar{\bar{S}}_3 + \bar{\bar{S}}_2S_3),$$

So, according to the biological facts, $K_i > R_i, i = 1,2$.

$$\frac{dA_5}{dt} < -\frac{\alpha_1}{h_1}(S_1-\bar{\bar{S}}_1)^2 - \frac{\alpha_2}{h_2}(S_2-\bar{\bar{S}}_2)^2 - e(S_1-\bar{\bar{S}}_1)(S_3-\bar{\bar{S}}_3) - C_1(S_1S_3 - \bar{\bar{S}}_1\bar{\bar{S}}_3)(S_1-\bar{\bar{S}}_1) + K_1\left(\frac{S_3\bar{\bar{S}}_1}{(n+S_1^2)} + \frac{\bar{\bar{S}}_3S_1}{(n+\bar{\bar{S}}_1^2)}\right) + K_2(S_2\bar{\bar{S}}_3 + \bar{\bar{S}}_2S_3),$$

Then $\frac{dA_5}{dt} = -\bar{\bar{N}}_1 + \bar{\bar{N}}_2$

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So, $\frac{dA_4}{dt} < 0$ according to conditions (3.g-3.i) .Thus m_6 is a globally asymptotically stable.

6. Numerical Simulation

In this section, we study the numerical simulation of the dynamic behavior of the proposed model (1) with the help of the program (MATLAB). To confirm the analytical results, calculations can be performed for a different set of parameters with different initial points. This gives us a good explanation of the effect of changing the values of system parameters.

$\alpha_i = 0.5, i = 1,2,\ h_i = 0.5, i = 1,2, C_1 = 0.1, n = 0.5, e = 0.1,$

$k_i = 0.5, i = 1,2, R_i = 0.3, i = ,1,2, d_i = 0.1, i = 1,2.$ (1.6)

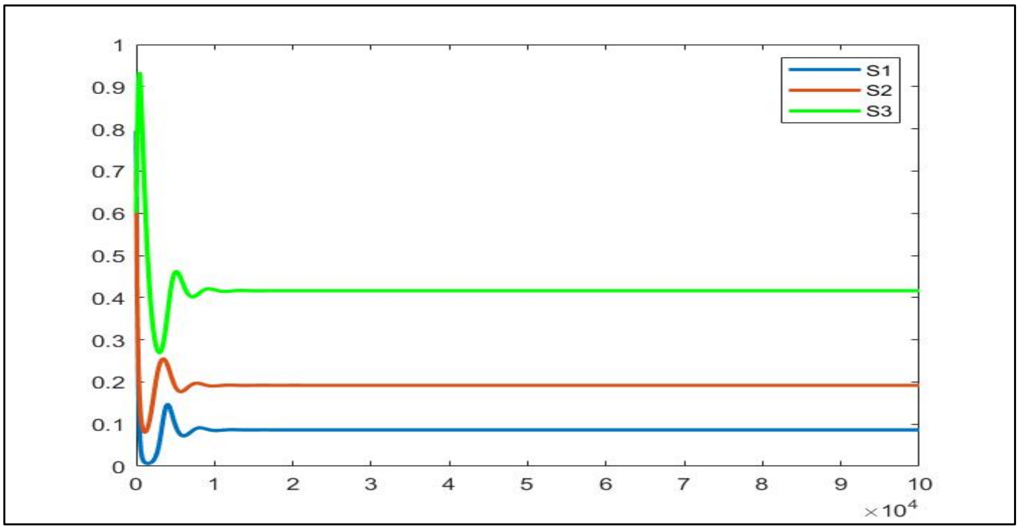

Figure 1: shows a graphical representation of the dynamics of the three species with the data provided by Eq. (1.6) for system (1), which is approximated to a positive equilibrium point $m_6 = (1.32,0.24,0.12)$.

Now, in order to discuss the effect of parameters on the dynamic behavior of system (1), the system was solved numerically with the data in equation (9), changing one parameter at a time, and the following results were obtained.The effect of the following parameters is summarized in Table 1.

Table 1:

Range of parameter	The stable point	Range of parameter	The stable point
$0.1 \le n < 0.2$ $0.2 \le n \le 1.5$	m_5 m_6	$0.3 \le K_1 \le 0.11$ $0.11 \le K_1 \le 0.18$ $0.18 \le K_1 < 2$	m_6 m_4 m_5
$0.1 \le C_1 \le 1.5$	m_6	$0.1 \le e < 1.5$	m_5
$0.1 \le d_1 < 0.4$ $0.4 \le d_1 \le 1$	m_6 m_4	$0.5 \le \alpha_1 < 0.26$ $0.26 \le \alpha_1 < 1$ $1 \le \alpha_1 \le 2$	m_4 m_5 m_6
$0.5 \le \alpha_2 < 0.10$	m_6	$0.3 \le K_2 \le 0.14$	m_1

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$0.10 \leq \alpha_2 < 1$ $1 \leq \alpha_2 < 2$	m_5 m_4	$0.14 \leq K_2 \leq 0.20$ $0.20 \leq K_2 \leq 2$	m_4 m_5
$0.1 \leq R_i < 0.4$ $, i = 1,2$	m_6	$0.1 \leq d_2 < 0.3$ $0.3 \leq d_2 \leq 1$	m_6 m_3

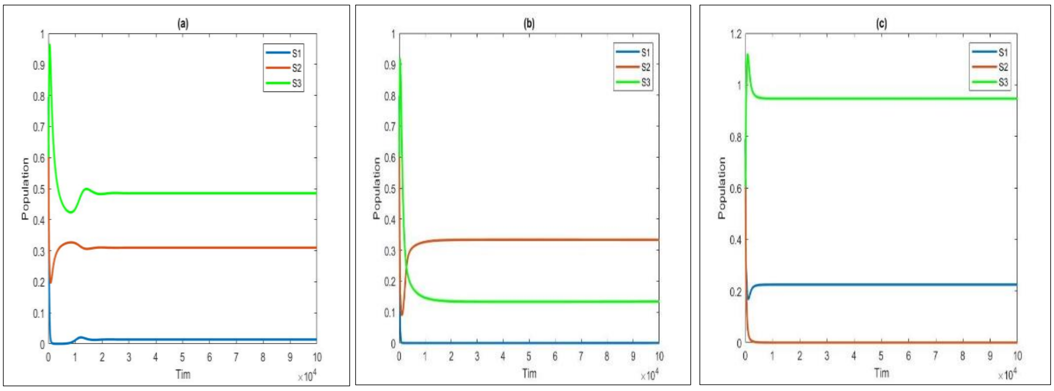

Figure 2: shows a graphical representation of the (α_2) parameter. When the parameter is changed in the range $0.5 \leq \alpha_2 < 0.10$, the solution approaches the positive equilibrium point m_6 as shown in the Fig. (a), and when there is a further increase in the range $0.10 \leq \alpha_2 < 1$, the solution approaches the equilibrium point m_5 (which shows the death of the first prey) as in the Fig.(b) but in $1 \leq \alpha_2 < 2$ the solution approaches the equilibrium point m_4 (which shows the death of the second prey) as shown in the Fig .(c)

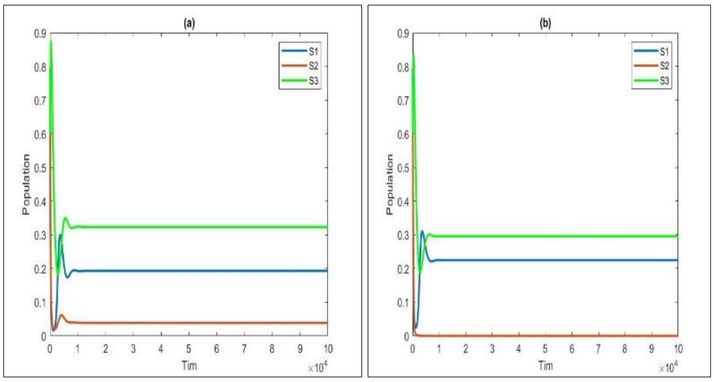

Figure 3: shows a graphical representation of the (d_2) parameter. When the parameter is changed in the range $0.1 \leq d_2 < 0.3$, the solution approaches the positive equilibrium point m_6 as shown in the Fig. (a), and when there is a further increase in the range $0.3 \leq d_2 < 1$, the solution approaches the equilibrium point m_3 (which shows the death of the predator) as in the Fig. (b).

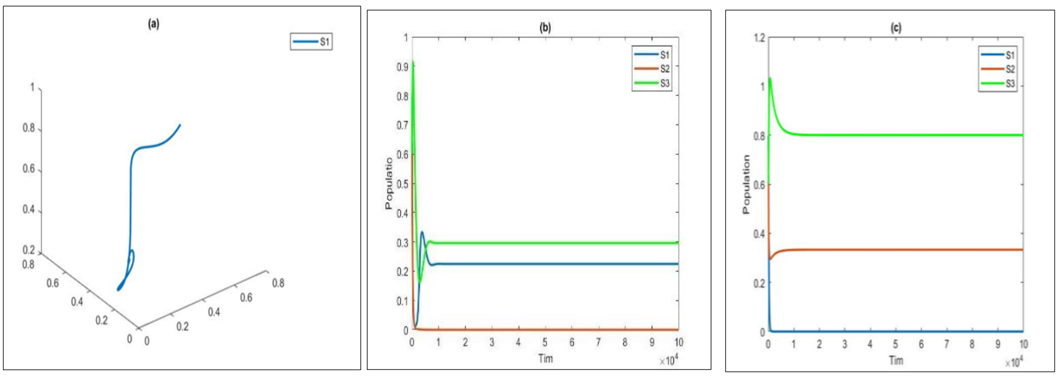

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Figure 4: shows a graphical representation of the (K_2) parameter. When the parameter is changed in the range $0.3 \leq K_2 < 0.14$, the solution approaches the equilibrium point m_1 (which represents the death of the second prey and predator) as shown in the Fig. (a), and when there is a further increase in the range $0.14 \leq K_2 < 0.20$, the solution approaches the equilibrium point m_4 (which shows the death of the second prey) as in the Fig.(b) but in $0.20 \leq K_2 \leq 2$ the solution approaches the equilibrium point m_5 (which shows the death of the first prey) as shown in the Fig .(c).

The Neutrosophic Model

To generalize our work to deal with neutrosophic numbers, we substitute every real number (a) with a neutrosophic real number $a + bI$. All equations and functions still have the same solutions but with neutrosophic real variables.

Table 2:

Range of parameter	The stable point	Range of parameter	The stable point
$0.1 \leq n + mI < 0.2$ $0.2 \leq n + mI \leq 1.5$	m_5 m_6	$0.3 \leq K_1 + K_3I \leq 0.11$ $0.11 \leq K_1 + K_3I \leq 0.18$ $0.18 \leq K_1 + K_3I < 2$	m_6 m_4 m_5
$0.1 \leq C_1 + C_2I \leq 1.5$	m_6	$0.1 \leq e < 1.5$	m_5
$0.1 \leq d_1 + d_2I < 0.4$ $0.4 \leq d_1 + d_2I \leq 1$	m_6 m_4	$0.5 \leq \alpha_1 + \alpha_4I < 0.26$ $0.26 \leq \alpha_1 + \alpha_4I < 1$ $1 \leq \alpha_1 + \alpha_4I \leq 2$	m_4 m_5 m_6
$0.5 \leq \alpha_2 + \alpha_3I < 0.10$ $0.10 \leq \alpha_2 + \alpha_3I < 1$ $1 \leq \alpha_2 + \alpha_3I < 2$	m_6 m_5 m_4	$0.3 \leq K_2 + K_4I \leq 0.14$ $0.14 \leq K_2 + K_4I_2 \leq 0.20$ $0.20 \leq K_2 + K_4I_2 \leq 2$	m_1 m_4 m_5
$0.1 \leq R_i + R_iI < 0.4$ $, i = 1,2$	m_6	$0.1 \leq d_2 + d_3I < 0.3$ $0.3 \leq d_2 + d_3I \leq 1$	m_6 m_3

7. Conclusions and Discussions

In this study, a neutrosophic mathematical model was proposed consisting of three nonlinear ordinary differential equations describing the interaction between two prey and a predator with the use of function response Holling's type IV and Lotka Volttra. The first prey appears to have a way of defending itself by directing the toxic substance directly at the predator, as well as the effect of the predator on the toxic substance. The conditions for the existence of the solution and the unity of the boundaries were discussed, and then the various equilibrium points and the stability of the system around the equilibrium points were analyzed. The Lyapunov function was used to study the global dynamics of this proposed model. It is solved numerically using the set of default parameter values given in Eq. (1.6) Thus, for system (1), we obtain the following results

- System (1) does not have periodic dynamics.
- Found that the parameters $(\alpha_2, K_2, K_1, \alpha_1, n, d_2, d_1)$ play an essential role in the system by changing these parameters, a fundamental change occurred in the behavior of the solution while the other parameters (R_1, R_2, C_1) have no effect on the behavior of the solution for system (1), which is still close to the positive equilibrium point $m_6 = \left(\bar{\bar{S}}_1, \bar{\bar{S}}_2, \bar{\bar{S}}_3\right)$, as the anti-predator rate (e) increases while keeping the other parameters as in equation (1.6), the solution of the system approaches a point $m_5 = \left(0, \dot{S}_2, \dot{S}_3\right)$.
- A Table of the neutrosophic results of our model is written.

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