# Extension of arithmetic and geometric aggregating operators using new type interval-valued neutrosophic sets

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**Abstract**

The purpose of this article is to present a novel approach to the $(\delta, \varepsilon)$ interval-valued neutrosophic set (IVNS). This is an extension of the IVNS. As a result of this article, we will discuss the concept of $(\delta, \varepsilon)$ interval-valued neutrosophic weighted averaging (IVNWA), $(\delta, \varepsilon)$ interval-valued neutrosophic weighted geometric (IVNWG), $(\delta, \varepsilon)$ generalized interval-valued neutrosophic weighted averaging (GIVNWA) and $(\delta, \varepsilon)$ generalized interval-valued neutrosophic weighted geometric (GIVNWG). Additionally, the $(\delta, \varepsilon)$ IVNS approach is characterized by idempotency, boundedness, commutativity and monotonicity.

**Keywords:** $(\delta, \varepsilon)$IVNWA, $(\delta, \varepsilon)$IVNWG, G$(\delta, \varepsilon)$IVNWA and G$(\delta, \varepsilon)$IVNWG.

## 1 Introduction

Every day, systems become more complex, which makes it harder for decision-makers to select the best option. A single goal is hard to achieve, but you can do it. For many businesses, it was challenging to inspire people, create objectives, and form attitudes. Therefore, many goals must be taken into account at the same time when making decisions, whether they are made by an individual or a group. After giving it some thought, it appears that the criteria are solved flexibly, which makes it challenging for any decision-maker to come up with the best answer possible for each of the relevant criteria. Reliable and adequate methodologies should be developed by decision-makers in order to identify the optimal solution. Generally, when dealing with ambiguity and uncertainty in decision-making, crisp methods don't work. To deal with the ambiguities, many uncertain theories including fuzzy set (FS),[1] intuitionistic FS (IFS),[2] Pythagorean FS (PFS)[3] and spherical FS (SFS).[4] A FS is a set of elements with membership grade (MD) in the given set values from zero to one; Later, Atanassov proposed the concept of an IFS that is divided into categories using non-membership grade (NMD), which cannot exceed one.[2] It is possible to convey a single problem to the decision-making (DM) when MD and NMD scores are greater than one. PFS is characterized by a square sum of MD and NMD less than one for an IFS that has a value less than one as determined by Yager.[3] Positive MD, neutral MD and negative MD are the three pointers that make up the picture FS concept developed by Cuong et al.[5] As a result, it has a

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number of advantages over PFS and IFS as well. A generalization of picture FS was examined by Liu et al.[6] using AOs. A generalized PFS based on AO and its applications has been proposed by Liu et al.[7] AO features based on PFS and interval values.[8] An AO-based picture FS was presented by Liu et al.[6] In the DM approach challenge, the sum of the positive, neutral and negative MD values rarely exceeds one. The concept of SFS is presented by Ashraf et al.[4] which makes sure the square sum of positive, neutral and negative grades doesn't exceed 1. The notion of SFS was examined by Fatmaa et al.[9]

Recently, Palanikumar et al. discussed many algebraic structures with applications by[10]-.[13] In 2002, Li et al.[14] proposed generalized ordered weighted averaging operators (GOWAs). Many researchers introduced new aggregating operators such as weighted and hybrid operators[15]-.[21] Palanikumar et al. discussed new aggregation operators with applications[22]-.[24] I will follow the following structure throughout the remainder of this paper. An introduction is found in section 1. PFS and NS were discussed in section 2. Section 3 discusses some of the operations on $(\delta, \varepsilon)$ IVNNs. Section 4 discusses the arithmetic and geometric aggregating operator $(\delta, \varepsilon)$ IVNNs.

## 2 Background

This section contains a number of important definitions that we must review for our further learning.

**Definition 2.1.** [8] Let $X$ be an universal. The PIVFS $\ell = \left\{\epsilon, \left\langle \widetilde{\Xi^t_\ell}(\epsilon), \widetilde{\Xi^f_\ell}(\epsilon) \right\rangle \Big| \epsilon \in X \right\}$, where $\widetilde{\Xi^t_\ell}, \widetilde{\Xi^f_\ell} : X \to Int([0,1])$ denote the MD and NMD of $\epsilon \in X$ to the set $\ell$, respectively, and $0 \leq (\Xi^t_\ell(\epsilon))^2 + (\Xi^f_\ell(\epsilon))^2 \leq 1$. For convenience, $\ell = \left\langle \left[\Xi^t_\ell, \Xi^t_\ell\right], \left[\Xi^f_\ell, \Xi^f_\ell\right] \right\rangle$ is called a Pythagorean interval-valued fuzzy number (PyIVFN).

**Definition 2.2.** The NS $\ell = \left\{x, \langle \Xi^t_\ell(\epsilon), \Xi^m_\ell(\epsilon), \Xi^f_\ell(\epsilon) \rangle \big| \epsilon \in X \right\}$, where $\Xi^t_\ell, \Xi^m_\ell, \Xi^f_\ell : X \to [0,1]$ is denote the positive MD, neutral MD and negative MD of $\epsilon \in X$, respectively and $0 \leq (\Xi^t_\ell(\epsilon)) + (\Xi^m_\ell(\epsilon)) + (\Xi^f_\ell(\epsilon)) \leq 2$. For $M = \langle \Xi^t_\ell, \Xi^m_\ell, \Xi^f_\ell \rangle$ is called a neutrosophic number (SFN).

**Definition 2.3.** The Pythagorean NS $\ell = \left\{\epsilon, \langle \Xi^t_\ell(\epsilon), \Xi^m_\ell(\epsilon), \Xi^f_\ell(\epsilon) \rangle \big| \epsilon \in X \right\}$, where $\Xi^t_\ell, \Xi^m_\ell, \Xi^f_\ell : X \to [0,1]$ is denote the positive MD, neutral MD and negative MD of $\epsilon \in X$, respectively and $0 \leq (\Xi^t_\ell(\epsilon))^2 + (\Xi^m_\ell(\epsilon))^2 + (\Xi^f_\ell(\epsilon))^2 \leq 2$. For $M = \langle \Xi^t_\ell, \Xi^m_\ell, \Xi^f_\ell \rangle$ is called a Pythagorean neutrosophic number (PySFN).

**Definition 2.4.** [9] The SFS $\ell$ in $X$ is given by $\ell = \left\{\epsilon, \langle \Xi^t_\ell(\epsilon), \Xi^m_\ell(\epsilon), \Xi^f_\ell(\epsilon) \rangle \big| \epsilon \in X \right\}$, where $\Xi^t_\ell, \Xi^m_\ell, \Xi^f_\ell : X \to [0,1]$ denote the truth, indeterminacy and falsity membership grade of $\epsilon \in X$ to $\ell$, respectively and $0 \leq (\Xi^t_\ell(\epsilon))^2 + (\Xi^m_\ell(\epsilon))^2 + (\Xi^f_\ell(\epsilon))^2 \leq 1$. For all $\epsilon \in X$, $\sqrt{1 - \left((\Xi^t_\ell(\epsilon))^2 + (\Xi^m_\ell(\epsilon))^2 + (\Xi^f_\ell(\epsilon))^2\right)}$ is called the grade of refusal of membership of $\epsilon$ in $\ell$. For convenience, $\ell = \langle \Xi^t_\ell, \Xi^m_\ell, \Xi^f_\ell \rangle$ is called a spherical fuzzy number (SFN).

**Definition 2.5.** [19] Let $\ell_1 = (a_1, b_1) \in N$and $\ell_2 = (a_2, b_2) \in N$. Then the distance between $\ell_1$ and $\ell_2$ is defined as $\mathcal{D}(\ell_1, \ell_2) = \sqrt{(a_1 - a_2)^2 + \frac{1}{2}(b_1 - b_2)^2}$, where $N$ is a natural number.

**Definition 2.6.** For any two IVNNs $\ell_1 = \left\langle \left( [\Xi_1^{t-}(\epsilon), \Xi_1^{t+}(\epsilon)], [\Xi_1^{m-}(\epsilon), \Xi_1^{m+}(\epsilon)], [\Xi_1^{f-}(\epsilon), \Xi_1^{f+}(\epsilon)] \right) \right\rangle$ and $\ell_2 = \left\langle \left( [\Xi_2^{t-}(\epsilon), \Xi_2^{t+}(\epsilon)], [\Xi_2^{m-}(\epsilon), \Xi_2^{m+}(\epsilon)], [\Xi_2^{f-}(\epsilon), \Xi_2^{f+}(\epsilon)] \right) \right\rangle$. Then

$$\mathcal{D}_E(\ell_1, \ell_2) = \frac{1}{2} \sqrt{\begin{array}{l} \left[ \begin{array}{l} \frac{1+(\Xi_1^{t-})^2+(\Xi_1^{t+})^2-((\Xi_1^{m-})^2+(\Xi_1^{m+})^2)-((\Xi_1^{f-})^2+(\Xi_1^{f+})^2)}{2} \\ -\frac{1+(\Xi_2^{t-})^2+(\Xi_2^{t+})^2-((\Xi_2^{m-})^2+(\Xi_2^{m+})^2)-((\Xi_2^{f-})^2+(\Xi_2^{f+})^2)}{2} \end{array} \right]^2 \\ +\frac{1}{2} \left[ \begin{array}{l} \frac{1+(\Xi_1^{t-})^2+(\Xi_1^{t+})^2-((\Xi_1^{m-})^2+(\Xi_1^{m+})^2)-((\Xi_1^{f-})^2+(\Xi_1^{f+})^2)}{2} \\ -\frac{1+(\Xi_2^{t-})^2+(\Xi_2^{t+})^2-((\Xi_2^{m-})^2+(\Xi_2^{m+})^2)-((\Xi_2^{f-})^2+(\Xi_2^{f+})^2)}{2} \end{array} \right]^2 \end{array}}$$

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where $\mathcal{D}_E(\ell_1, \ell_2)$ is called the ED between $\ell_1$ and $\ell_2$.

$$\mathcal{D}_H(\ell_1,\ell_2)=\frac{1}{2}\left[\begin{array}{c}\left|\begin{array}{c}\frac{1+(\Xi_1^{t-})^2+(\Xi_1^{t+})^2-((\Xi_1^{m-})^2+(\Xi_1^{m+})^2)-((\Xi_1^{f-})^2+(\Xi_1^{f+})^2)}{2}\\ -\frac{1+(\Xi_2^{t-})^2+(\Xi_2^{t+})^2-((\Xi_2^{m-})^2+(\Xi_2^{m+})^2)-((\Xi_2^{f-})^2+(\Xi_2^{f+})^2)}{2}\end{array}\right|\\ +\frac{1}{2}\left|\begin{array}{c}\frac{1+(\Xi_1^{t-})^2+(\Xi_1^{t+})^2-((\Xi_1^{m-})^2+(\Xi_1^{m+})^2)-((\Xi_1^{f-})^2+(\Xi_1^{f+})^2)}{2}\\ -\frac{1+(\Xi_2^{t-})^2+(\Xi_2^{t+})^2-((\Xi_2^{m-})^2+(\Xi_2^{m+})^2)-((\Xi_2^{f-})^2+(\Xi_2^{f+})^2)}{2}\end{array}\right|\end{array}\right]$$

where $\mathcal{D}_H(\ell_1, \ell_2)$ is called the HD between $\ell_1$ and $\ell_2$.

## 3 Operations for $(\delta, \varepsilon)$ IVNN

We discuss the concept of $(\delta, \varepsilon)$ interval-valued neutrosophic number (IVNN). As a result, the $(\delta, \varepsilon)$ IVNN and its operations were defined.

**Definition 3.1.** The $(\delta, \varepsilon)$ NS $\ell = \left\{\epsilon, \left\langle \left( [\Xi_\ell^{t-}(\epsilon), \Xi_\ell^{t+}(\epsilon)], [\Xi_\ell^{m-}(\epsilon), \Xi_\ell^{m+}(\epsilon)], [\Xi_\ell^{f-}(\epsilon), \Xi_\ell^{f+}(\epsilon)] \right) \right\rangle \middle| \epsilon \in X \right\}$, where $\Xi_\ell^t, \Xi_\ell^m, \Xi_\ell^f : X \to [0,1]$ denote the PMD, neutral MD and NMD of $\epsilon \in X$ to $\ell$, respectively and $0 \le (\Xi_\ell^{t+}(\epsilon))^\delta + (\Xi_\ell^{m+}(\epsilon))^{lcm(\delta,\varepsilon)} + (\Xi_\ell^{f+}(\epsilon))^\varepsilon \le 1$. For convenience, $\ell = \left\langle \left( \Xi_\ell^{t+}, \Xi_\ell^{m+}, \Xi_\ell^{f+} \right) \right\rangle$ is represent a $(\delta, \varepsilon)$ IVNN.

**Definition 3.2.** Let $\ell = \left\langle \left( [\Xi_\ell^{t-}, \Xi_\ell^{t+}], [\Xi_\ell^{m-}, \Xi_\ell^{m+}], [\Xi_\ell^{f-}, \Xi_\ell^{f+}] \right) \right\rangle$, $\ell_1 = \left\langle \left( [\Xi_1^{t-}, \Xi_1^{t+}], [\Xi_1^{m-}, \Xi_1^{m+}], [\Xi_1^{f-}, \Xi_1^{f+}] \right) \right\rangle$ and $\ell_2 = \left\langle \left( [\Xi_2^{t-}, \Xi_2^{t+}], [\Xi_2^{m-}, \Xi_2^{m+}], [\Xi_2^{f-}, \Xi_2^{f+}] \right) \right\rangle$ be any three $(\delta, \varepsilon)$ IVNNs, and $\Re > 0$. Then

1. $$\ell_1 \bigvee \ell_2 = \left[\begin{array}{c}\left[\begin{array}{c}\sqrt[\delta]{(\Xi_1^{t-})^\delta + (\Xi_2^{t-})^\delta - (\Xi_1^{t-})^\delta \cdot (\Xi_2^{t-})^\delta},\\ \sqrt[\delta]{(\Xi_1^{t+})^\delta + (\Xi_2^{t+})^\delta + (\Xi_1^{t+})^\delta \cdot (\Xi_2^{t+})^\delta}\end{array}\right]\\ \left[\begin{array}{c}\sqrt[lcm(\delta,\varepsilon)]{(\Xi_1^{m-})^{lcm(\delta,\varepsilon)} + (\Xi_2^{m-})^{lcm(\delta,\varepsilon)} - (\Xi_1^{m-})^{lcm(\delta,\varepsilon)} \cdot (\Xi_2^{m-})^{lcm(\delta,\varepsilon)}},\\ \sqrt[lcm(\delta,\varepsilon)]{(\Xi_1^{m+})^{lcm(\delta,\varepsilon)} + (\Xi_2^{m+})^{lcm(\delta,\varepsilon)} + (\Xi_1^{m+})^{lcm(\delta,\varepsilon)} \cdot (\Xi_2^{m+})^{lcm(\delta,\varepsilon)}}\end{array}\right]\\ [(\Xi_1^{f-})^\varepsilon \cdot (\Xi_2^{f-})^\varepsilon, (\Xi_1^{f+})^\varepsilon \cdot (\Xi_2^{f+})^\varepsilon]\end{array}\right],$$

2. $$\ell_1 \bigwedge \ell_2 = \left[\begin{array}{c}[(\Xi_1^{t-})^\delta \cdot (\Xi_2^{t-})^\delta, (\Xi_1^{t+})^\delta \cdot (\Xi_2^{t+})^\delta]\\ \left[\begin{array}{c}\sqrt[lcm(\delta,\varepsilon)]{(\Xi_1^{m-})^{lcm(\delta,\varepsilon)} + (\Xi_2^{m-})^{lcm(\delta,\varepsilon)} - (\Xi_1^{m-})^{lcm(\delta,\varepsilon)} \cdot (\Xi_2^{m-})^{lcm(\delta,\varepsilon)}},\\ \sqrt[lcm(\delta,\varepsilon)]{(\Xi_1^{m+})^{lcm(\delta,\varepsilon)} + (\Xi_2^{m+})^{lcm(\delta,\varepsilon)} + (\Xi_1^{m+})^{lcm(\delta,\varepsilon)} \cdot (\Xi_2^{m+})^{lcm(\delta,\varepsilon)}}\end{array}\right]\\ \left[\begin{array}{c}\sqrt[\varepsilon]{(\Xi_1^{f-})^\varepsilon + (\Xi_2^{f-})^\varepsilon - (\Xi_1^{f-})^\varepsilon \cdot (\Xi_2^{f-})^\varepsilon},\\ \sqrt[\varepsilon]{(\Xi_1^{f+})^\varepsilon + (\Xi_2^{f+})^\varepsilon + (\Xi_1^{f+})^\varepsilon \cdot (\Xi_2^{f+})^\varepsilon}\end{array}\right]\end{array}\right]$$

3. $$\Re \cdot \ell = \left[\begin{array}{c}[\sqrt[\delta]{1 - \left(1 - (\Xi^{t-})^\delta\right)^\Re}, \sqrt[\delta]{1 + \left(1 + (\Xi^{t+})^\delta\right)^\Re}]\\ [\sqrt[lcm(\delta,\varepsilon)]{1 - \left(1 - (\Xi^{m-})^{lcm(\delta,\varepsilon)}\right)^\Re}, \sqrt[lcm(\delta,\varepsilon)]{1 + \left(1 + (\Xi^{m+})^{lcm(\delta,\varepsilon)}\right)^\Re}]\\ [((\Xi^{f-})^\varepsilon)^\Re, ((\Xi^{f+})^\varepsilon)^\Re]\end{array}\right],$$

4. $$\ell^\Re = \left[\begin{array}{c}[((\Xi^{t-})^\delta)^\Re, ((\Xi^{t+})^\delta)^\Re]\\ [\sqrt[lcm(\delta,\varepsilon)]{1 - \left(1 - (\Xi^{m-})^{lcm(\delta,\varepsilon)}\right)^\Re}, \sqrt[lcm(\delta,\varepsilon)]{1 + \left(1 + (\Xi^{m+})^{lcm(\delta,\varepsilon)}\right)^\Re}]\\ [\sqrt[\varepsilon]{1 - \left(1 - (\Xi^{f-})^\varepsilon\right)^\Re}, \sqrt[\varepsilon]{1 + \left(1 + (\Xi^{f+})^\varepsilon\right)^\Re}]\end{array}\right].$$

## 4 AOs based on $(\delta, \varepsilon)$ IVNN

Here we describe the AOs using $(\delta, \varepsilon)$ IVNWA, $(\delta, \varepsilon)$ IVNWG, G$(\delta, \varepsilon)$ IVNWA, and G$(\delta, \varepsilon)$ IVNWG.

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### 4.1 $(\delta,\varepsilon)IVNWA$

**Definition 4.1.** Let $\ell_i = \left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ be the $(\delta,\varepsilon)$IVNNs, $W=(\kappa_1,\kappa_2,...,\kappa_n)$ be the weight of $\ell_i$, $\kappa_i \geq 0$ and $\bigvee_{i=1}^{n}\kappa_i = 1$. Then $(\delta,\varepsilon)$ IVNWA $(\ell_1,\ell_2,...,\ell_n)=\bigvee_{i=1}^{n}\kappa_i\ell_i$.

**Theorem 4.2.** *Let* $\ell_i = \left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ *be the* $(\delta,\varepsilon)$ *IVNNs. Then* $(\delta,\varepsilon)IVNWA(\ell_1,\ell_2,...,\ell_n)$

$$=\begin{bmatrix}\left[\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{t-})^{\delta}\right)^{\kappa_i}},\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{t+})^{\delta}\right)^{\kappa_i}}\right]\\ \left[\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}},\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right]\\ \left[\bigcirc_{i=1}^{n}((\Xi_i^{f-})^{\varepsilon})^{\kappa_i},\bigcirc_{i=1}^{n}((\Xi_i^{f+})^{\varepsilon})^{\kappa_i}\right]\end{bmatrix}.$$

**Proof.** If $n=2$, then $(\delta,\varepsilon)$ IVNWA$(\ell_1,\ell_2)=\kappa_1\ell_1\bigvee\kappa_2\ell_2$, where

$$\kappa_1\ell_1=\begin{bmatrix}\left[\sqrt[\delta]{1-\left(1-(\Xi_1^{t-})^{\delta}\right)^{\kappa_1}},\sqrt[\delta]{1-\left(1-(\Xi_1^{t+})^{\delta}\right)^{\kappa_1}}\right]\\ \left[\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-(\Xi_1^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}},\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-(\Xi_1^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right]\\ \left[((\Xi_1^{f-})^{\varepsilon})^{\kappa_1},((\Xi_1^{f+})^{\varepsilon})^{\kappa_1}\right]\end{bmatrix}$$

$$\kappa_2\ell_2=\begin{bmatrix}\left[\sqrt[\delta]{1-\left(1-(\Xi_2^{t-})^{\delta}\right)^{\kappa_2}},\sqrt[\delta]{1-\left(1-(\Xi_2^{t+})^{\delta}\right)^{\kappa_2}}\right]\\ \left[\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-(\Xi_2^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_2}},\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-(\Xi_2^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_2}}\right]\\ \left[((\Xi_2^{f-})^{\varepsilon})^{\kappa_2},((\Xi_2^{f+})^{\varepsilon})^{\kappa_2}\right]\end{bmatrix}.$$

Now,

$$\kappa_1\ell_1\bigvee\kappa_2\ell_2 = \begin{bmatrix}\left[\begin{array}{c}\sqrt[\delta]{\begin{array}{c}\left(1-\left(1-(\Xi_1^{t-})^{\delta}\right)^{\kappa_1}\right)+\left(1-\left(1-(\Xi_2^{t-})^{\delta}\right)^{\kappa_2}\right)\\ -\left(1-\left(1-(\Xi_1^{t-})^{\delta}\right)^{\kappa_1}\right)\cdot\left(1-\left(1-(\Xi_2^{t-})^{\delta}\right)^{\kappa_2}\right)\end{array}},\\ \sqrt[\delta]{\begin{array}{c}\left(1-\left(1-(\Xi_1^{t+})^{\delta}\right)^{\kappa_1}\right)+\left(1-\left(1-(\Xi_2^{t+})^{\delta}\right)^{\kappa_2}\right)\\ +\left(1-\left(1-(\Xi_1^{t+})^{\delta}\right)^{\kappa_1}\right)\cdot\left(1-\left(1-(\Xi_2^{t+})^{\delta}\right)^{\kappa_2}\right)\end{array}},\end{array}\right]\\ \left[\begin{array}{c}\sqrt[lcm(\delta,\varepsilon)]{\begin{array}{c}\left(1-\left(1-(\Xi_1^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}\right)+\left(1-\left(1-(\Xi_2^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_2}\right)\\ -\left(1-\left(1-(\Xi_1^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}\right)\cdot\left(1-\left(1-(\Xi_2^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_2}\right)\end{array}},\\ \sqrt[lcm(\delta,\varepsilon)]{\begin{array}{c}\left(1-\left(1-(\Xi_1^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}\right)+\left(1-\left(1-(\Xi_2^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_2}\right)\\ +\left(1-\left(1-(\Xi_1^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}\right)\cdot\left(1-\left(1-(\Xi_2^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_2}\right)\end{array}},\end{array}\right]\\ \left[((\Xi_1^{f-})^{\varepsilon})^{\kappa_1}\cdot((\Xi_2^{f-})^{\varepsilon})^{\kappa_2}\right]\end{bmatrix}$$

$$=\begin{bmatrix}\left[\sqrt[\delta]{1-\left(1-(\Xi_1^{t-})^{\delta}\right)^{\kappa_1}\left(1-(\Xi_2^{t-})^{\delta}\right)^{\kappa_2}},\sqrt[\delta]{1-\left(1-(\Xi_1^{t+})^{\delta}\right)^{\kappa_1}\left(1-(\Xi_2^{t+})^{\delta}\right)^{\kappa_2}}\right]\\ \left[\begin{array}{c}\sqrt[\delta]{1-\left(1-(\Xi_1^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}\left(1-(\Xi_2^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_2}},\\ \sqrt[\delta]{1-\left(1-(\Xi_1^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}\left(1-(\Xi_2^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_2}}\end{array}\right]\\ \left[((\Xi_1^{f-})^{\varepsilon})^{\kappa_1}\cdot((\Xi_2^{f-})^{\varepsilon})^{\kappa_2},((\Xi_1^{f+})^{\varepsilon})^{\kappa_1}\cdot((\Xi_2^{f+})^{\varepsilon})^{\kappa_2}\right]\end{bmatrix}$$


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Hence, $(\delta,\varepsilon)IVNWA(\ell_1,\ell_2)$

$$=\begin{bmatrix}\left[\sqrt[\delta]{1-\bigcirc_{i=1}^{2}\left(1-(\Xi_i^{t-})^{\delta}\right)^{\kappa_i}},\sqrt[\delta]{1-\bigcirc_{i=1}^{2}\left(1-(\Xi_i^{t+})^{\delta}\right)^{\kappa_i}}\right]\\ \left[\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{2}\left(1-(\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}},\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{2}\left(1-(\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right]\\ \left[\bigcirc_{i=1}^{2}((\Xi_i^{f-})^{\varepsilon})^{\kappa_i},\bigcirc_{i=1}^{2}((\Xi_i^{f+})^{\varepsilon})^{\kappa_i}\right]\end{bmatrix}.$$

It valid for $n\geq 3$,
Thus, $(\delta,\varepsilon)IVNWA(\ell_1,\ell_2,...,\ell_l)$

$$=\begin{bmatrix}\left[\sqrt[\delta]{1-\bigcirc_{i=1}^{l}\left(1-(\Xi_i^{t-})^{\delta}\right)^{\kappa_i}},\sqrt[\delta]{1-\bigcirc_{i=1}^{l}\left(1-(\Xi_i^{t+})^{\delta}\right)^{\kappa_i}}\right]\\ \left[\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l}\left(1-(\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}},\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l}\left(1-(\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right]\\ \left[\bigcirc_{i=1}^{l}((\Xi_i^{f-})^{\varepsilon})^{\kappa_i},\bigcirc_{i=1}^{l}((\Xi_i^{f+})^{\varepsilon})^{\kappa_i}\right]\end{bmatrix}.$$

If $n=l+1$, then $(\delta,\varepsilon)$ IVNWA $(\ell_1,\ell_2,...,\ell_l,\ell_{l+1})$

$$=\begin{bmatrix}\left[\begin{array}{c}\sqrt[\delta]{\begin{array}{c}\bigvee_{i=1}^{l}\left(1-\left(1-(\Xi_i^{t-})^{\delta}\right)^{\kappa_i}\right)+\left(1-\left(1-(\Xi_{l+1}^{t-})^{\delta}\right)^{\kappa_{l+1}}\right)\\ -\bigcirc_{i=1}^{l}\left(1-\left(1-(\Xi_i^{t-})^{\delta}\right)^{\kappa_i}\right)\cdot\left(1-\left(1-(\Xi_{l+1}^{t-})^{\delta}\right)^{\kappa_{l+1}}\right)\end{array}},\\ \sqrt[\delta]{\begin{array}{c}\bigvee_{i=1}^{l}\left(1-\left(1-(\Xi_i^{t+})^{\delta}\right)^{\kappa_i}\right)+\left(1-\left(1-(\Xi_{l+1}^{t+})^{\delta}\right)^{\kappa_{l+1}}\right)\\ +\bigcirc_{i=1}^{l}\left(1-\left(1-(\Xi_i^{t+})^{\delta}\right)^{\kappa_i}\right)\cdot\left(1-\left(1-(\Xi_{l+1}^{t+})^{\delta}\right)^{\kappa_{l+1}}\right)\end{array}},\end{array}\right]\\ \left[\begin{array}{c}\sqrt[lcm(\delta,\varepsilon)]{\begin{array}{c}\bigvee_{i=1}^{l}\left(1-\left(1-(\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}\right)+\left(1-\left(1-(\Xi_{l+1}^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_{l+1}}\right)\\ -\bigcirc_{i=1}^{l}\left(1-\left(1-(\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}\right)\cdot\left(1-\left(1-(\Xi_{l+1}^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_{l+1}}\right)\end{array}},\\ \sqrt[lcm(\delta,\varepsilon)]{\begin{array}{c}\bigvee_{i=1}^{l}\left(1-\left(1-(\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}\right)+\left(1-\left(1-(\Xi_{l+1}^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_{l+1}}\right)\\ +\bigcirc_{i=1}^{l}\left(1-\left(1-(\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}\right)\cdot\left(1-\left(1-(\Xi_{l+1}^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_{l+1}}\right)\end{array}},\end{array}\right]\\ \left[\bigcirc_{i=1}^{l}((\Xi_i^{f-})^{\varepsilon})^{\kappa_i}\cdot((\Xi_{l+1}^{f-})^{\varepsilon})^{\kappa_{l+1}},\bigcirc_{i=1}^{l}((\Xi_i^{f+})^{\varepsilon})^{\kappa_i}\cdot((\Xi_{l+1}^{f+})^{\varepsilon})^{\kappa_{l+1}}\right]\end{bmatrix}$$

$$=\begin{bmatrix}\left[\sqrt[\delta]{1-\bigcirc_{i=1}^{l+1}\left(1-(\Xi_i^{t-})^{\delta}\right)^{\kappa_i}},\sqrt[\delta]{1-\bigcirc_{i=1}^{l+1}\left(1-(\Xi_i^{t+})^{\delta}\right)^{\kappa_i}}\right]\\ \left[\begin{array}{c}\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l+1}\left(1-(\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}},\\ \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l+1}\left(1-(\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\end{array}\right]\\ \left[\bigcirc_{i=1}^{l+1}((\Xi_i^{f-})^{\varepsilon})^{\kappa_i},\bigcirc_{i=1}^{l+1}((\Xi_i^{f+})^{\varepsilon})^{\kappa_i}\right]\end{bmatrix}.$$

**Theorem 4.3.** *Let* $\ell_i=\left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ *be the* $(\delta,\varepsilon)$ *IVNNs. Then* $(\delta,\varepsilon)$ *IVNWA* $(\ell_1,\ell_2,...,\ell_n)=\ell$ *(idempotency property).*

**Proof.** Since $\Xi_i^{t-}=\Xi^{t-}$ , $\Xi_i^{m-}=\Xi^{m-}$ and $\Xi_i^{f-}=\Xi^{f-}$, $\Xi_i^{t+}=\Xi^{t+}$ , $\Xi_i^{m+}=\Xi^{m+}$ and $\Xi_i^{f+}=\Xi^{f+}$ and

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$\bigvee_{i=1}^{n}\kappa_i = 1$. Now, $(\delta,\varepsilon)IVNWA(\ell_1,\ell_2,...,\ell_n)$

$$
\begin{aligned}
&= \left[\begin{array}{c}
\left[\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{t-})^{\delta}\right)^{\kappa_i}}, \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{t+})^{\delta}\right)^{\kappa_i}}\right] \\
\left[\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}, \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right] \\
\left[\bigcirc_{i=1}^{n}((\Xi_i^{f-})^{\varepsilon})^{\kappa_i}, \bigcirc_{i=1}^{n}((\Xi_i^{f+})^{\varepsilon})^{\kappa_i}\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\left[\sqrt[\delta]{1-\left(1-(\Xi^{t-})^{\delta}\right)^{\bigvee_{i=1}^{n}\kappa_i}}, \sqrt[\delta]{1-\left(1-(\Xi^{t+})^{\delta}\right)^{\bigvee_{i=1}^{n}\kappa_i}}\right] \\
\left[\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-(\Xi^{m-})^{lcm(\delta,\varepsilon)}\right)^{\bigvee_{i=1}^{n}\kappa_i}}, \sqrt[lcm(\delta,\varepsilon)]{1-\left(1-(\Xi^{m+})^{lcm(\delta,\varepsilon)}\right)^{\bigvee_{i=1}^{n}\kappa_i}}\right] \\
\left[((\Xi^{f-})^{\varepsilon})^{\bigvee_{i=1}^{n}\kappa_i}, ((\Xi^{f+})^{\varepsilon})^{\bigvee_{i=1}^{n}\kappa_i}\right]
\end{array}\right] \\
&= \left[\begin{array}{c}
\left[\sqrt[\delta]{1-\left(1-(\Xi^{t-})^{\delta}\right)}, \sqrt[\delta]{1-\left(1-(\Xi^{t+})^{\delta}\right)}\right] \\
\left[\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-(\Xi^{m-})^{lcm(\delta,\varepsilon)}\right)}, \sqrt[lcm(\delta,\varepsilon)]{1-\left(1-(\Xi^{m+})^{lcm(\delta,\varepsilon)}\right)}\right] \\
\left[(\Xi^{f-})^{\varepsilon}, (\Xi^{f+})^{\varepsilon}\right]
\end{array}\right] \\
&= \ell.
\end{aligned}
$$

**Theorem 4.4.** *Let $\ell_i = \left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ be the $(\delta,\varepsilon)$ IVNNs. Then $(\delta,\varepsilon)$ IVNWA$(\ell_1,\ell_2,...,\ell_n)$, where $\underbrace{\Xi^{t-}} = \min \Xi_{ij}^{t-}$, $\overbrace{\Xi^{t-}} = \max \Xi_{ij}^{t-}$, $\underbrace{\Xi^{m-}} = \min \Xi_{ij}^{m-}$, $\overbrace{\Xi^{m-}} = \max \Xi_{ij}^{m-}$, $\underbrace{\Xi^{f-}} = \min \Xi_{ij}^{f-}$, $\overbrace{\Xi^{f-}} = \max \Xi_{ij}^{f-}$, $\underbrace{\Xi^{t+}} = \min \Xi_{ij}^{t+}$, $\overbrace{\Xi^{t+}} = \max \Xi_{ij}^{t+}$, $\underbrace{\Xi^{m+}} = \min \Xi_{ij}^{m+}$, $\overbrace{\Xi^{m+}} = \max \Xi_{ij}^{m+}$, $\underbrace{\Xi^{f+}} = \min \Xi_{ij}^{f+}$, $\overbrace{\Xi^{f+}} = \max \Xi_{ij}^{f+}$ and where $1 \leq i \leq n$, $j = 1,2,...,i_j$. Then, $\left\langle[\underbrace{\Xi^{t-}},\underbrace{\Xi^{t+}}],[\underbrace{\Xi^{m-}},\underbrace{\Xi^{m+}}],[\overbrace{\Xi^{f-}},\overbrace{\Xi^{f+}}]\right\rangle$*

$$
\begin{aligned}
&\leq (\delta,\varepsilon)IVNWA(\ell_1,\ell_2,...,\ell_n) \\
&\leq \left\langle[\overbrace{\Xi^{t-}},\overbrace{\Xi^{t+}}],[\overbrace{\Xi^{m-}},\overbrace{\Xi^{m+}}],[\underbrace{\Xi^{f-}},\underbrace{\Xi^{f+}}]\right\rangle.
\end{aligned}
$$

*(Boundedness property).*

**Proof.** Since, $\underbrace{\Xi^{t-}} = \min \Xi_{ij}^{t-}$, $\overbrace{\Xi^{t-}} = \max \Xi_{ij}^{t-}$ and $\underbrace{\Xi^{t-}} \leq \Xi_{ij}^{t-} \leq \overbrace{\Xi^{t-}}$.

Now, $\underbrace{\Xi^{t-}} = \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\underbrace{\Xi^{t-}})^{\delta}\right)^{\kappa_i}} \leq \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ij}^{t-})^{\delta}\right)^{\kappa_i}} \leq \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\overbrace{\Xi^{t-}})^{\delta}\right)^{\kappa_i}} = \overbrace{\Xi^{t-}}$.

Since $\underbrace{\Xi^{t+}} = \min \Xi_{ij}^{t+}$, $\overbrace{\Xi^{t+}} = \max \Xi_{ij}^{t+}$ and $\underbrace{\Xi^{t+}} \leq \Xi_{ij}^{t+} \leq \overbrace{\Xi^{t+}}$. Now,

$\underbrace{\Xi^{t+}} = \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\underbrace{\Xi^{t+}})^{\delta}\right)^{\kappa_i}} \leq \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ij}^{t+})^{\delta}\right)^{\kappa_i}} \leq \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\overbrace{\Xi^{t+}})^{\delta}\right)^{\kappa_i}} = \overbrace{\Xi^{t+}}$.

Since, $\underbrace{\Xi^{m-}} = \min \Xi_{ij}^{m-}$, $\overbrace{\Xi^{m-}} = \max \Xi_{ij}^{m-}$ and $\underbrace{\Xi^{m-}} \leq \Xi_{ij}^{m-} \leq \overbrace{\Xi^{m-}}$.

Now, $\underbrace{\Xi^{m-}} = \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\underbrace{\Xi^{m-}})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} \leq \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ij}^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} \leq \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\overbrace{\Xi^{m-}})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} = \overbrace{\Xi^{m-}}$.

Since, $\underbrace{\Xi^{m+}} = \min \Xi_{ij}^{m+}$, $\overbrace{\Xi^{m+}} = \max \Xi_{ij}^{m+}$ and $\underbrace{\Xi^{m+}} \leq \Xi_{ij}^{m+} \leq \overbrace{\Xi^{m+}}$.

Now, $\underbrace{\Xi^{m+}} = \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\underbrace{\Xi^{m+}})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} \leq \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ij}^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} \leq$

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$\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\overbrace{\Xi^{m+}})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} = \overbrace{\Xi^{m+}}$.

Since, $\underbrace{(\Xi^{f-})^{\varepsilon}} = \min(\Xi_{ij}^{f-})^{\varepsilon}$, $\overbrace{(\Xi^{f-})^{\varepsilon}} = \max(\Xi_{ij}^{f-})^{\varepsilon}$ and $\underbrace{(\Xi^{f-})^{\varepsilon}} \leq (\Xi_{ij}^{f-})^{\varepsilon} \leq \overbrace{(\Xi^{f-})^{\varepsilon}}$. We have, $\underbrace{(\Xi^{f-})^{\varepsilon}} = \bigcirc_{i=1}^{n}(\underbrace{(\Xi^{f-})^{\varepsilon}})^{\kappa_i} \leq \bigcirc_{i=1}^{n}((\Xi_{ij}^{f-})^{\varepsilon})^{\kappa_i} \leq \bigcirc_{i=1}^{n}(\overbrace{(\Xi^{f-})^{\varepsilon}})^{\kappa_i} = \overbrace{(\Xi^{f-})^{\varepsilon}}$.

Since, $\underbrace{(\Xi^{f+})^{\varepsilon}} = \min(\Xi_{ij}^{f+})^{\varepsilon}$, $\overbrace{(\Xi^{f+})^{\varepsilon}} = \max(\Xi_{ij}^{f+})^{\varepsilon}$ and $\underbrace{(\Xi^{f+})^{\varepsilon}} \leq (\Xi_{ij}^{f+})^{\varepsilon} \leq \overbrace{(\Xi^{f+})^{\varepsilon}}$. We have, $\underbrace{(\Xi^{f+})^{\varepsilon}} = \bigcirc_{i=1}^{n}(\underbrace{(\Xi^{f+})^{\varepsilon}})^{\kappa_i} \leq \bigcirc_{i=1}^{n}((\Xi_{ij}^{f+})^{\varepsilon})^{\kappa_i} \leq \bigcirc_{i=1}^{n}(\overbrace{(\Xi^{f+})^{\varepsilon}})^{\kappa_i} = \overbrace{(\Xi^{f+})^{\varepsilon}}$.

Therefore,

$$\frac{1}{2}\times\left[\begin{array}{c}\dfrac{\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\underbrace{\Xi^{t-}})^{\delta}\right)^{\kappa_i}}\right)^2+\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\underbrace{\Xi^{t+}})^{\delta}\right)^{\kappa_i}}\right)^2}{2} \\ -\dfrac{\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\underbrace{\Xi^{m-}})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2+\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\underbrace{\Xi^{m+}})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2}{2} \\ +1-\dfrac{\left(\bigcirc_{i=1}^{n}((\overbrace{\Xi^{f-}})^{\varepsilon})^{\kappa_i}\right)^2+\left(\bigcirc_{i=1}^{n}((\overbrace{\Xi^{f+}})^{\varepsilon})^{\kappa_i}\right)^2}{2}\end{array}\right]$$

$$\leq \frac{1}{2}\times\left[\begin{array}{c}\dfrac{\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ij}^{t-})^{\delta}\right)^{\kappa_i}}\right)^2+\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ij}^{t+})^{\delta}\right)^{\kappa_i}}\right)^2}{2} \\ -\dfrac{\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ij}^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2+\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ij}^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2}{2} \\ +1-\dfrac{\left(\bigcirc_{i=1}^{n}((\Xi_{ij}^{f-})^{\varepsilon})^{\kappa_i}\right)^2+\left(\bigcirc_{i=1}^{n}((\Xi_{ij}^{f+})^{\varepsilon})^{\kappa_i}\right)^2}{2}\end{array}\right]$$

$$\leq \frac{1}{2}\times\left[\begin{array}{c}\dfrac{\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\overbrace{\Xi^{t-}})^{\delta}\right)^{\kappa_i}}\right)^2+\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\overbrace{\Xi^{t+}})^{\delta}\right)^{\kappa_i}}\right)^2}{2} \\ -\dfrac{\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\overbrace{\Xi^{m-}})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2+\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\overbrace{\Xi^{m+}})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2}{2} \\ +1-\dfrac{\left(\bigcirc_{i=1}^{n}((\underbrace{\Xi^{f-}})^{\varepsilon})^{\kappa_i}\right)^2+\left(\bigcirc_{i=1}^{n}((\underbrace{\Xi^{f+}})^{\varepsilon})^{\kappa_i}\right)^2}{2}\end{array}\right].$$

Hence, $\left\langle [\underbrace{\Xi^{t-}}, \underbrace{\Xi^{t+}}], [\underbrace{\Xi^{m-}}, \underbrace{\Xi^{m+}}], [\overbrace{\Xi^{f-}}, \overbrace{\Xi^{f+}}] \right\rangle \leq (\delta,\varepsilon)IVNWA(\ell_1, \ell_2, ..., \ell_n)$
$\leq \left\langle [\overbrace{\Xi^{t-}}, \overbrace{\Xi^{t+}}], [\overbrace{\Xi^{m-}}, \overbrace{\Xi^{m+}}], [\underbrace{\Xi^{f-}}, \underbrace{\Xi^{f+}}] \right\rangle$.

**Theorem 4.5.** *Let* $\ell_i = \left\langle \left( [\Xi_{k_{ij}}^{t-}, \Xi_{k_{ij}}^{t+}], [\Xi_{k_{ij}}^{m-}, \Xi_{k_{ij}}^{m+}], [\Xi_{k_{ij}}^{f-}, \Xi_{k_{ij}}^{f+}] \right) \right\rangle$ *and* $W_i = \left\langle \left( [\Xi_{h_{ij}}^{t-}, \Xi_{h_{ij}}^{t+}], [\Xi_{h_{ij}}^{m-}, \Xi_{h_{ij}}^{m+}], [\Xi_{h_{ij}}^{f-}, \Xi_{h_{ij}}^{f+}] \right) \right\rangle$ *be the* $(\delta,\varepsilon)$ *IVNWAs. For any* $i$, *if there is* $(\Xi_{k_{ij}}^{t-})^2 \leq (\Xi_{h_{ij}}^{t-})^2$ *and* $(\Xi_{k_{ij}}^{m-})^2 \leq (\Xi_{h_{ij}}^{m-})^2$ *and* $(\Xi_{k_{ij}}^{f-})^2 \geq (\Xi_{h_{ij}}^{f-})^2$ *and* $(\Xi_{k_{ij}}^{t+})^2 \leq (\Xi_{h_{ij}}^{t+})^2$ *and* $(\Xi_{k_{ij}}^{m+})^2 \leq (\Xi_{h_{ij}}^{m+})^2$ *and* $(\Xi_{k_{ij}}^{f+})^2 \geq (\Xi_{h_{ij}}^{f+})^2$ *or* $\ell_i \leq W_i$. *Prove that* $(\delta,\varepsilon)IVNWA(\ell_1, \ell_2, ..., \ell_n) \leq (\delta,\varepsilon)IVNWA(W_1, W_2, ..., W_n)$, *where* $(i = 1, 2, ..., n); (j = 1, 2, ..., i_j)$ *(monotonicity property).*

**Proof.** For any $i$, $(\Xi_{k_{ij}}^{t-})^2 \leq \left(\Xi_{h_{ij}}^{t-}\right)^2$. Therefore, $1-(\Xi_{k_i}^{t-})^2 \geq 1-(\Xi_{h_i}^{t-})^2$.
Hence, $\bigcirc_{i=1}^{n}\left(1-(\Xi_{k_i}^{t-})^2\right)^{\kappa_i} \geq \bigcirc_{i=1}^{n}\left(1-(\Xi_{h_i}^{t-})^2\right)^{\kappa_i}$
and $\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{k_i}^{t-})^{\delta}\right)^{\kappa_i}} \leq \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{h_i}^{t-})^{\delta}\right)^{\kappa_i}}$.
For any $i$, $(\Xi_{k_{ij}}^{t+})^2 \leq \left(\Xi_{h_{ij}}^{t+}\right)^2$. Therefore, $1-(\Xi_{k_i}^{t+})^2 \geq 1-(\Xi_{h_i}^{t+})^2$.
Hence, $\bigcirc_{i=1}^{n}\left(1-(\Xi_{k_i}^{t+})^2\right)^{\kappa_i} \geq \bigcirc_{i=1}^{n}\left(1-(\Xi_{h_i}^{t+})^2\right)^{\kappa_i}$


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and $\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-\left(\Xi_{k_i}^{t+}\right)^{\delta}\right)^{\kappa_i}} \le \sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-\left(\Xi_{h_i}^{t+}\right)^{\delta}\right)^{\kappa_i}}$.

For any $i$, $(\Xi_{k_{ij}}^{m-})^{lcm(\delta,\varepsilon)} \le \left(\Xi_{h_{ij}}^{m-}\right)^{lcm(\delta,\varepsilon)}$. Therefore, $1-\left(\Xi_{k_i}^{m-}\right)^{lcm(\delta,\varepsilon)} \ge 1-\left(\Xi_{h_i}^{m-}\right)^{lcm(\delta,\varepsilon)}$.

Hence, $\bigcirc_{i=1}^{n}\left(1-\left(\Xi_{k_i}^{m-}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i} \ge \bigcirc_{i=1}^{n}\left(1-\left(\Xi_{h_i}^{m-}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}$.

This implies that $\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-\left(\Xi_{k_i}^{m-}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} \le \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-\left(\Xi_{h_i}^{m-}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}$.

For any $i$, $(\Xi_{k_{ij}}^{m+})^{lcm(\delta,\varepsilon)} \le \left(\Xi_{h_{ij}}^{m+}\right)^{lcm(\delta,\varepsilon)}$. Therefore, $1-\left(\Xi_{k_i}^{m+}\right)^{lcm(\delta,\varepsilon)} \ge 1-\left(\Xi_{h_i}^{m+}\right)^{lcm(\delta,\varepsilon)}$.

Hence, $\bigcirc_{i=1}^{n}\left(1-\left(\Xi_{k_i}^{m+}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i} \ge \bigcirc_{i=1}^{n}\left(1-\left(\Xi_{h_i}^{m+}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}$.

This implies that $\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-\left(\Xi_{k_i}^{m+}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} \le \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-\left(\Xi_{h_i}^{m+}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}$.

For any $i$, $\left(\Xi_{k_{ij}}^{f-}\right)^2 \ge \left(\Xi_{h_{ij}}^{f-}\right)^2$ and $\left(\Xi_{k_{ij}}^{f-}\right)^{\varepsilon} \ge \left(\Xi_{h_{ij}}^{f-}\right)^{\varepsilon}$. Therefore, $1-\frac{\left(\bigcirc_{i=1}^{n}\Xi_{k_{ij}}^{f-}\right)^{\varepsilon}}{2} \le 1-\frac{\left(\bigcirc_{i=1}^{n}\Xi_{h_{ij}}^{f-}\right)^{\varepsilon}}{2}$.

For any $i$, $\left(\Xi_{k_{ij}}^{f+}\right)^2 \ge \left(\Xi_{h_{ij}}^{f+}\right)^2$ and $\left(\Xi_{k_{ij}}^{f+}\right)^{\varepsilon} \ge \left(\Xi_{h_{ij}}^{f+}\right)^{\varepsilon}$. Therefore, $1-\frac{\left(\bigcirc_{i=1}^{n}\Xi_{k_{ij}}^{f+}\right)^{\varepsilon}}{2} \le 1-\frac{\left(\bigcirc_{i=1}^{n}\Xi_{h_{ij}}^{f+}\right)^{\varepsilon}}{2}$.

$$\begin{aligned}
&\frac{1}{2}\times\left[\begin{array}{c}\frac{\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ti}^{t-})^{\delta}\right)^{\kappa_i}}\right)^2+\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ti}^{t+})^{\delta}\right)^{\kappa_i}}\right)^2}{2}\\ -\frac{\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ti}^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2+\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{ti}^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2}{2}\\ +1-\frac{\left(\bigcirc_{i=1}^{n}(\Xi_{tij}^{f-})^{\varepsilon}\right)^2+\left(\bigcirc_{i=1}^{n}(\Xi_{tij}^{f+})^{\varepsilon}\right)^2}{2}\end{array}\right]\\
\le\;&\frac{1}{2}\times\left[\begin{array}{c}\frac{\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{hi}^{t-})^{\delta}\right)^{\kappa_i}}\right)^2+\left(\sqrt[\delta]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{hi}^{t+})^{\delta}\right)^{\kappa_i}}\right)^2}{2}\\ -\frac{\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{hi}^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2+\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_{hi}^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^2}{2}\\ +1-\frac{\left(\bigcirc_{i=1}^{n}(\Xi_{hij}^{f-})^{\varepsilon}\right)^2+\left(\bigcirc_{i=1}^{n}(\Xi_{hij}^{f+})^{\varepsilon}\right)^2}{2}\end{array}\right].
\end{aligned}$$

Hence, $(\delta,\varepsilon)IVNWA\left(\ell_1,\ell_2,...,\ell_n\right) \le (\delta,\varepsilon)IVNWA\left(W_1,W_2,...,W_n\right)$.

### 4.2 $(\delta,\varepsilon)$ IVNWG

**Definition 4.6.** Let $\ell_i=\left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ be the $(\delta,\varepsilon)$IVNNs. Then $(\delta,\varepsilon)$IVNWG $(\ell_1,\ell_2,...,\ell_n)=\bigcirc_{i=1}^{n}\ell_i^{\kappa_i}$.

**Theorem 4.7.** *Let $\ell_i=\left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ be the $(\delta,\varepsilon)$ IVNNs. Then $(\delta,\varepsilon)$ IVNWG $(\ell_1,\ell_2,...,\ell_n)$*

$$=\left[\begin{array}{c}\left[\bigcirc_{i=1}^{n}((\Xi_i^{t-})^{\delta})^{\kappa_i},\bigcirc_{i=1}^{n}((\Xi_i^{t+})^{\delta})^{\kappa_i}\right]\\ \left[\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}},\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right]\\ \left[\sqrt[\varepsilon]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{f-})^{\varepsilon}\right)^{\kappa_i}},\sqrt[\varepsilon]{1-\bigcirc_{i=1}^{n}\left(1-(\Xi_i^{f+})^{\varepsilon}\right)^{\kappa_i}}\right]\end{array}\right].$$

**Theorem 4.8.** *Let $\ell_i=\left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ be the $(\delta,\varepsilon)$IVNNs and all are equal. Then $(\delta,\varepsilon)$IVNWG$(\ell_1,\ell_2,...,\ell_n)=\ell$.*

**Remark 4.9.** It has other properties, including boundedness and monotonicity, as well as having $(\delta,\varepsilon)$IVNWG.

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### 4.3 Generalized $(\delta,\varepsilon)$ IVNWA (G$(\delta,\varepsilon)$IVNWA)

**Definition 4.10.** Let $\ell_i = \left\langle \left( [\Xi_i^{t-}, \Xi_i^{t+}], [\Xi_i^{m-}, \Xi_i^{m+}], [\Xi_i^{f-}, \Xi_i^{f+}] \right) \right\rangle$ be the $(\delta,\varepsilon)$IVNN. Then G$(\delta,\varepsilon)$ IVNWA $(\ell_1, \ell_2, ..., \ell_n) = \left( \bigvee_{i=1}^{n} \kappa_i \ell_i^{\Re} \right)^{1/\Re}$.

**Theorem 4.11.** *Let $\ell_i = \left\langle \left( [\Xi_i^{t-}, \Xi_i^{t+}], [\Xi_i^{m-}, \Xi_i^{m+}], [\Xi_i^{f-}, \Xi_i^{f+}] \right) \right\rangle$ be the $(\delta,\varepsilon)$ IVNNs. Then G$(\delta,\varepsilon)$ IVNWA* $(\ell_1, \ell_2, ..., \ell_n)$

$$= \begin{bmatrix} \left[ \left( \sqrt[\delta]{1 - \bigcirc_{i=1}^{n} \left( 1 - \left( (\Xi_i^{t-})^{\delta} \right)^{\delta} \right)^{\kappa_i}} \right)^{1/\delta}, \left( \sqrt[\delta]{1 - \bigcirc_{i=1}^{n} \left( 1 - \left( (\Xi_i^{t+})^{\delta} \right)^{\delta} \right)^{\kappa_i}} \right)^{1/\delta} \right] \\ \left[ \begin{array}{c} \left( \sqrt[lcm(\delta,\varepsilon)]{1 - \bigcirc_{i=1}^{n} \left( 1 - \left( (\Xi_i^{m-})^{lcm(\delta,\varepsilon)} \right)^{lcm(\delta,\varepsilon)} \right)^{\kappa_i}} \right)^{1/lcm(\delta,\varepsilon)}, \\ \left( \sqrt[lcm(\delta,\varepsilon)]{1 - \bigcirc_{i=1}^{n} \left( 1 - \left( (\Xi_i^{m+})^{lcm(\delta,\varepsilon)} \right)^{lcm(\delta,\varepsilon)} \right)^{\kappa_i}} \right)^{1/lcm(\delta,\varepsilon)} \end{array} \right] \\ \left[ \begin{array}{c} \sqrt[\varepsilon]{1 - \left( 1 - \left( \bigcirc_{i=1}^{n} \left( \sqrt[\varepsilon]{1 - \left( 1 - (\Xi_i^{f-})^{\varepsilon} \right)^{\varepsilon}} \right)^{\kappa_i} \right)^{\varepsilon} \right)^{1/\varepsilon}}, \\ \sqrt[\varepsilon]{1 - \left( 1 - \left( \bigcirc_{i=1}^{n} \left( \sqrt[\varepsilon]{1 - \left( 1 - (\Xi_i^{f+})^{\varepsilon} \right)^{\varepsilon}} \right)^{\kappa_i} \right)^{\varepsilon} \right)^{1/\varepsilon}} \end{array} \right] \end{bmatrix}.$$

**Proof.** We can prove this first by demonstrating that,

$$\bigvee_{i=1}^{n} \kappa_i \ell_i^{\delta} = \begin{bmatrix} \left[ \sqrt[\delta]{1 - \bigcirc_{i=1}^{n} \left( 1 - \left( (\Xi_i^{t-})^{\delta} \right)^{\delta} \right)^{\kappa_i}}, \sqrt[\delta]{1 - \bigcirc_{i=1}^{n} \left( 1 - \left( (\Xi_i^{t+})^{\delta} \right)^{\delta} \right)^{\kappa_i}} \right] \\ \left[ \begin{array}{c} \sqrt[lcm(\delta,\varepsilon)]{1 - \bigcirc_{i=1}^{n} \left( 1 - \left( (\Xi_i^{m-})^{lcm(\delta,\varepsilon)} \right)^{lcm(\delta,\varepsilon)} \right)^{\kappa_i}}, \\ \sqrt[lcm(\delta,\varepsilon)]{1 - \bigcirc_{i=1}^{n} \left( 1 - \left( (\Xi_i^{m+})^{lcm(\delta,\varepsilon)} \right)^{lcm(\delta,\varepsilon)} \right)^{\kappa_i}} \end{array} \right] \\ \left[ \bigcirc_{i=1}^{n} \left( \sqrt[\varepsilon]{1 - \left( 1 - (\Xi_i^{f-})^{\varepsilon} \right)^{\varepsilon}} \right)^{\kappa_i}, \bigcirc_{i=1}^{n} \left( \sqrt[\varepsilon]{1 - \left( 1 - (\Xi_i^{f+})^{\varepsilon} \right)^{\varepsilon}} \right)^{\kappa_i} \right] \end{bmatrix}.$$

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Put $n = 2$, $\kappa_1\ell_1 \bigvee \kappa_2\ell_2$

$$= \left[\begin{array}{c} \left[\begin{array}{c} \sqrt[\delta]{\begin{array}{c} \left(\sqrt[\delta]{1-\left(1-\left((\Xi_1^{t-})^{\delta}\right)^{\delta}\right)^{\kappa_1}}\right)^{\delta} + \left(\sqrt[\delta]{1-\left(1-\left((\Xi_2^{t-})^{\delta}\right)^{\delta}\right)^{\kappa_1}}\right)^{\delta}, \\ -\left(\sqrt[\delta]{1-\left(1-\left((\Xi_1^{t-})^{\delta}\right)^{\delta}\right)^{\kappa_1}}\right)^{\delta} \cdot \left(\sqrt[\delta]{1-\left(1-\left((\Xi_2^{t-})^{\delta}\right)^{\delta}\right)^{\kappa_1}}\right)^{\delta} \end{array}} \\ \sqrt[\delta]{\begin{array}{c} \left(\sqrt[\delta]{1-\left(1-\left((\Xi_1^{t+})^{\delta}\right)^{\delta}\right)^{\kappa_1}}\right)^{\delta} + \left(\sqrt[\delta]{1-\left(1-\left((\Xi_2^{t+})^{\delta}\right)^{\delta}\right)^{\kappa_1}}\right)^{\delta}, \\ -\left(\sqrt[\delta]{1-\left(1-\left((\Xi_1^{t+})^{\delta}\right)^{\delta}\right)^{\kappa_1}}\right)^{\delta} \cdot \left(\sqrt[\delta]{1-\left(1-\left((\Xi_2^{t+})^{\delta}\right)^{\delta}\right)^{\kappa_1}}\right)^{\delta} \end{array}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[lcm(\delta,\varepsilon)]{\begin{array}{c} \left(\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-\left((\Xi_1^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right)^{lcm(\delta,\varepsilon)} + \\ \left(\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-\left((\Xi_2^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right)^{lcm(\delta,\varepsilon)} \\ -\left(\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-\left((\Xi_1^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right)^{lcm(\delta,\varepsilon)} \cdot \\ \left(\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-\left((\Xi_2^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right)^{lcm(\delta,\varepsilon)} \end{array}} \\ , \\ \sqrt[lcm(\delta,\varepsilon)]{\begin{array}{c} \left(\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-\left((\Xi_1^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right)^{lcm(\delta,\varepsilon)} + \\ \left(\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-\left((\Xi_2^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right)^{lcm(\delta,\varepsilon)} \\ -\left(\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-\left((\Xi_1^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right)^{lcm(\delta,\varepsilon)} \cdot \\ \left(\sqrt[lcm(\delta,\varepsilon)]{1-\left(1-\left((\Xi_2^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_1}}\right)^{lcm(\delta,\varepsilon)} \end{array}} \end{array}\right] \\ \left[\begin{array}{c} \left(\sqrt[\varepsilon]{1-\left(1-(\Xi_1^{f-})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_1} \cdot \left(\sqrt[\varepsilon]{1-\left(1-(\Xi_2^{f-})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_1}, \\ \left(\sqrt[\varepsilon]{1-\left(1-(\Xi_1^{f+})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_1} \cdot \left(\sqrt[\varepsilon]{1-\left(1-(\Xi_2^{f+})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_1} \end{array}\right] \end{array}\right]$$

$$= \left[\begin{array}{c} \left[\sqrt[\delta]{1-\bigcirc_{i=1}^{2}\left(1-\left((\Xi_1^{t-})^{\delta}\right)^{\delta}\right)^{\kappa_i}}, \sqrt[\delta]{1-\bigcirc_{i=1}^{2}\left(1-\left((\Xi_1^{t+})^{\delta}\right)^{\delta}\right)^{\kappa_i}}\right] \\ \left[\begin{array}{c} \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{2}\left(1-\left((\Xi_1^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}, \\ \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{2}\left(1-\left((\Xi_1^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}} \end{array}\right] \\ \left[\bigcirc_{i=1}^{2}\left(\sqrt[\varepsilon]{1-\left(1-(\Xi_i^{f-})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_i}, \bigcirc_{i=1}^{2}\left(\sqrt[\varepsilon]{1-\left(1-(\Xi_i^{f+})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_i}\right] \end{array}\right].$$

Hence,

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$$\bigvee_{i=1}^{l}\kappa_i\ell_i^{\Re}=\left[\begin{array}{c}\left[\sqrt[\delta]{1-\bigcirc_{i=1}^{l}\left(1-\left((\Xi_1^{t-})^{\delta}\right)^{\delta}\right)^{\kappa_i}},\sqrt[\delta]{1-\bigcirc_{i=1}^{l}\left(1-\left((\Xi_1^{t+})^{\delta}\right)^{\delta}\right)^{\kappa_i}}\right]\\ \left[\begin{array}{c}\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l}\left(1-\left((\Xi_1^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}},\\ \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l}\left(1-\left((\Xi_1^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\end{array}\right]\\ \left[\bigcirc_{i=1}^{l}\left(\sqrt[\varepsilon]{1-\left(1-(\Xi_i^{f-})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_i},\bigcirc_{i=1}^{l}\left(\sqrt[\varepsilon]{1-\left(1-(\Xi_i^{f+})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_i}\right]\end{array}\right].$$

If $n=l+1$, then $\bigvee_{i=1}^{l}\kappa_i\ell_i^{\Re}+\kappa_{l+1}\ell_{l+1}^{\Re}=\bigvee_{i=1}^{l+1}\kappa_i\ell_i^{\Re}$.
Now,$\bigvee_{i=1}^{l}\kappa_i\ell_i^{\Re}+\kappa_{l+1}\ell_{l+1}^{\Re}=\kappa_1\ell_1^{\Re}\bigvee\kappa_2\ell_2^{\Re}\bigvee...\bigvee\kappa_l\ell_l^{\Re}\bigvee\kappa_{l+1}\ell_{l+1}^{\Re}$

$$\bigvee_{i=1}^{l+1}\kappa_i\ell_i^{\Re}=\left[\begin{array}{c}\left[\sqrt[\delta]{1-\bigcirc_{i=1}^{l+1}\left(1-\left((\Xi_i^{t-})^{\delta}\right)^{\delta}\right)^{\kappa_i}},\sqrt[\delta]{1-\bigcirc_{i=1}^{l+1}\left(1-\left((\Xi_i^{t+})^{\delta}\right)^{\delta}\right)^{\kappa_i}}\right]\\ \left[\begin{array}{c}\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l+1}\left(1-\left((\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}},\\ \sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l+1}\left(1-\left((\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\end{array}\right]\\ \left[\bigcirc_{i=1}^{l+1}\left(\sqrt[\varepsilon]{1-\left(1-(\Xi_i^{f-})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_i},\bigcirc_{i=1}^{l+1}\left(\sqrt[\varepsilon]{1-\left(1-(\Xi_i^{f+})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_i}\right]\end{array}\right].$$

$$\left(\bigvee_{i=1}^{l+1}\kappa_i\ell_i^{\Re}\right)^{1/\Re}=\left[\begin{array}{c}\left[\left(\sqrt[\Re]{1-\bigcirc_{i=1}^{l+1}\left(1-\left((\Xi_i^{t-})^{\delta}\right)^{\delta}\right)^{\kappa_i}}\right)^{1/\delta},\left(\sqrt[\Re]{1-\bigcirc_{i=1}^{l+1}\left(1-\left((\Xi_i^{t+})^{\delta}\right)^{\delta}\right)^{\kappa_i}}\right)^{1/\delta}\right]\\ \left[\begin{array}{c}\left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l+1}\left(1-\left((\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^{1/lcm(\delta,\varepsilon)},\\ \left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{l+1}\left(1-\left((\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^{1/lcm(\delta,\varepsilon)}\end{array}\right]\\ \left[\begin{array}{c}\sqrt[\varepsilon]{1-\left(1-\left(\bigcirc_{i=1}^{l+1}\left(\sqrt[\varepsilon]{1-\left(1-(\Xi_i^{f-})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_i}\right)^{2}\right)^{1/\varepsilon}},\\ \sqrt[\varepsilon]{1-\left(1-\left(\bigcirc_{i=1}^{l+1}\left(\sqrt[\varepsilon]{1-\left(1-(\Xi_i^{f+})^{\varepsilon}\right)^{\varepsilon}}\right)^{\kappa_i}\right)^{2}\right)^{1/\varepsilon}}\end{array}\right]\end{array}\right].$$

It is valid for any $l$.

**Remark 4.12.** An operator modified from the G$(\delta,\varepsilon)$ IVNWA operator to the $(\delta,\varepsilon)$ IVNWA operator is performed if $\Re=1$.

**Theorem 4.13.** *If all $\ell_i=\left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ and all are equal. Then G$(\delta,\varepsilon)$ IVNWA$(\ell_1,\ell_2,...,\ell_n)=\ell$.*

**Remark 4.14.** In the G$(\delta,\varepsilon)$ IVNWA operator, boundedness and monotonicity are satisfied.

### 4.4 Generalized $(\delta,\varepsilon)$IVNWG ( G$(\delta,\varepsilon)$IVNWG)

**Definition 4.15.** Let $\ell_i=\left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ be the $(\delta,\varepsilon)$IVNNs. Then G$(\delta,\varepsilon)$IVNWG $(\ell_1,\ell_2,...,\ell_n)=\frac{1}{\Re}\left(\bigcirc_{i=1}^{n}(\Re\ell_i)^{\kappa_i}\right)$.

**Theorem 4.16.** *Let $\ell_i=\left\langle\left([\Xi_i^{t-},\Xi_i^{t+}],[\Xi_i^{m-},\Xi_i^{m+}],[\Xi_i^{f-},\Xi_i^{f+}]\right)\right\rangle$ be the $(\delta,\varepsilon)$ IVNNs. Then G$(\delta,\varepsilon)$IVNWG$(\ell_1,\ell_2,...,\ell_n)$*


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$$= \begin{bmatrix} \begin{bmatrix} \sqrt[\delta]{1-\left(1-\left(\bigcirc_{i=1}^{n}\left(\sqrt[\delta]{1-\left(1-(\Xi_i^{t-})^{\delta}\right)^{\delta}}\right)^{\kappa_i}\right)^{\delta}\right)^{1/\delta}}, \\ \sqrt[\delta]{1-\left(1-\left(\bigcirc_{i=1}^{n}\left(\sqrt[\delta]{1-\left(1-(\Xi_i^{t+})^{\delta}\right)^{\delta}}\right)^{\kappa_i}\right)^{\delta}\right)^{1/\delta}} \end{bmatrix} \\ \begin{bmatrix} \left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-\left((\Xi_i^{m-})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^{1/lcm(\delta,\varepsilon)}, \\ \left(\sqrt[lcm(\delta,\varepsilon)]{1-\bigcirc_{i=1}^{n}\left(1-\left((\Xi_i^{m+})^{lcm(\delta,\varepsilon)}\right)^{lcm(\delta,\varepsilon)}\right)^{\kappa_i}}\right)^{1/lcm(\delta,\varepsilon)} \end{bmatrix} \\ \left[\left(\sqrt[\varepsilon]{1-\bigcirc_{i=1}^{n}\left(1-\left((\Xi_i^{f-})^{\varepsilon}\right)^{\varepsilon}\right)^{\kappa_i}}\right)^{1/\varepsilon}, \left(\sqrt[\varepsilon]{1-\bigcirc_{i=1}^{n}\left(1-\left((\Xi_i^{f+})^{\varepsilon}\right)^{\varepsilon}\right)^{\kappa_i}}\right)^{1/\varepsilon}\right] \end{bmatrix}.$$

**Remark 4.17.** There is a conversion that takes place when $\Re = 1$, which converts the G$(\delta, \varepsilon)$ IVNWG into the $(\delta, \varepsilon)$IVNWG.

**Remark 4.18.** Boundness and monotonicity properties that are satisfied by G$(\delta, \varepsilon)$IVNWG operators.

**Theorem 4.19.** *If all* $\ell_i = \left\langle\left([\Xi_i^{t-}, \Xi_i^{t+}], [\Xi_i^{m-}, \Xi_i^{m+}], [\Xi_i^{f-}, \Xi_i^{f+}]\right)\right\rangle$ *are equal.*
*Then* $G(\delta,\varepsilon)IVNWG(\ell_1, \ell_2, ..., \ell_n) = \ell$.

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