



## On the Compactness and Continuity of Uryson's Operator in Orlicz Spaces

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### Abstract

Uryson's operators are very famous in the theory of fuzzy functional analysis. This paper is dedicated to studying and generalizing many results about the compactness and the continuity of Uryson's operator in two-variables defined with integral equation on  $G$  with the norm

$$\|u\|_f = \sup_{\rho(v,f) \leq} \left| \int_G u(x)v(x)dx \right| ; v(x) \in L_g^* , \quad u(x) \in L_f^*.$$

Also, we study the convergence of Uryson's sequences  $K_n$  defined with the family of functions  $K_n(x, y ; u)$  by using the convergence with respect to the defined measure and Caratheodory Condition.

**Keywords:** Fuzzy functional analysis; Uryson's operator; Orlicz space; measure.

### 1. Introduction and basic concepts

Linear and non-linear integral effects are considered among the important topics that depend in their study on convex, continuous, and compact rows of functions, which play an important role in various branches of mathematics and in the development of the theory of spaces in general and the Orlicz's space (regular space, and in particular the  $L^P$  spaces) and its applications in fuzzy function analysis. Due to their special importance, nonlinear integral equations of the form were studied:

$$\lambda \varphi(x) \int_0^1 k(x, y) f[\varphi(y)] dy.$$

As  $f(u)$  is a steadily increasing function for any positive function  $u$  since the effects defined on the right side of the integral equation are not defined in any  $L^P$  space. Therefore, studying the integral equations using the non-linear method in the function analysis showed its difficulty, then resorting to Studying rows of nonlinear equations through nonlinear problems of integral equations for those effects with two variables  $(x, y \in G, -\infty < u < \infty)$ .

These integral and nonlinear effects were studied by many mathematicians: [2,3,5,6,7,11,14,16,17,18,20,21]. In this research, we will study the complete continuity and compactness of one of the most important nonlinear integral operators, which is the Uryson Operator in the Orlicz space.

We will consider that the operator  $K$  is defined on the Orlicz space, which consists of the rows of real functions. For this reason, we begin by mentioning some basic concepts that we rely on in our study, including the finite closed set  $G$  in the finite-dimensional regional space, and we will introduce it to the measure and the set  $\hat{G}$ , which is the topological product.

$G \times G$  has a measure, and the concept of measurement continuity means that there is a subset in each group such that:

$$G_1 \subset G ; \mu(G_1) = \frac{1}{2}\mu(G).$$

The function  $f(u)$  is called  $N$ -function if it can be represented by

$$f(u) = \int_0^{|u|} P(t)dt,$$

where  $p(t)$  is positive for  $t > 0$  and continuous from the right when  $t \geq 0$  and has the property:

$$P(0) = 0, P(\infty) = \lim_{t \rightarrow \infty} P(t).$$

The functions  $f(u), g(v)$  defined with

$$f(u) = \int_0^{|u|} P(t)dt, \quad g(v) = \int_0^{|v|} q(s)ds ; q(s) = \sup_{P(t) \leq s} t,$$

are called  $N$ -complements of each other.

The  $N$ -function  $f(u)$  has the property  $\Delta$  for big values of  $u$  if there exists  $f(2u) \leq kf(u)$ , ( $u \geq u_0$ ).

We denote by  $L_g(G)$  to the class of functions defined on  $G$  with:

$$\rho(u, g) = \int_G g[u(x)]dx < \infty.$$

We denote by  $L_g^*$  to the set of functions with the condition:

$$\langle u, v \rangle = \int_G u(x)v(x)dx < \infty, \forall v(x) \in L_g(G).$$

For the definitions of Orlicz's norm and Caratheodory conditions, see [8,6]. For more details on fuzzy functional analysis, check [15-19].

## 2. Main Discussion

**Remark 2.1:** Uryson's operator is defined as follows:

$$Ku(x) = \int_G K[x, y, u(y)]dy. \quad (1)$$

If the function  $k(x, y, u)$  is a function with Caratheodory condition, then it is measurable and continuous almost everywhere, then we have:

$$|k(x, y, u)| \leq k(x, y)[\varphi(x) + R(|u|)] \quad (x, y \in G, -\infty < u < +\infty) \quad (2)$$

$$\iint_G g[k(x, y)]dxdy \leq b < +\infty. \quad (3)$$

Conditions (2) and (3) implies that:

$$\|Ku\|_\phi \leq c, (\|u\|_\phi \leq r). \quad (4)$$

Where  $c$  is a positive constant and  $K(x)$  is a non-negative function with the condition:

$$\int_G g[k(x)]dx \leq \frac{b}{\mu(G)}.$$

The operator defined with:

$$ku(x) = \int_G \frac{k(x) + k(y)}{2} R(|u(y)|) dy,$$

has the properties (2) and (3), and from (4) we get the formula:

$$\left\| \int_G k(y)R(|u(y)|)dy + k(x) \int_G R(|u(y)|)dy \right\|_{\emptyset} \leq 2c. \tag{5}$$

**Theorem 2.1.** Assume that  $f$  is an  $N$ -function with the following conditions  $(\Delta)$  (2), (3) and:

$$\emptyset(\alpha u) < f(u) \quad ; \quad \text{function} - N \emptyset, \tag{6}$$

$$f[\beta R(\gamma u)] < k\emptyset(\alpha u). \tag{7}$$

Then there exists a sequence of closed sets  $\{G_n\}_{n=1}^{\infty}$  that has the property  $\hat{G}_n \subset \hat{G}$ , and  $(\hat{G} \setminus \hat{G}_n) < \frac{1}{n}$  such that the function  $k(x, y, u)$  is continuous on  $\hat{G}_n \times ]-\infty, +\infty[$ , and the functions  $k(x, y)\varphi(y), k(x, y)$  are continuous on  $\hat{G}_n$ , therefore we can define the sequence

$$k_n(x, y, u) = \begin{cases} (x, y, u) & ; \{x, y\} \in \hat{G}_n \\ 0 & ; \{x, y\} \notin \hat{G}_n \end{cases}$$

with,

$$\begin{aligned} Ku(x) - k_n u(x) &= \int k_n[x, y, u(y)]X(x, y; \hat{G} \setminus \hat{G}_n) dy, \\ \|Ku - k_n u\|_{\emptyset} &\leq \left\| \int k(x, y)X(x, y; \hat{G} \setminus \hat{G}_n)[\varphi(x) + R(|u(y)|)] dy \right\|_{\emptyset} \leq \\ &c \|k(x, y)X(x, y; \hat{G} \setminus \hat{G}_n)\|_{\hat{g}} \quad \left( \|u\|_{\emptyset} \leq \frac{\gamma}{\alpha} \right). \end{aligned}$$

**Remark 2.2:**

For an  $N$ -function  $f$ , we have:

$$f[\beta R(\gamma u)] < kg(u), \tag{8}$$

$$\|Ku\|_{\emptyset} \leq c \|k(x, y)\|_{\hat{g}}, \|u\|_{\emptyset} \leq \frac{\gamma}{\alpha}. \tag{9}$$

**Theorem 2.2**

The continuity and compactness properties for an arbitrary operator  $K$  equivalents with the properties of Uryson's operator

$$Ku(x) = \int_G k[x, y, u(y)] dy.$$

**Proof:**

Since  $k(x, y, u)$  is compact, then we have:

$$|k(x, y)| \leq \varphi(x) + R|u(y)|; (x, y \in G, -\infty < u < +\infty) \tag{10}$$

Thus we can define:

$$\begin{cases} k(x, y, u) & ; \|u\| \leq n \\ k(x, y, u)(N + 1 - u) & ; n < u < n + 1 \\ k(x, y, -n)(u + n + 1) & ; -n - 1 < u < -n \\ 0 & ; |u| \geq n + 1 \end{cases}$$

$$\lim_{n \rightarrow \infty} \sup_{\|u\|_{\phi} \leq \frac{\gamma}{\alpha}} \|Ku - k_n u\|_{\phi} = 0 \tag{11}$$

so that for  $u(x) \in T\left(\theta, \frac{\gamma}{\alpha}; L_{\phi}^*\right)$  and  $G' = G\{|u(x)| > n\}$ ,

$$\mu(G') \leq \frac{1}{\phi\left(\frac{\alpha n}{\gamma}\right)} \int_G \phi\left[\frac{\alpha u(x)}{\gamma}\right] dx \leq \frac{1}{\phi\left(\frac{\alpha n}{\gamma}\right)} \left\| \frac{\alpha}{\gamma} u(x) \right\|_{\phi} \leq \frac{1}{\phi\left(\frac{\alpha n}{\gamma}\right)},$$

$$\begin{aligned} |ku(x) - k_n u(x)| &\leq \int_G |k[x, y, u(x)] - k_n[x, y, u(x)]| dy \\ &\leq \int_{G'} |k[x, y, u(x)]| dy - \int_{G'} |k_n[x, y, u(x)]| dy, \end{aligned}$$

$$\Psi(x) = nX(x, G') \text{sing } u(x) \quad ; \|\Psi\|_{\phi} \leq \|u\|_{\phi} \leq \gamma/\alpha,$$

$$|ku(x) - k_n u(x)| \leq \int_G X(x, y; \hat{G})[\varphi + R(|u(y)|)] dy + \int_G X(x, y; \hat{G})[\varphi + R(|\Psi(y)|)] dy;$$

$$\hat{G} = G \times G',$$

thus

$$\|Ku - k_n u\|_{\phi} \leq \frac{2C\mu(G)}{\phi\left(\frac{\alpha n}{\gamma}\right)} f^{-1}\left[\phi\left(\frac{\alpha n}{\gamma}\right)\right] \quad ; \left(\|u\|_{\phi} \leq \frac{\gamma}{\alpha}\right),$$

which implies the proof.

Also, we can easily regard that:

$$|k(x, y, u)| \leq d; (x, y \in G, -\infty < u < +\infty) \tag{12}$$

$$k(x, y, u) \equiv 0 \quad (|u| \geq u_0 \quad \& \quad u_0 = \text{const} > 0) \tag{13}$$

**Theorem 2.3**

Uryson's operators  $k_n$  defined with  $k_n(x, y, u)$  are compact and continuous.

**Proof:**

Since  $k_n(x, y, u)$  are compact and continuous, and they make together an Orlicz space, then we can write:

$$k_n u_0(x) = \int_G K[x, y, u(y)] dy = \lim_{n \rightarrow \infty} \int_G K[x, y, u_i(y)] dy = \lim_{i \rightarrow \infty} k_n u_i(x)$$

So that

$$k_n \in \{L_\emptyset^* \rightarrow c; a. e\} \Rightarrow k_n \in \{L_\emptyset^* \rightarrow L_\emptyset^*; a. e.\} \forall u(x) \in L_\emptyset^*,$$

thus

$$|Ku - k_n u(x)| \leq 2d \int_G X(x, y; \hat{G}_n \setminus \hat{G}) dy$$

$$\|Ku - k_n u\|_\emptyset \leq c_1 \|x, y; \hat{G}_n \setminus \hat{G}\|_\emptyset.$$

**Theorem 2.4**

Let  $f(u), g(u)$  be two  $N$ -functions and complements to each other, and the second one has the property  $\Delta_1$ , and let

$$|k(x, y, u)| \leq k(x, y)[\varphi(x) + R(|u|)]; k(x, y \in G, -\infty < u < \infty),$$

where  $\varphi(x) \in L_f^*$ , and  $k(x, y) \in E_g^*$ , then we can find the positive constants  $\beta, \gamma, c$  such that  $f[\beta R(\gamma u)] \leq c g(u)$ ,

$$(14)$$

and the operator defined with

$$ku(x) = \int_G k[x, y, u(y)] dy, \tag{15}$$

has the property  $Ku \in \{T(\theta, \gamma; \rightarrow L_\emptyset^*) \rightarrow L_\emptyset^*; a. e.\}$ , i.e. we can find an  $N$ -function  $\emptyset(u)$  with  $f[\beta R(\gamma u)] \leq c \emptyset(u)$ .

$$(16)$$

**Proof:**

It holds by adding a condition on the definition of the operator  $K$ , not only on  $T(\theta, \gamma, L_\emptyset^*)$ , but over the all of the space  $L_\emptyset^*$ , then by using (14),(16) and making the constant  $\gamma$  very large, we get the desired proof.

In the special case that  $K$  is defied on  $L_\emptyset$ , we get  $f[\beta R(2^8 u)] \leq c \emptyset(u) \leq c_1 g(u)$ , and for large values of  $u$ , we can see that  $f[\emptyset(u)] \leq \emptyset(au)$ , and

$$R[\alpha \gamma u] \leq c \emptyset(u) \leq c g(u), \tag{17}$$

which implies:

$$[R(\gamma u)] \leq f \left[ c \emptyset \left( \frac{u}{\alpha} \right) \right] \leq c_1 \emptyset(u) \leq c_1 g(u).$$

**Theorem 2.5**

Let  $f(u), g(u)$  be two  $N$ -functions and complements to each other, and the second has the property

$$|k(x, y, u)| \leq k(x, y)[\varphi(x) + R(|u|)] (x, y \in G, -\infty < u < +\infty), \tag{18}$$

$$R(u), k(x, y) \in L_g^* = L_g^*, \varphi(x) \in L_f^*,$$

then we can find  $c = \text{const} > 0$  such that

$$R(u) < c \frac{g(u)}{u}. \tag{19}$$

Also, the function  $R(u)$  is defined on Orlicz space and Uryson's operator

$$Ku(y) = \int_G k[x, y, u(y)] dy, \tag{20}$$

is completely continuous.

**Proof:**

Since the function  $g[k(x, y)]$  is bounded on  $\hat{G}$ , we can find an  $N$ -function  $\phi(u)$  with the conditions  $\Delta_1, \Delta$  and

$$\iint_{\hat{G}} \phi\{g[k(x, y)]\} dx dy < \infty.$$

From (20) we get

$$f[\beta R(\gamma u)] \leq \frac{1}{\gamma} u, \quad (21)$$

for  $u_0 \leq u, u(x) \in L_{\phi}^* = L_{\phi}$  with

$$\begin{aligned} \|\varphi(x) + R(|u(x)|)\|_f &\leq \|\varphi(x)\|_f + \frac{1}{\beta} \|\beta R(|u(x)|)\|_f \leq \|\varphi(x)\|_f + \frac{1}{\beta} \left\{ 1 + \int_{\hat{G}} f[\beta R(|u(x)|)] dx \right\} \\ \|\varphi(x) + R(|u(x)|)\|_f &\leq \|\varphi(x)\|_f + \frac{1}{\beta} \left\{ 1 + f[\beta R(u_0)] \mu(G) + \frac{1}{\gamma} \int_{\hat{G}} |u(x)| dx \right\} \leq \\ &\|\varphi(x)\|_f + \frac{1}{\beta} \{ 1 + f[\beta R(u_0)] \mu(G) + \lambda \|u\|_{\phi} \} \quad ; \lambda = \text{const}, \end{aligned}$$

hence if  $\|u\|_{\phi} \leq r$  we get:

$$\|\varphi(x) + R(|u(x)|)\|_f \leq c(r), \quad (22)$$

so that by using the integral operator

$$\begin{aligned} Av(x) &= \int_{\hat{G}} k(x, y) v(y) dy, \\ \|Av\|_{\phi} &\leq 2l \|k(x, y)\|_{\Psi} \|v\|_f \quad ; \Psi(u) = \phi[g(u)]. \end{aligned} \quad (23)$$

According to (20), we find that:

$$\begin{aligned} |Ku(x)| &\leq A[\varphi(x) + R(|u(x)|)] \\ \|Ku\|_{\phi} &\leq 2l \|k(x, y)\|_{\Psi} \|\varphi(x) + R(|u(x)|)\|_f \leq 2lc(r) \|k(x, y)\|_{\Psi}. \end{aligned}$$

**3. Conclusion**

Linear and non-linear effects have been of particular importance in studying issues of complementarity from an analytical point of view in various types of dependent analysis spaces, which depend in their study on rows of specific, continuous functions, in particular on the properties of the  $N$ -function in the Orlicz's space and their importance in forming the sequence of continuous and compact effects, and the study of various Types of convergence, such as convergence by regularity and convergence by analogy in that space. The question that can be asked is if the series of non-linear integral operators are convergent by measure, then what is the condition that these operators must fulfill in order for convergence to the mean to be achieved, and is the normality of the groups defined by these operators necessary? What is the role of the  $N$ -dependent in this, and can the Carlson measurement be used in this?

**References**

- [1] Luxemburg W . A.J. and Zaanen A . C. :Some remarks on Banach function space , Nederl . Akad . Wetensch.Proc.ser.A.59-Indag.Math.56( 1984).
- [2] Hatamleh, R., " On the Form of Correlation Function for a Class of Nonstationary Field with a Zero Spectrum, " *Rocky Mountain Journal of Mathematics*, 33(1),2003.
- [ 3 ] D.L.Cohn :,Measure Theory ,Birk hauser ,Boston,p.138-212. (1980)

- [4 ] D.Girela and J.A.P elaez:, Carlson Measures ,multipliers and integration operators for space of Dirichlet Type,J.Anal.Math .Malaga Spain,1-15. ( 2006)
- [5] Hatamleh, R., Zolotarev, V. A., “Triangular Models of Commutative Systems of Linear Operators Close to Unitary Operators,” *Ukrainian Mathematical Journal*,68(5),791–811,2006.
- [ 6 ] K. Hoffman :,Banach spaces of Analytic function ,Dover Publications, Inc, Minneola New York p.379-417, ( 2007).
- [7] Heilat, A. S., Zureigat, H., Hatamleh, R., Batiha, B., “New Spline Method for Solving Linear Two-Point Boundar Value Problems,” *European Journal of Pure and Applied Mathematics*, 14(4), 1283–1294,2021.
- [8 ] Milnes H.W. :,Convexity of Orlicz spaces,Pacif.J.Math,7.3(1957).
- [9 ] H.L.Royden : Real Analysis THIRD EDITION.p.118-130,( 1988).
- [10 ] WALTER RUDIN: Principles of Mathematical Analysis McGRAW-HILL INTERNATIONAL EDITIONS(Mathematics Series).p.300-325,( 1976).
- [11] T.Qawasmeh,R.Hatamleh, “A new contraction based on H-simulation functions in the frame of extended b-metric spaces and application,”*International Journal of Electrical and Computer Engineering*, 13 (4),4212-4221,2023.
- [12] Walter Rudin: FUNCTIONAL ANALYSIS McGRAW-HILL INTERNATIONAL EDITIONS(Mathematics Serie),p.292-363 .(1980).
- [13] Erwin Kreyszig: INTRODUCTORY FUNCTINAL ANALYSIS WITH APPLICATIONS University of Windsor,p .129-197.(1978).
- [14] Ayman Hazaymeh, Rania Saadeh, Raed Hatamleh, Mohammad W Alomari, Ahmad Qazza, “A Perturbed Milne’s Quadrature Rule for n-Times Differentiable Functions with Lp-Error Estimates,” *Axioms-MDPI*, ,12(9),803,2023.
- [15] Bayramov and C. Gunduz, “Soft locally compact spaces and soft paracompact spaces,” *Journal of Mathematics and System*, vol. 3, pp. 122–130, 2013.
- [16] S. Das and S. K. Samanta, “Soft Metric,” *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 1, pp. 77–94, 2013.
- [17] M. I. Yazar, T. Bilgin, S. Bayramov, and C. Gunduz (Aras), “A new view on soft normed spaces,” *International Mathematical Forum*, vol. 9, no. 24, pp. 1149–1159, 2014.
- [18] Hatamleh, R., Zolotarev, V. A., “On Two-Dimensional Model Representations of One Class of Commuting Operators,” *Ukrainian Mathematical Journal*, 66(1), 122–144,2014.
- [19] S. Das and S. K. Samanta, “On soft inner product spaces,” *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 1, pp. 151–170, 2013.
- [20] Hatamleh, R., & Zolotarev, V. A., “ On Model Representations of Non-Selfadjoint Operators with Infinitely Dimensional Imaginary Component,” *Zurnal Matematicheskoy Fiziki, Analiza, Geometrii*, 11(2), 174–186,2015.
- [21] Ayman Hazaymeh, Ahmad Qazza, Raed Hatamleh, Mohammad W Alomari, Rania Saadeh, “On Further Refinements of Numerical Radius, Inequalities,” *Axiom-MDPI*, 12(9),807,2023.