RETRACTED ARTICLE: On the Characterization of Some m-Plithogenic Vector Spaces and Their AH-Substructures Under the Condition $6 \leq dim\, SP_V \leq 10$

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Abstract

This paper is concerned with studying symbolic m-plithogenic vector spaces with finite orders between 6 and 10, where it defines and characterizes the AH-subspaces, AH-kernels, and AH- linear transformations in five different symbolic m-plithogenic spaces (6-plithogenic, 7-plithogenic,10-plithogenic vector spaces). Also, we prove many theorems that describe the computation of the kernels and direct images of the plithogenic AH-linear transformations.

Keywords: 6-plithogenic vector space; 7-plithogenic AH-kernel; 8-plithogenic AH-image; 10-plithgenic space.

1. Introduction

The study of plithogenic vector spaces is a new generalization of the concept of vector spaces, as they differ from classical vector spaces in that they are not vector spaces but modules over rings [1-3,19]. The generalization of vector spaces and the study of the properties associated with it is a broad algebraic field in which many deep results and concepts play an important role in the rest of the branches of algebra [4-12].

AH-substructures were defined in [20] as an applied study on neutrosophic and refined neutrosophic vector spaces [21], where these substructures were studied, and several results related to them were proved [22]. Many studies have also been applied to other types of algebraic structures such as matrices and integers [13-16, 24-28].

The development of plithogenic spaces of various orders starting at order 2 and ending at 5 was carried out by many researchers [3-6], where the structures of the type (AH) related to this type of spaces were studied in addition to many algebraic dependencies represented by them [23].

Proceeding from this importance, we expanded the study and generalized the results to include five different types of plithogenic spaces, which are 6-plithogenic, 7-plithogenic, 8-plithogenic, 9-plithogenic, and 10-plithogenic vector spaces.

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2. Main Results

Definition:
Let V a vector space over the field F, and let $6-SP_F$ be the corresponding symbolic 6-plithogenic field defined as follows:
$6-SP_F=\{d_0+\sum_{i=1}^{6} d_iP_i\,;a_i\in F\}$, we define the symbolic 6-plithogenic vector space as follows:
$$6-SP_V=\left\{q_0+\sum_{i=1}^{6} q_iP_i\,;t_i\in V\right\}$$
Definition.
The addition on $6-SP_V$ is defined as follows:
$(d_0+\sum_{i=1}^{6} d_iP_i)+(c_0+\sum_{i=1}^{6} c_iP_i)=(d_0+c_0)+\sum_{i=1}^{6}(d_i+c_i)P_i\,;d_j,c_j\in V.$
multiplication on $6-SP_V$ is defined as follows:
$(d_0+\sum_{i=1}^{6} d_iP_i)\times(c_0+\sum_{i=1}^{6} c_iP_i)=d_0c_0+\sum_{i,j=1}^{6} d_ic_iP_{max(i,j)}$, where $a_i\in F,t_i\in V$.
Theorem
$(6-SP_V,+,.)$ Is module over $6-SP_F$.
Proof :
Let $X=x_0+\sum_{i=1}^{6} x_iP_i\,,Y=y_0+\sum_{i=1}^{6} y_iP_i\,,Z=z_0+\sum_{i=1}^{6} z_iP_i\in 6-SP_V$.
$$X+Y=(x_0+y_0)+\sum_{i=1}^{6}(x_i+y_i)P_i=(y_0+x_0)+\sum_{i=1}^{6}(y_i+x_i)P_i=Y+X$$
$X.1=X,-X=-x_0+\sum_{i=1}^{6}(-x_i)P_i$ such that $X+(-X)=O$.
$X+O=O+X=X$.
Let $A=a_0+\sum_{i=1}^{6} a_iP_i\,,B=b_0+\sum_{i=1}^{6} b_iP_i\in 5-SP_R$, then:
$$(A+B).X=\left[(a_0+b_0)+\sum_{i=1}^{6}(a_i+b_i)\,P_i\right]\left(x_0+\sum_{i=1}^{6} x_iP_i\right)=(a_0+b_0)x_0+\sum_{i,j=1}^{6}(a_i+b_i)x_j\,P_{max(i,j)}$$
$$=AX+BX$$
$$A.(X+Y)=\left(a_0+\sum_{i=1}^{6} a_iP_i\right)\left[(x_0+y_0)+\sum_{i=1}^{6}(x_i+y_i)\,P_i\right]=a_0(x_0+y_0)+\sum_{i,j=1}^{6} a_i\big(x_j+y_j\big)\,P_{max(i,j)}$$
$$=A.X+A.Y$$
$$(A.B)X=\left(a_0b_0+\sum_{i,j=1}^{6} a_ib_j\,P_{max(i,j)}\right)\left(x_0+\sum_{i=1}^{6} x_iP_i\right)=a_0b_0x_0+\sum_{i,j,k=1}^{6} a_ib_j\,x_kP_{max(i,j)}=A(B.X)$$
Definition.
Let $w_j;0\le j\le 6$ be subspaces of V, then:
$W=\{x_0+\sum_{i=1}^{6} x_iP_i\,;x_i\in w_i\}$ is called symbolic 6-plithogenic AH-subspace.
If $w_j=w_s$ for all $0\le j,s\le 6$, then W is called AHS-subspace.
Theorem
Let $W=w_0+\sum_{i=1}^{6} w_iP_i$ be an AHS-subspace of $6-SP_V$, hence W is submodule of $6-SP_V$.
Proof
Let $X=x_0+\sum_{i=1}^{6} x_iP_i\,,Y=y_0+\sum_{i=1}^{6} y_iP_i\in W$, and $A=a_0+\sum_{i=1}^{6} a_iP_i\in 6-SP_R$, then:
$X-Y=(x_0-y_0)+\sum_{i=1}^{6}(x_i-y_i)\,P_i;x_j-y_j\in w_j;0\le j\le 6$, hence $X-Y\in W$.
$A.X=a_0x_0+\sum_{i,j=1}^{6} a_ix_j\,P_{mx(i,j)}\in W$, that is because $a_ix_j\in w_j$.
Definition.
Let $L_j:V\to T$ be linear transformation between $V,T;0\le j\le 6$, we define the AH-linear transformation s follows:
$L=L_0+\sum_{i=1}^{6} L_iP_i:6-SP_V\to 6-SP_T$ such that:
$L(x_0+\sum_{i=1}^{6} x_iP_i)=L_0(x_0)+\sum_{i=1}^{6} L_i(x_i)P_i$.
If $L_j=L_k$ for all $0\le j,k\le 6$, then L is called AHS-linear transformation.
Definition.
Let $L=L_0+\sum_{i=1}^{6} L_iP_i:6-SP_V\to 6-SP_T$ be an AHS-linear transformation, we define:
1). $AH-ker(L)=ker(L_0)+\sum_{i=1}^{6} ker(L_i)P_i$.
2). $AH-Im(L)=Im(L_0)+\sum_{i=1}^{6} Im(L_i)P_i$
If L is an AHS-liner transformation, then we get the AHS-kernel and AHS image.
Theorem
Let L be an AHS-liner transformation such that $L:6-SP_V\to 6-SP_T$, then L is a module homomorphism.
Proof

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Let $L = l_0 + \sum_{i=1}^{6} l_i P_i$ be an AHS-linear transformation, let $X = x_0 + \sum_{i=1}^{6} x_i P_i, Y = y_0 + \sum_{i=1}^{6} y_i P_i \in 6 - SP_V, A = a_0 + \sum_{i=1}^{6} a_i P_i \in 6 - SP_R$, then:

$$L(X+Y) = L\left[(x_0+y_0) + \sum_{i=1}^{6}(x_i+y_i)\,P_i\right] = L_0(x_0+y_0) + \sum_{i=1}^{6} L_0(x_i+y_i)\,P_i$$

$$= \left[L_0(x_0) + \sum_{i=1}^{6} L_0(x_i)\,P_i\right] + \left[L_0(y_0) + \sum_{i=1}^{6} L_0(y_i)\,P_i\right] = L(X) + L(Y)$$

$$L(A.X) = L\left[(a_0x_0) + \sum_{i,j=1}^{6} a_i x_j\,P_{max(i,j)}\right] = L_0(a_0x_0) + \sum_{i=1}^{6} L_0\big(a_i x_j\big)\,P_{max(i,j)}$$

$$= a_0 L_0(x_0) + \sum_{i=1}^{6} a_i L_0\big(x_j\big)\,P_{max(i,j)} = A.L(X)$$

Theorem

Let L be an AH-liner transformation, then:

1). $AH - ker(L)$ is n AH-subspace of $6 - SP_V$.

2). $AH - Im(L)$ is n AH-subspace of $6 - SP_T$.

Proof

1). Since $ker(L_i)$ is a subspace of V, then:

$AH - ker(L) = ker(L_0) + \sum_{i=1}^{6} ker(L_i)\,P_i$ is an AH-subspace of $6 - SP_V$.

2). Since $Im(L_i)$ is a subspace of T, then:

$AH - Im(L) = Im(L_0) + \sum_{i=1}^{6} Im(L_i)\,P_i$ is an AH-subspace of $6 - SP_T$.

Definition:

Let V a vector space over the field F, and let $7 - SP_F$ be the corresponding symbolic 7-plithogenic field defined as follows:

$7 - SP_F = \{d_0 + \sum_{i=1}^{7} d_i P_i\,; a_i \in F\}$, we define the symbolic 6-plithogenic vector space as follows:

$$7 - SP_V = \left\{q_0 + \sum_{i=1}^{7} q_i P_i\,; t_i \in V\right\}$$

Definition.

The addition on $7 - SP_V$ is defined as follows:

$(d_0 + \sum_{i=1}^{7} d_i P_i) + (c_0 + \sum_{i=1}^{7} c_i P_i) = (d_0 + c_0) + \sum_{i=1}^{7}(d_i + c_i)P_i\,; d_j, c_j \in V$.

multiplication on $7 - SP_V$ is defined as follows:

$(d_0 + \sum_{i=1}^{7} d_i P_i) \times (c_0 + \sum_{i=1}^{7} c_i P_i) = d_0 c_0 + \sum_{i,j=1}^{7} d_i c_i P_{max(i,j)}$, where $a_i \in F, t_i \in V$.

Theorem

$(7 - SP_V, +, .)$ Is module over $7 - SP_F$.

Proof :

Let $X = x_0 + \sum_{i=1}^{7} x_i P_i, Y = y_0 + \sum_{i=1}^{7} y_i P_i, Z = z_0 + \sum_{i=1}^{7} z_i P_i \in 7 - SP_V$.

$$X + Y = (x_0+y_0) + \sum_{i=1}^{7}(x_i+y_i)P_i = (y_0+x_0) + \sum_{i=1}^{7}(y_i+x_i)P_i = Y + X$$

$X.1 = X, -X = -x_0 + \sum_{i=1}^{7}(-x_i)P_i$ such that $X + (-X) = O$.

$X + O = O + X = X$.

Let $A = a_0 + \sum_{i=1}^{7} a_i P_i, B = b_0 + \sum_{i=1}^{7} b_i P_i \in 7 - SP_R$, then:

$$(A+B).X = \left[(a_0+b_0) + \sum_{i=1}^{7}(a_i+b_i)\,P_i\right]\left(x_0 + \sum_{i=1}^{7} x_i P_i\right) = (a_0+b_0)x_0 + \sum_{i,j=1}^{7}(a_i+b_i)x_j\,P_{max(i,j)}$$

$$= AX + BX$$

$$A.(X+Y) = \left(a_0 + \sum_{i=1}^{7} a_i P_i\right)\left[(x_0+y_0) + \sum_{i=1}^{7}(x_i+y_i)\,P_i\right] = a_0(x_0+y_0) + \sum_{i,j=1}^{7} a_i\big(x_j+y_j\big)\,P_{max(i,j)}$$

$$= A.X + A.Y$$

$$(A.B)X = \left(a_0b_0 + \sum_{i,j=1}^{7} a_i b_j\,P_{max(i,j)}\right)\left(x_0 + \sum_{i=1}^{7} x_i P_i\right) = a_0b_0x_0 + \sum_{i,j,k=1}^{7} a_i b_j\,x_k P_{max(i,j)} = A(B.X)$$

Definition.

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Let $w_j; 0 \le j \le 7$ be subspaces of V, then:
$W = \{x_0 + \sum_{i=1}^{7} x_i P_i \,; x_i \in w_i\}$ is called symbolic 7-plithogenic AH-subspace.
If $w_j = w_s$ for all $0 \le j, s \le 7$, then W is called AHS-subspace.

Theorem

Let $W = w_0 + \sum_{i=1}^{7} w_i P_i$ be an AHS-subspace of $7 - SP_V$, hence W is submodule of $7 - SP_V$.

Proof

Let $X = x_0 + \sum_{i=1}^{7} x_i P_i \,, Y = y_0 + \sum_{i=1}^{7} y_i P_i \in W$, and $A = a_0 + \sum_{i=1}^{7} a_i P_i \in 7 - SP_R$, then:
$X - Y = (x_0 - y_0) + \sum_{i=1}^{7}(x_i - y_i)\, P_i; x_j - y_j \in w_j; 0 \le j \le 7$, hence $X - Y \in W$.
$A.X = a_0 x_0 + \sum_{i,j=1}^{7} a_i x_j \, P_{mx(i,j)} \in W$, that is because $a_i x_j \in w_j$.

Definition.

Let $L_j: V \to T$ be linear transformation between $V, T; 0 \le j \le 7$, we define the AH-linear transformation s follows:
$L = L_0 + \sum_{i=1}^{7} L_i P_i : 7 - SP_V \to 7 - SP_T$ such that:
$L(x_0 + \sum_{i=1}^{7} x_i P_i) = L_0(x_0) + \sum_{i=1}^{7} L_i(x_i) P_i$.
If $L_j = L_k$ for all $0 \le j, k \le 7$, then L is called AHS-linear transformation.

Definition.

Let $L = L_0 + \sum_{i=1}^{7} L_i P_i : 7 - SP_V \to 7 - SP_T$ be an AHS-linear transformation, we define:
1). $AH - ker(L) = ker(L_0) + \sum_{i=1}^{7} ker(L_i) P_i$.
2). $AH - Im(L) = Im(L_0) + \sum_{i=1}^{7} Im(L_i) P_i$
If L is an AHS-liner transformation, then we get the AHS-kernel and AHS image.

Theorem

Let L be an AHS-liner transformation such that $L: 7 - SP_V \to 7 - SP_T$, then L is a module homomorphism.

Proof

Let $L = l_0 + \sum_{i=1}^{7} l_i P_i$ be an AHS-linear transformation, let $X = x_0 + \sum_{i=1}^{7} x_i P_i \,, Y = y_0 + \sum_{i=1}^{7} y_i P_i \in 7 - SP_V, A = a_0 + \sum_{i=1}^{6} a_i P_i \in 7 - SP_R$, then:

$$L(X + Y) = L\left[(x_0 + y_0) + \sum_{i=1}^{7}(x_i + y_i)\, P_i\right] = L_0(x_0 + y_0) + \sum_{i=1}^{7} L_0(x_i + y_i)\, P_i$$
$$= \left[L_0(x_0) + \sum_{i=1}^{7} L_0(x_i)\, P_i\right] + \left[L_0(y_0) + \sum_{i=1}^{7} L_0(y_i)\, P_i\right] = L(X) + L(Y)$$
$$L(A.X) = L\left[(a_0 x_0) + \sum_{i,j=1}^{7} a_i x_j \, P_{max(i,j)}\right] = L_0(a_0 x_0) + \sum_{i=1}^{7} L_0(a_i x_j)\, P_{max(i,j)}$$
$$= a_0 L_0(x_0) + \sum_{i=1}^{7} a_i L_0(x_j)\, P_{max(i,j)} = A.L(X)$$

Theorem

Let L be an AH-liner transformation, then:
1). $AH - ker(L)$ is n AH-subspace of $7 - SP_V$.
2). $AH - Im(L)$ is n AH-subspace of $7 - SP_T$.

Proof

1). Since $ker(L_i)$ is a subspace of V, then:
$AH - ker(L) = ker(L_0) + \sum_{i=1}^{7} ker(L_i)\, P_i$ is an AH-subspace of $7 - SP_V$.
2). Since $Im(L_i)$ is a subspace of T, then:
$AH - Im(L) = Im(L_0) + \sum_{i=1}^{6} Im(L_i)\, P_i$ is an AH-subspace of $7 - SP_T$.

Definition:

Let V a vector space over the field F, and let $8 - SP_F$ be the corresponding symbolic 8-plithogenic field defined as follows:
$8 - SP_F = \{d_0 + \sum_{i=1}^{8} d_i P_i \,; a_i \in F\}$, we define the symbolic 8-plithogenic vector space as follows:

$$8 - SP_V = \left\{q_0 + \sum_{i=1}^{8} q_i P_i \,; t_i \in V\right\}$$

Definition.

The addition on$8 - SP_V$ is defined as follows:
$(d_0 + \sum_{i=1}^{8} d_i P_i) + (c_0 + \sum_{i=1}^{8} c_i P_i) = (d_0 + c_0) + \sum_{i=1}^{8}(d_i + c_i) P_i \,; d_j, c_j \in V$.
multiplication on $8 - SP_V$ is defined as follows:

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$(d_0+\sum_{i=1}^{8} d_iP_i)\times(c_0+\sum_{i=1}^{8} c_iP_i)=d_0c_0+\sum_{i,j=1}^{8} d_ic_iP_{max(i,j)}$, where $a_i\in F, t_i\in V$.

Theorem

$(8-SP_V,+,.)$ Is module over $8-SP_F$.

Proof :

Let $X=x_0+\sum_{i=1}^{8} x_iP_i, Y=y_0+\sum_{i=1}^{8} y_iP_i, Z=z_0+\sum_{i=1}^{8} z_iP_i\in 8-SP_V$.

$$X+Y=(x_0+y_0)+\sum_{i=1}^{8}(x_i+y_i)P_i=(y_0+x_0)+\sum_{i=1}^{8}(y_i+x_i)P_i=Y+X$$

$X.1=X, -X=-x_0+\sum_{i=1}^{8}(-x_i)P_i$ such that $X+(-X)=O$.

$X+O=O+X=X$.

Let $A=a_0+\sum_{i=1}^{8} a_iP_i, B=b_0+\sum_{i=1}^{8} b_iP_i\in 8-SP_R$, then:

$$(A+B).X=\left[(a_0+b_0)+\sum_{i=1}^{8}(a_i+b_i)P_i\right]\left(x_0+\sum_{i=1}^{8}x_iP_i\right)=(a_0+b_0)x_0+\sum_{i,j=1}^{8}(a_i+b_i)x_jP_{max(i,j)}$$
$$=AX+BX$$

$$A.(X+Y)=\left(a_0+\sum_{i=1}^{8}a_iP_i\right)\left[(x_0+y_0)+\sum_{i=1}^{8}(x_i+y_i)P_i\right]=a_0(x_0+y_0)+\sum_{i,j=1}^{8}a_i(x_j+y_j)P_{max(i,j)}$$
$$=A.X+A.Y$$

$$(A.B)X=\left(a_0b_0+\sum_{i,j=1}^{8}a_ib_jP_{max(i,j)}\right)\left(x_0+\sum_{i=1}^{8}x_iP_i\right)=a_0b_0x_0+\sum_{i,j,k=1}^{8}a_ib_jx_kP_{max(i,j)}=A(B.X)$$

Definition.

Let $w_j; 0\le j\le 8$ be subspaces of V, then:

$W=\{x_0+\sum_{i=1}^{8} x_iP_i; x_i\in w_i\}$ is called symbolic 8-plithogenic AH-subspace.

If $w_j=w_s$ for all $0\le j,s\le 8$, then W is called AHS-subspace.

Theorem

Let $W=w_0+\sum_{i=1}^{8} w_iP_i$ be an AHS-subspace of $8-SP_V$, hence W is submodule of $8-SP_V$.

Proof

Let $X=x_0+\sum_{i=1}^{8} x_iP_i, Y=y_0+\sum_{i=1}^{8} y_iP_i\in W$, and $A=a_0+\sum_{i=1}^{8} a_iP_i\in 8-SP_R$, then:

$X-Y=(x_0-y_0)+\sum_{i=1}^{8}(x_i-y_i)P_i; x_j-y_j\in w_j; 0\le j\le 8$, hence $X-Y\in W$.

$A.X=a_0x_0+\sum_{i,j=1}^{8} a_ix_jP_{mx(i,j)}\in W$, that is because $a_ix_j\in w_j$.

Definition.

Let $L_j: V\to T$ be linear transformation between $V, T; 0\le j\le 8$, we define the AH-linear transformation s follows:

$L=L_0+\sum_{i=1}^{8} L_iP_i: 8-SP_V\to 8-SP_T$ such that:

$L(x_0+\sum_{i=1}^{8} x_iP_i)=L_0(x_0)+\sum_{i=1}^{8} L_i(x_i)P_i$.

If $L_j=L_k$ for all $0\le j,k\le 8$, then L is called AHS-linear transformation.

Definition.

Let $L=L_0+\sum_{i=1}^{8} L_iP_i: 8-SP_V\to 8-SP_T$ be an AHS-linear transformation, we define:

1). $AH-ker(L)=ker(L_0)+\sum_{i=1}^{8} ker(L_i)P_i$.

2). $AH-Im(L)=Im(L_0)+\sum_{i=1}^{8} Im(L_i)P_i$

If L is an AHS-liner transformation, then we get the AHS-kernel and AHS image.

Theorem

Let L be an AHS-liner transformation such that $L: 8-SP_V\to 8-SP_T$, then L is a module homomorphism.

Proof

Let $L=l_0+\sum_{i=1}^{8} l_iP_i$ be an AHS-linear transformation, let $X=x_0+\sum_{i=1}^{8} x_iP_i, Y=y_0+\sum_{i=1}^{8} y_iP_i\in 8-SP_V, A=a_0+\sum_{i=1}^{8} a_iP_i\in 8-SP_R$, then:

$$L(X+Y)=L\left[(x_0+y_0)+\sum_{i=1}^{8}(x_i+y_i)P_i\right]=L_0(x_0+y_0)+\sum_{i=1}^{8}L_0(x_i+y_i)P_i$$
$$=\left[L_0(x_0)+\sum_{i=1}^{8}L_0(x_i)P_i\right]+\left[L_0(y_0)+\sum_{i=1}^{8}L_0(y_i)P_i\right]=L(X)+L(Y)$$

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$$L(A.X) = L\left[(a_0x_0) + \sum_{i,j=1}^{8} a_ix_j\, P_{max(i,j)}\right] = L_0(a_0x_0) + \sum_{i=1}^{8} L_0\big(a_ix_j\big)\, P_{max(i,j)}$$
$$= a_0L_0(x_0) + \sum_{i=1}^{8} a_iL_0\big(x_j\big)\, P_{max(i,j)} = A.L(X)$$

Theorem

Let L be an AH-liner transformation, then:

1). $AH - ker(L)$ is n AH-subspace of $8 - SP_V$.

2). $AH - Im(L)$ is n AH-subspace of $8 - SP_T$.

Proof

1). Since $ker(L_i)$ is a subspace of V, then:

$AH - ker(L) = ker(L_0) + \sum_{i=1}^{8} ker(L_i)\, P_i$ is an AH-subspace of $8 - SP_V$.

2). Since $Im(L_i)$ is a subspace of T, then:

$AH - Im(L) = Im(L_0) + \sum_{i=1}^{8} Im(L_i)\, P_i$ is an AH-subspace of $8 - SP_T$.

Definition:

Let V a vector space over the field F, and let $9 - SP_F$ be the corresponding symbolic 9-plithogenic field defined as follows:

$9 - SP_F = \{d_0 + \sum_{i=1}^{9} d_iP_i\, ; a_i \in F\}$, we define the symbolic 8-plithogenic vector space as follows:

$$9 - SP_V = \left\{q_0 + \sum_{i=1}^{9} q_iP_i\, ; t_i \in V\right\}$$

Definition.

The addition on$9 - SP_V$ is defined as follows:

$(d_0 + \sum_{i=1}^{9} d_iP_i) + (c_0 + \sum_{i=1}^{9} c_iP_i) = (d_0 + c_0) + \sum_{i=1}^{9}(d_i + c_i)P_i\, ; d_j, c_j \in V$.

multiplication on $9 - SP_V$ is defined as follows:

$(d_0 + \sum_{i=1}^{9} d_iP_i) \times (c_0 + \sum_{i=1}^{9} c_iP_i) = d_0c_0 + \sum_{i,j=1}^{9} d_ic_iP_{max(i,j)}$, where $a_i \in F, t_i \in V$.

Theorem

$(9 - SP_V, +, .)$ Is module over $9 - SP_F$.

Proof :

Let $X = x_0 + \sum_{i=1}^{9} x_iP_i\, , Y = y_0 + \sum_{i=1}^{9} y_iP_i\, , Z = z_0 + \sum_{i=1}^{9} z_iP_i \in 9 - SP_V$.

$$X + Y = (x_0 + y_0) + \sum_{i=1}^{9}(x_i + y_i)P_i = (y_0 + x_0) + \sum_{i=1}^{9}(y_i + x_i)P_i = Y + X$$

$X.1 = X, -X = -x_0 + \sum_{i=1}^{9}(-x_i)P_i$ such that $X + (-X) = O$.

$X + O = O + X = X$.

Let $A = a_0 + \sum_{i=1}^{9} a_iP_i\, , B = b_0 + \sum_{i=1}^{9} b_iP_i \in 9 - SP_R$, then:

$$(A + B).X = \left[(a_0 + b_0) + \sum_{i=1}^{9}(a_i + b_i)\, P_i\right]\left(x_0 + \sum_{i=1}^{9} x_iP_i\right) = (a_0 + b_0)x_0 + \sum_{i,j=1}^{9}(a_i + b_i)x_j\, P_{max(i,j)}$$
$$= AX + BX$$

$$A.(X + Y) = \left(a_0 + \sum_{i=1}^{9} a_iP_i\right)\left[(x_0 + y_0) + \sum_{i=1}^{9}(x_i + y_i)\, P_i\right] = a_0(x_0 + y_0) + \sum_{i,j=1}^{9} a_i\big(x_j + y_j\big)\, P_{max(i,j)}$$
$$= A.X + A.Y$$

$$(A.B)X = \left(a_0b_0 + \sum_{i,j=1}^{9} a_ib_j\, P_{max(i,j)}\right)\left(x_0 + \sum_{i=1}^{9} x_iP_i\right) = a_0b_0x_0 + \sum_{i,j,k=1}^{9} a_ib_j\, x_kP_{max(i,j)} = A(B.X)$$

Definition.

Let $w_j; 0 \le j \le 9$ be subspaces of V, then:

$W = \{x_0 + \sum_{i=1}^{9} x_iP_i\, ; x_i \in w_i\}$ is called symbolic 9-plithogenic AH-subspace.

If $w_j = w_s$ for all $0 \le j, s \le 9$, then W is called AHS-subspace.

Theorem

Let $W = w_0 + \sum_{i=1}^{9} w_iP_i$ be an AHS-subspace of $9 - SP_V$, hence W is submodule of $9 - SP_V$.

Proof

Let $X = x_0 + \sum_{i=1}^{9} x_iP_i\, , Y = y_0 + \sum_{i=1}^{9} y_iP_i \in W$, and $A = a_0 + \sum_{i=1}^{9} a_iP_i \in 9 - SP_R$, then:

$X - Y = (x_0 - y_0) + \sum_{i=1}^{9}(x_i - y_i)\, P_i; x_j - y_j \in w_j; 0 \le j \le 9$, hence $X - Y \in W$.

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$A.X = a_0x_0 + \sum_{i,j=1}^{9} a_ix_j\, P_{mx(i,j)} \in W$, that is because $a_ix_j \in w_j$.

Definition.

Let $L_j: V \to T$ be linear transformation between $V, T; 0 \le j \le 9$, we define the AH-linear transformation s follows:

$L = L_0 + \sum_{i=1}^{9} L_iP_i : 9 - SP_V \to 9 - SP_T$ such that:

$L(x_0 + \sum_{i=1}^{9} x_iP_i) = L_0(x_0) + \sum_{i=1}^{9} L_i(x_i)P_i$.

If $L_j = L_k$ for all $0 \le j, k \le 9$, then L is called AHS-linear transformation.

Definition.

Let $L = L_0 + \sum_{i=1}^{9} L_iP_i : 9 - SP_V \to 9 - SP_T$ be an AHS-linear transformation, we define:

1). $AH - ker(L) = ker(L_0) + \sum_{i=1}^{9} ker(L_i)P_i$.

2). $AH - Im(L) = Im(L_0) + \sum_{i=1}^{9} Im(L_i)P_i$

If L is an AHS-liner transformation, then we get the AHS-kernel and AHS image.

Theorem

Let L be an AHS-liner transformation such that $L: 9 - SP_V \to 9 - SP_T$, then L is a module homomorphism.

Proof

Let $L = l_0 + \sum_{i=1}^{9} l_iP_i$ be an AHS-linear transformation, let $X = x_0 + \sum_{i=1}^{9} x_iP_i\, , Y = y_0 + \sum_{i=1}^{9} y_iP_i \in 9 - SP_V, A = a_0 + \sum_{i=1}^{9} a_iP_i \in 9 - SP_R$, then:

$$L(X+Y) = L\left[(x_0 + y_0) + \sum_{i=1}^{9}(x_i + y_i)\, P_i\right] = L_0(x_0 + y_0) + \sum_{i=1}^{9} L_0(x_i + y_i)\, P_i$$

$$= \left[L_0(x_0) + \sum_{i=1}^{9} L_0(x_i)\, P_i\right] + \left[L_0(y_0) + \sum_{i=1}^{9} L_0(y_i)\, P_i\right] = L(X) + L(Y)$$

$$L(A.X) = L\left[(a_0x_0) + \sum_{i,j=1}^{9} a_ix_j\, P_{max(i,j)}\right] = L_0(a_0x_0) + \sum_{i=1}^{9} L_0(a_ix_j)\, P_{max(i,j)}$$

$$= a_0L_0(x_0) + \sum_{i=1}^{9} a_iL_0(x_j)\, P_{max(i,j)} = A.L(X)$$

Theorem

Let L be an AH-liner transformation, then:

1). $AH - ker(L)$ is n AH-subspace of $9 - SP_V$.

2). $AH - Im(L)$ is n AH-subspace of $9 - SP_T$.

Proof

1). Since $ker(L_i)$ is a subspace of V, then:

$AH - ker(L) = ker(L_0) + \sum_{i=1}^{9} ker(L_i)\, P_i$ is an AH-subspace of $9 - SP_V$.

2). Since $Im(L_i)$ is a subspace of T, then:

$AH - Im(L) = Im(L_0) + \sum_{i=1}^{9} Im(L_i)\, P_i$ is an AH-subspace of $9 - SP_T$.

Definition:

Let V a vector space over the field F, and let $10 - SP_F$ be the corresponding symbolic 10-plithogenic field defined as follows:

$10 - SP_F = \{d_0 + \sum_{i=1}^{10} d_iP_i\, ; a_i \in F\}$, we define the symbolic 10-plithogenic vector space as follows:

$$10 - SP_V = \left\{q_0 + \sum_{i=1}^{10} q_iP_i\, ; t_i \in V\right\}$$

Definition.

The addition on$10 - SP_V$ is defined as follows:

$(d_0 + \sum_{i=1}^{10} d_iP_i) + (c_0 + \sum_{i=1}^{10} c_iP_i) = (d_0 + c_0) + \sum_{i=1}^{10}(d_i + c_i)P_i\, ; d_j, c_j \in V$.

multiplication on $10 - SP_V$ is defined as follows:

$(d_0 + \sum_{i=1}^{10} d_iP_i) \times (c_0 + \sum_{i=1}^{10} c_iP_i) = d_0c_0 + \sum_{i,j=1}^{10} d_ic_iP_{max(i,j)}$, where $a_i \in F, t_i \in V$.

Theorem

$(10 - SP_V, +, .)$ Is module over $10 - SP_F$.

Proof :

Let $X = x_0 + \sum_{i=1}^{10} x_iP_i\, , Y = y_0 + \sum_{i=1}^{10} y_iP_i\, , Z = z_0 + \sum_{i=1}^{10} z_iP_i \in 10 - SP_V$.

$$X + Y = (x_0 + y_0) + \sum_{i=1}^{10}(x_i + y_i)P_i = (y_0 + x_0) + \sum_{i=1}^{10}(y_i + x_i)P_i = Y + X$$

$X.1 = X, -X = -x_0 + \sum_{i=1}^{10}(-x_i)P_i$ such that $X + (-X) = O$.

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$X + O = O + X = X$.

Let $A = a_0 + \sum_{i=1}^{10} a_i P_i\,, B = b_0 + \sum_{i=1}^{10} b_i P_i \in 10 - SP_R$, then:

$$(A + B).X = \left[(a_0 + b_0) + \sum_{i=1}^{10}(a_i + b_i)\,P_i\right]\left(x_0 + \sum_{i=1}^{10} x_i P_i\right) = (a_0 + b_0)x_0 + \sum_{i,j=1}^{10}(a_i + b_i)x_j\,P_{max(i,j)}$$
$$= AX + BX$$
$$A.(X + Y) = \left(a_0 + \sum_{i=1}^{10} a_i P_i\right)\left[(x_0 + y_0) + \sum_{i=1}^{10}(x_i + y_i)\,P_i\right] = a_0(x_0 + y_0) + \sum_{i,j=1}^{10} a_i\left(x_j + y_j\right)P_{max(i,j)}$$
$$= A.X + A.Y$$
$$(A.B)X = \left(a_0 b_0 + \sum_{i,j=1}^{10} a_i b_j\,P_{max(i,j)}\right)\left(x_0 + \sum_{i=1}^{10} x_i P_i\right) = a_0 b_0 x_0 + \sum_{i,j,k=1}^{10} a_i b_j\,x_k P_{max(i,j)} = A(B.X)$$

Definition.

Let $w_j; 0 \le j \le 10$ be subspaces of V, then:

$W = \{x_0 + \sum_{i=1}^{10} x_i P_i\,; x_i \in w_i\}$ is called symbolic 10-plithogenic AH-subspace.

If $w_j = w_s$ for all $0 \le j, s \le 10$, then W is called AHS-subspace.

Theorem

Let $W = w_0 + \sum_{i=1}^{10} w_i P_i$ be an AHS-subspace of $10 - SP_V$, hence W is submodule of $10 - SP_V$.

Proof

Let $X = x_0 + \sum_{i=1}^{10} x_i P_i\,, Y = y_0 + \sum_{i=1}^{10} y_i P_i \in W$, and $A = a_0 + \sum_{i=1}^{10} a_i P_i \in 10 - SP_R$, then:

$X - Y = (x_0 - y_0) + \sum_{i=1}^{10}(x_i - y_i)\,P_i; x_j - y_j \in w_j; 0 \le j \le 10$, hence $X - Y \in W$.

$A.X = a_0 x_0 + \sum_{i,j=1}^{10} a_i x_j\,P_{mx(i,j)} \in W$, that is because $a_i x_j \in w_j$.

Definition.

Let $L_j: V \to T$ be linear transformation between $V, T; 0 \le j \le 10$, we define the AH-linear transformation s follows:

$L = L_0 + \sum_{i=1}^{10} L_i P_i : 10 - SP_V \to 10 - SP_T$ such that:

$L(x_0 + \sum_{i=1}^{10} x_i P_i) = L_0(x_0) + \sum_{i=1}^{10} L_i(x_i) P_i$.

If $L_j = L_k$ for all $0 \le j, k \le 10$, then L is called AHS-linear transformation.

Definition.

Let $L = L_0 + \sum_{i=1}^{10} L_i P_i : 10 - SP_V \to 10 - SP_T$ be an AHS-linear transformation, we define:

1). $AH - ker(L) = ker(L_0) + \sum_{i=1}^{10} ker(L_i) P_i$.

2). $AH - Im(L) = Im(L_0) + \sum_{i=1}^{10} Im(L_i) P_i$

If L is an AHS-liner transformation, then we get the AHS-kernel and AHS image.

Theorem

Let L be an AHS-liner transformation such that $L: 10 - SP_V \to 10 - SP_T$, then L is a module homomorphism.

Proof

Let $L = l_0 + \sum_{i=1}^{10} l_i P_i$ be an AHS-linear transformation, let $X = x_0 + \sum_{i=1}^{10} x_i P_i\,, Y = y_0 + \sum_{i=1}^{10} y_i P_i \in 10 - SP_V, A = a_0 + \sum_{i=1}^{10} a_i P_i \in 10 - SP_R$, then:

$$L(X + Y) = L\left[(x_0 + y_0) + \sum_{i=1}^{10}(x_i + y_i)\,P_i\right] = L_0(x_0 + y_0) + \sum_{i=1}^{10} L_0(x_i + y_i)\,P_i$$
$$= \left[L_0(x_0) + \sum_{i=1}^{10} L_0(x_i)\,P_i\right] + \left[L_0(y_0) + \sum_{i=1}^{10} L_0(y_i)\,P_i\right] = L(X) + L(Y)$$
$$L(A.X) = L\left[(a_0 x_0) + \sum_{i,j=1}^{10} a_i x_j\,P_{max(i,j)}\right] = L_0(a_0 x_0) + \sum_{i=1}^{10} L_0\left(a_i x_j\right)P_{max(i,j)}$$
$$= a_0 L_0(x_0) + \sum_{i=1}^{10} a_i L_0\left(x_j\right)P_{max(i,j)} = A.L(X)$$

Theorem

Let L be an AH-liner transformation, then:

1). $AH - ker(L)$ is n AH-subspace of $10 - SP_V$.

2). $AH - Im(L)$ is n AH-subspace of $10 - SP_T$.

Proof

1). Since $ker(L_i)$ is a subspace of V, then:

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$AH - ker(L) = ker(L_0) + \sum_{i=1}^{10} ker(L_i)\, P_i$ is an AH-subspace of $10 - SP_V$.
2). Since $Im(L_i)$ is a subspace of T, then:
$AH - Im(L) = Im(L_0) + \sum_{i=1}^{10} Im(L_i)\, P_i$ is an AH-subspace of $10 - SP_T$.

3. Conclusion

In this paper, we have studied symbolic m-plithogenic vector spaces with finite orders between 6 and 10, where we defined and characterized the AH-subspaces, AH-kernels, and AH- linear transformations in five different symbolic m-plithogenic spaces (6-plithogenic, 7-plithogenic,10-plithogenic vector spaces). Also, we proved many theorems that describe the computation of the kernels and direct images of the plithogenic AH-linear transformations.

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