Generalized p-Transmuted Neutrosophic Distributions: Theory and its Applications

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Abstract

The study of neutrosophy offers a fresh approach for handling uncertain data with adaptability. This article explores the application of neutrosophic probability distribution in constructing a transmuted neutrosophic framework. Specifically, it introduces a generalized transmuted neutrosophic distribution. Building upon this generalization, quadratic and cubic transmuted distributions are developed and examined alongside certain lifetime distributions serving as foundational neutrosophic models. Additionally, an empirical investigation is conducted to assess the practicality and versatility of these distributions in real-world contexts.

Keywords: Generalized p-transmuted distributions; neutrosophic distribution; life time distributions; empirical study

1 Introduction

Probability distributions play a crucial role across various domains, with standard distributions being extensively employed for a considerable time. The efficacy of statistical data analysis procedures heavily relies on the assumed probability models or distributions of phenomena, leading to the development of numerous standard distributions and associated statistical methodologies by researchers. However, practical challenges persist where real-world data diverges from standard probability distributions. Consequently, there has been a surge in interest among scholars towards extending and generalizing probability distributions, particularly in recent decades, as evidenced by applications in biomedical sciences, engineering, environmental studies, finance, insurance, economics, and climatology. In these fields, continuous univariate distributions are often exceptions rather than the norm, necessitating the development of modified, extended, and generalized distributions to address pertinent issues.

These adaptations typically involve introducing transformations or additional parameters to existing well-known baseline probability distributions, thereby improving their suitability for real-life data and enhancing model flexibility. Noteworthy examples include the Beta-G distributions,[4] Transmuted family of distributions,[7] Gamma-G (type 1) distributions,[6] Kumaraswamy-G distributions,[8] McDonald-G distributions,[9] Gamma-G (type 2) distributions,[10] Gamma-G (type 3) distributions,[11] and T-X family distributions.[12]

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The transmuted family of distributions received a lot of attention by several notable authors in the past years. The quadratic transmutation map to generate new probability distribution using any baseline distribution to solve problems related to financial mathematics has been introduced.[9] The quadratic transmuted family of distributions is the composition of the cumulative distribution and its quantile function of the baseline distribution.

1.1 Generalized p-Transmuted Family of Distributions

To accommodate the intricacies of complex data, Rahman et al.,[13] introduced the concept of general transmuted families (GT-G) of distributions. This family, denoted as the p-transmuted family, is defined as follows:

Consider a random variable X with a cumulative density function (CDF) F(x). Then, the p-transmuted family is characterized by the following definition:

$$F(x) = G(x) + [1 - G(x)] \sum_{i=1}^{p} \lambda_i \left[G(x)\right]^i ; x \in \mathbb{R}, \tag{1}$$

$with\ \lambda_i \in [-1,1]\ for\ i = 1,2,...,p\ and\ -p \leq \sum_{i=1}^{p} \lambda_i \leq 1$
where, λ is the transmutation parameter and $G(x)$ is the CDF of baseline distribution.

The general transmuted family reduces to the base distribution for $\lambda_i = 0 \quad \forall \quad i = 1,2,...,p$ The PDF corresponding to Eqn.(1) is,

$$f(x) = g(x) \left[1 - \sum_{i=1}^{p} \lambda_i G^i(x) + [1 - G(x)] \sum_{i=1}^{p} i\lambda_i G^{i-1}(x)\right] \tag{2}$$

where $g(x)$ is the probability density function (PDF) of the base distribution.

The cumulative density function (CDF) of the quadratic family of transmuted distributions, as proposed by Shaw et al.,,[7] is derived by setting p=1 in Eqn.(1).

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x);\ x \in \mathbb{R}\ and\ \lambda \in [-1,1] \tag{3}$$

The probability density function (PDF) of the quadratic transmuted distribution corresponding to Eqn.(3) is expressed as:

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)]; x \in \mathbb{R}\ and\ \lambda \in [-1,1] \tag{4}$$

When dealing with multi-modal distributions, the quadratic transmuted distribution may not always suffice. Thus, Rahman et al.,[13] introduced the cubic transmuted distribution family. The cumulative density function (CDF) of this family is derived by setting p=2 in Eqn.(1), resulting in the following definition:

$$F(x) = (1+\lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x) \tag{5}$$

$with\ \lambda_1 \in [-1,1], \lambda_2 \in [-1,1]\ and\ -2 \leq \lambda_1 + \lambda_2 \leq 1$

The PDF of the above family corresponding to Eqn. (5) is,

$$f(x) = g(x) \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1) G(x) - 3\lambda_2 G^2(x)\right] \tag{6}$$

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1.2 Neutrosophic Approach

Neutrosophic statistics, introduced by Smarandache in 1995, extends classical statistics and serves as a philosophical branch. It is presented as a generalization of fuzzy logic[1] and intuitionistic fuzzy logic,[2] addressing issues arising from aberrant, unclear, vague, imprecise, incomplete, extreme, and indeterminate data.[14] Smarandache (1999) introduced neutrosophic logic as a generalized version of fuzzy logic, offering a framework that encompasses various existing logics.[3] Neutrosophic logic operates within a three-dimensional neutrosophic space, describing each proposition in terms of its degree of truth, indeterminacy, and falsity.[15] The neutrosophic number assumes a standard form depicted as: $X_N = D + i$ where D is the determined part of data and i is the indeterminacy part of data.

1.3 Some Existing Neutrosophic Lifetime Probability Distributions

1. **Neutrosophic Exponential Distribution**
The Neutrosophic Exponential Distribution (NED) serves as a generalization of the classical exponential distribution.[14] For a neutrosophic random variable $X_N \sim exp(\theta_N)$, the CDF of NED as:
$$G(X, \theta_N) = 1 - e^{-\theta_N X}; X, \theta_N > 0 \tag{7}$$
where, θ_N is the neutrosophic rate parameter.
Also, the PDF of NED is:
$$g(X, \theta_N) = \theta_N e^{-\theta_N X}; X, \theta_N > 0 \tag{8}$$

2. **Neutrosophic Weibull Distribution**
A Neutrosophic Weibull Distribution (NWD) of a continuous variable X is a classical Weibull distribution of x, but such that its parameters α or β are indeterminate.[16] Then the CDF is:
$$G(X, \alpha_N, \beta_N) = 1 - e^{-\left(\frac{X}{\alpha_N}\right)^{\beta_N}}; X, \alpha \; and \; \beta > 0 \tag{9}$$
where, α_N and β_N are the scale and shape parameter. The PDF of NWD is given below:
$$g(X, \alpha_N, \beta_N) = \frac{\beta_N}{\alpha_N^{\beta_N}} X^{\beta_N - 1} e^{-\left(\frac{X}{\alpha_N}\right)^{\beta_N}}; X, \alpha \; and \; \beta > 0 \tag{10}$$

3. **Neutrosophic Rayleigh Distribution**
The Neutrosophic model of the Rayleigh distribution (NRD) of a continuous random variable X, having indeterminacy in the scale parameter θ_N has the CDF:[21]
$$G(X, \theta_N) = 1 - e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}; X, \theta_N > 0 \tag{11}$$
and the PDF is:
$$g(X, \theta_N) = \frac{X}{\theta_N^2} e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}; X, \theta_N > 0 \tag{12}$$

4. **Neutrosophic Pareto Distribution**
Let X be a continuous random variable is said to be Neutrosophic Pareto Distribution (NPD), with neutrosophic shape and scale parameters,α_N and θ_N respectively, then the neutrosophic distribution function of NPD is given by,[?]
$$G(X, \alpha_N, \theta_N) = 1 - \left(\frac{\theta_N}{X}\right)^{\alpha_N}; \; X, \alpha_N \; and \; \theta_N > 0 \tag{13}$$
So, the PDF of NPD is of the form;
$$g(X, \alpha_N, \theta_N) = \frac{\alpha_N \theta_N^{\alpha_N}}{X^{\alpha_N + 1}}; \; X, \alpha_N \; and \; \theta_N > 0 \tag{14}$$

In this paper, focused on and studies the Generalized p-Transmuted Neutrosophic distribution and discusses some properties of these distributions, illustrated through examples and graphs. In section 3, an estimation of the neutrosophic parameters and a real-life application are presented. Finally, in Section 4, some conclusions alongwith future works are given.

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2 Generalized p-Transmuted Neutrosophic Distribution

A Generalized p-transmuted Neutrosophic Distribution (G-TND) of a continuous variable X is a classical Generalized p-transmuted distribution of x, but such that the parameters of the base distributions are unclear or imprecise. Then, the CDF and PDF of the distribution using Eqn. (1) and Eqn. (2) is:

$$F_N(x) = G(x,\alpha_N) + [1 - G(x,\alpha_N)] \sum_{i=1}^{p} \lambda_i \left[G(x,\alpha_N)\right]^i ; x \in \mathbb{R}, \tag{15}$$

$with\ \lambda_i \in [-1,1]\ \ for\ i = 1,2,...,p\ and\ -p \leq \sum_{i=1}^{p} \lambda_i \leq 1$

$$f_N(X) = g(x,\alpha_N) \left[1 - \sum_{i=1}^{p} \lambda_i G^i(x,\alpha_N) + [1 - G(x,\alpha_N)] \sum_{i=1}^{p} i\lambda_i G^{i-1}(x,\alpha_N)\right] \tag{16}$$

where, $\alpha_N = (\alpha_{1N}, \alpha_{2N}, ..., \alpha_{kN})$ are the neutrosophic parameter and λ is the transmutation parameter $with\ \lambda_i \in [-1,1]\ \ for\ i = 1,2,...,p\ and\ -p \leq \sum_{i=1}^{p} \lambda_i \leq 1$
where, G(x,α_N) and g(x,α_N) is the CDF and PDF of the neutrosophic baseline distribution.

2.1 Quadratic Transmuted Neutrosophic Distributions(QTND)

The CDF and PDF of quadratic transmuted neutrosophic distribution is obtained by setting p=1 in Eqn. (15) and Eqn. (16) and is given as,

$$F_N(x) = (1+\lambda)G(x,\alpha_N) - \lambda G^2(x,\alpha_N);\ x \in \mathbb{R}\ and\ \lambda \in [-1,1] \tag{17}$$

$$f_N(x) = g(x,\alpha_N)[1 + \lambda - 2\lambda G(x,\alpha_N)];\ x \in \mathbb{R}\ and\ \lambda \in [-1,1] \tag{18}$$

In the following, we discussed some instances of quadratic transmuted neutrosophic distributions for different choices of baseline neutrosophic distributions with some functions in reliability analysis.

1. **Quadratic Transmuted Neutrosophic Exponential Distribution**

   The CDF of Quadratic Transmuted Neutrosophic Exponential Distribution (QTNED) is obtained by using Eqn. (7) in Eqn. (17) as follows:

$$F_N(X) = \left(1 - e^{-\theta_N X}\right)\left[1 + \lambda e^{-\theta_N X}\right] \tag{19}$$

   Hence, the PDF is;

$$f_N(X) = \theta_N e^{-\theta_N X}\left[1 - \lambda + 2\lambda e^{-\theta_N X}\right] \tag{20}$$

   The density and CDF curve of QTNED for $\theta_N \in [0.5, 1.5]$ and λ =0.5 is depicted in Figure 1.

   a. Reliability Properties Of QTNED

   - The reliability function for the quadratic transmuted neutrosophic exponential random variables is given by:

$$R_N(X) = \lambda e^{-2\theta_N X} + (1-\lambda)e^{-\theta_N X}$$

   - The hazard rate function is:

$$h_N(X) = \frac{\theta_N[1 - \lambda + 2\lambda e^{-\theta_N X}]}{[1 - \lambda + \lambda e^{-\theta_N X}]}$$

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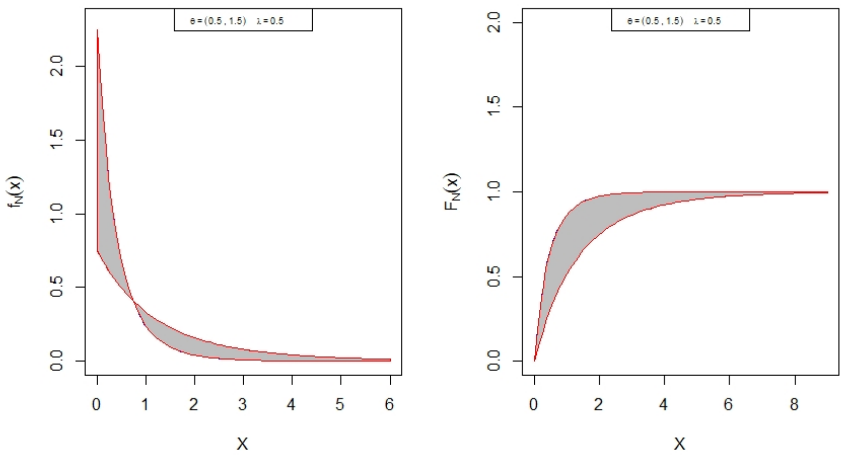

Figure 1.

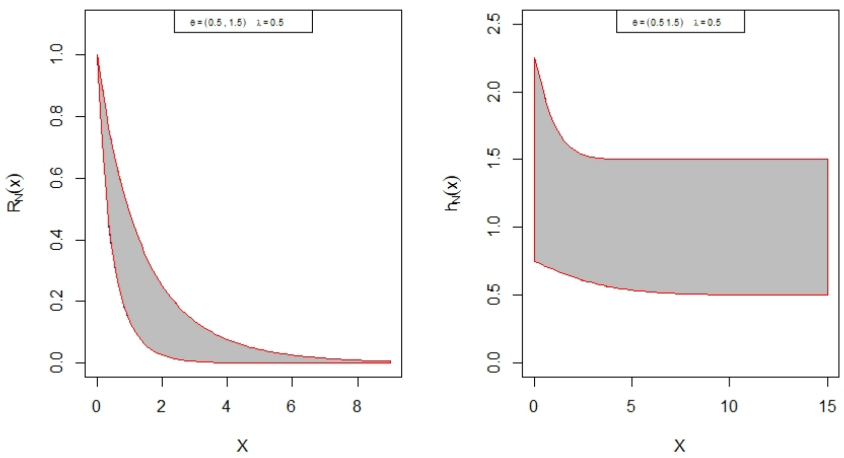

Figure 2.

Figure 2 illustrates the reliability function and hazard rate function of the proposed QTNED for the model parameters $\theta_N \in [0.5, 1.5]$ and λ =0.5

b. Quantile Function Of QTNED

The quantile function for QTNED is obtained by solving Eqn.(19) for x and is obtained as,

$$x_q = \frac{1}{\theta_N}\left[-ln\left(1-\left[\frac{(\lambda+1)-\sqrt{(\lambda+1)^2-4\lambda u}}{2\lambda}\right]\right)\right]$$

2. **Quadratic Transmuted Neutrosophic Weibull Distribution**

Now, using Eqn. (9) in Eqn. (17), the CDF of Quadratic Transmuted Neutrosophic Weibull Distribution (QTNWD) is as follows:

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$$F_N(X) = \left[1 - e^{-(\frac{X}{\alpha_N})^{\beta_N}}\right]\left[1 + \lambda e^{-(\frac{X}{\alpha_N})^{\beta_N}}\right] \quad (21)$$

and its PDF as given below

$$f_N(X) = \frac{\beta_N}{\alpha_N^{\beta_N}} X^{\beta_N - 1} e^{-(\frac{X}{\alpha_N})^{\beta_N}} \left[1 - \lambda + 2\lambda e^{-(\frac{X}{\alpha_N})^{\beta_N}}\right] \quad (22)$$

Figure 3. illustrates the shape of PDF and CDF of QTNWD keeping $\alpha_N \in [1, 1.5]$, $\beta_N \in [1.5, 2]$ and $\lambda = 0.5$

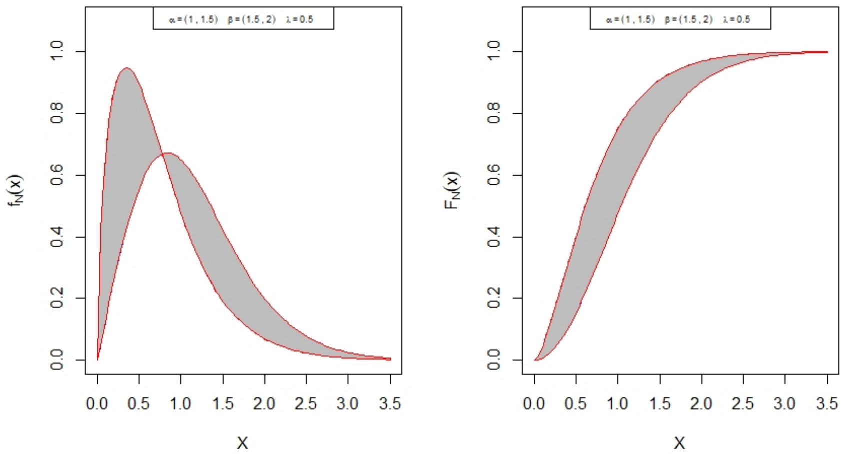

Figure 3.

a. Reliability Properties Of QTNWD

- The reliability function is:

$$R_N(X) = e^{-(\frac{X}{\alpha_N})^{\beta_N}}\left[1 - \lambda + \lambda e^{-(\frac{X}{\alpha_N})^{\beta_N}}\right]$$

- The hazard rate function for QTNWD is given by:

$$h_N(X) = \frac{\frac{\beta_N}{\alpha_N^{\beta_N}} X^{\beta_N - 1}\left[1 - \lambda + 2\lambda e^{-(\frac{X}{\alpha_N})^{\beta_N}}\right]}{\left[1 - \lambda + \lambda e^{-(\frac{X}{\alpha_N})^{\beta_N}}\right]}$$

The reliability and hazard rate behavior of QTNWD is given in Figure 4 for the parameter values $\alpha_N \in [1, 1.5]$, $\beta_N \in [1.5, 2]$ and $\lambda = 0.5$ and $\alpha_N \in [0.9, 1.2]$, $\beta_N \in [1.5, 5]$ and $\lambda = 1$ respectively.

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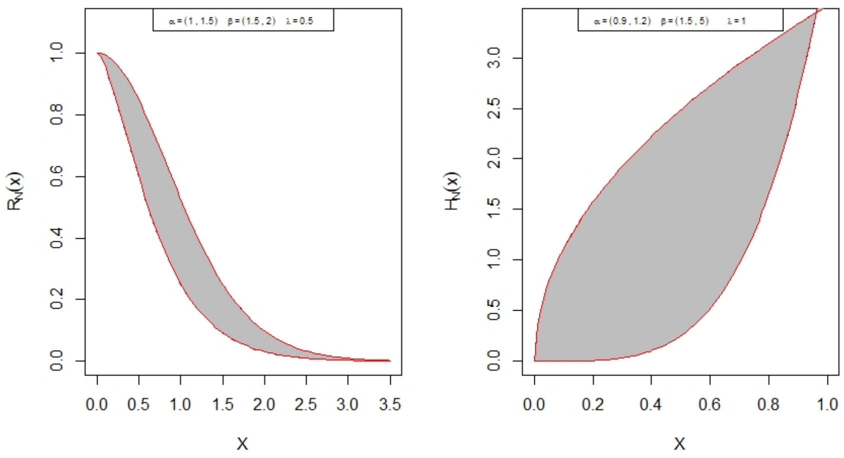

Figure 4.

b. Quantile Function Of QTNWD

The quantile function for the proposed distribution QTNWD is obtained by solving Eqn. (21) for x and is obtained as,

$$x_q = \alpha_N \left[-ln \left(1 - \left[\frac{(\lambda+1) - \sqrt{(\lambda+1)^2 - 4\lambda u}}{2\lambda} \right] \right) \right]^{\frac{1}{\beta}}$$

3. **Quadratic Transmuted Neutrosophic Rayleigh Distribution**

If a random variable X has a Neutrosophic Rayleigh distribution with the cumulative distribution function given in Eqn.(11) and the probability density function given in Eqn.(12). The corresponding Quadratic Transmuted Neutrosophic Rayleigh Distribution (QTNRD), using the quadratic rank transmutation map is given by,

$$F_N(X) = \left(1 - e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}\right)\left(1 + \lambda e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}\right) \tag{23}$$

and the corresponding PDF is:

$$f_N(X) = \frac{X}{\theta_N^2} e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}\left[1 - \lambda + 2\lambda e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}\right] \tag{24}$$

The graphical representation of PDF and CDF for QTNRD with neutrosophic parameter θ_N=[0.5,0.75] and transmuted parameter $\lambda = -0.5$ is given in Figure 5.

a. Reliability Properties Of QTNRD

- The reliability function is given by:

$$R_N(X) = e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}\left[1 - \lambda + \lambda e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}\right]$$

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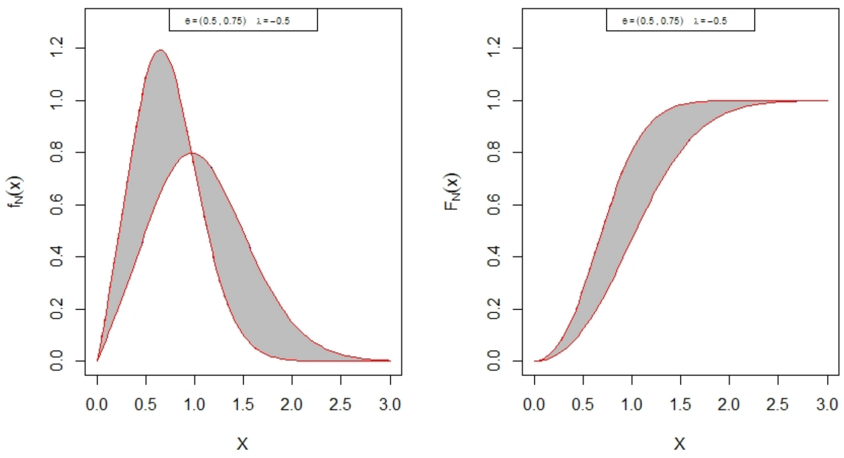

Figure 5.

- The hazard rate function is:

$$h_N(X) = \frac{X}{\theta_N^2} \frac{\left[1 - \lambda + 2\lambda e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}\right]}{\left[1 - \lambda + \lambda e^{-\frac{1}{2}\left(\frac{X}{\theta_N}\right)^2}\right]}$$

The following graphs (Figure 6.) show the shape of the survival and hazard rate function of QTNRD.

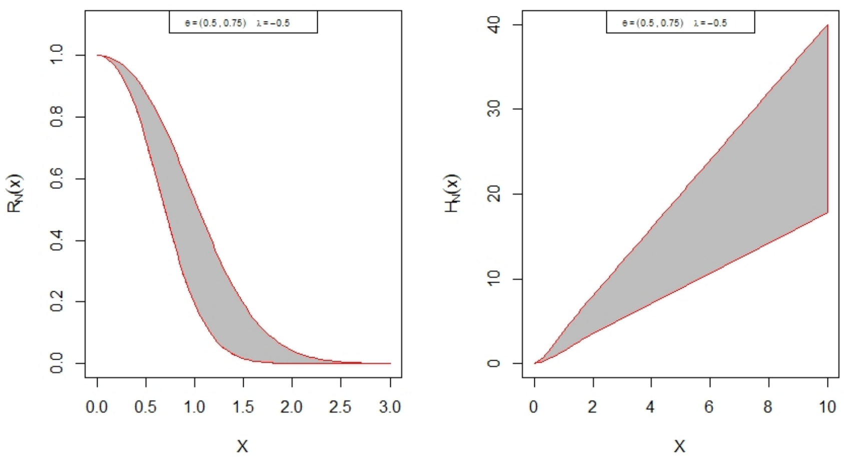

Figure 6.

b. Quantile Function Of QTNRD

The quantile function of QTNRD is the solution of the following equation

$$x_q = \theta_N \sqrt{-2ln\left[\frac{(\lambda - 1) + \sqrt{(\lambda + 1)^2 - 4\lambda u}}{2\lambda}\right]}$$

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4. **Quadratic Transmuted Neutrosophic Pareto Distribution**
A random variable X is said to follow the QTNPD distribution, if its CDF is given by

$$F_N(X) = \left[1 - \left(\frac{\theta_N}{X}\right)^{\alpha_N}\right]\left[1 + \lambda\left(\frac{\theta_N}{X}\right)^{\alpha_N}\right] \tag{25}$$

and the PDF is;

$$f_N(X) = \frac{\alpha_N \theta_N^{\alpha_N}}{X^{\alpha_N+1}}\left[1 - \lambda + 2\lambda\left(\frac{\theta_N}{X}\right)^{\alpha_N}\right] \tag{26}$$

where, $\theta_N = min(X)$ and $\alpha_N > 0$

Figure 7 shows the shape of PDF and CDF of QTNPD for the parameter values, $\alpha_N \in [4, 6]$, $\theta_N \in [1, 1]$ and $\lambda = 1$

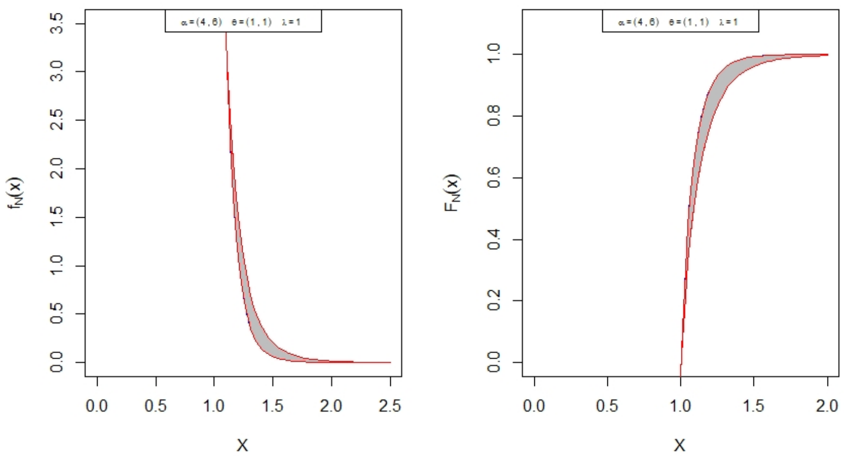

Figure 7.

a. Reliability Properties Of QTNPD

- The reliability function is given by:

$$R_N(X) = \left(\frac{\theta_N}{X}\right)^{\alpha_N}\left[1 - \lambda + \lambda\left(\frac{\theta_N}{X}\right)^{\alpha_N}\right]$$

- The hazard rate function is:

$$h_N(X) = \frac{\alpha_N}{X}\frac{\left[1 - \lambda + 2\lambda\left(\frac{\theta_N}{X}\right)^{\alpha_N}\right]}{\left[1 - \lambda + \lambda\left(\frac{\theta_N}{X}\right)^{\alpha_N}\right]}$$

The plots for the reliability function and hazard rate function are given in Figure 8.

b. Quantile Function Of QTNPD
The proposed QTNPD qth quantile x_q is obtained by using (2.11) as follows

$$x_q = \theta_N\left[\frac{(\lambda - 1) + \sqrt{(1-\lambda)^2 - 4\lambda(u-1)}}{2\lambda}\right]^{\frac{-1}{\alpha}}$$

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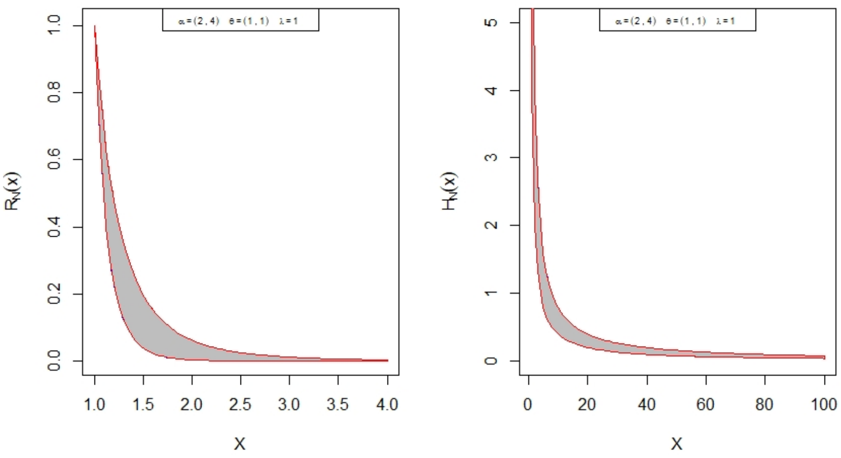

Figure 8.

2.2 Cubic Transmuted Neutrosophic Distributions (CTND)

The cubic transmuted neutrosophic family of distributions is obtained by letting p = 2 in Eqn. (15) and is given as;

$$F_N(x) = (1+\lambda_1)G(x) + (\lambda_2-\lambda_1)G^2(x) - \lambda_2 G^3(x) \tag{27}$$

where, λ_1 and $\lambda_2 \in [-1,1]$ are transmutation parameters and obey the condition $-2 \le \lambda_1+\lambda_2 \le 1$ The PDF corresponding to Eqn. (27) is:

$$f_N(x) = g(x)\left[1+\lambda_1+2(\lambda_2-\lambda_1)G(x)-3\lambda_2 G^2(x)\right] \tag{28}$$

For example, some of the cubic transmuted neutrosophic distributions for different choices of baseline neutrosophic distributions are discussed in the following:

1. **Cubic Transmuted Neutrosophic Exponential Distribution**
   Suppose that the random variable X has neutrosophic exponential distribution, then the CDF and PDF of cubic transmuted neutrosophic exponential distribution (CTNED) are given, respectively, by

$$F_N(X) = 1+(\lambda_1+\lambda_2-1)e^{-\theta_N X} - (\lambda_1+2\lambda_2)e^{-2\theta_N X} + \lambda_2 e^{-3\theta_N X} \tag{29}$$

and

$$f_N(X) = \theta_N e^{-\theta_N X}\left[1+\lambda_1+2(\lambda_2-\lambda_1)\left(1-e^{-\theta_N X}\right)-3\lambda_2\left(1-e^{-\theta_N X}\right)^2\right] \tag{30}$$

Figure 9. represents the captured behaviour of PDF and CDF of CTNED for the values $\theta \in [0.5, 1.5], \lambda_1 = 0.1\ and\ \lambda_2 = 0.5$

a. Reliability Properties of CTNED

- The reliability function,

$$R_N(X) = (\lambda_1+\lambda_2-1)e^{-\theta_N X} - (\lambda_1+2\lambda_2)e^{-2\theta_N X} + \lambda_2 e^{-3\theta_N X}$$

- The hazard rate function,

$$h_N(X) = \frac{\theta_N e^{-\theta_N X}\left[1+\lambda_1+2(\lambda_2-\lambda_1)\left(1-e^{-\theta_N X}\right)-3\lambda_2\left(1-e^{-\theta_N X}\right)^2\right]}{(\lambda_1+\lambda_2-1)e^{-\theta_N X} - (\lambda_1+2\lambda_2)e^{-2\theta_N X} + \lambda_2 e^{-3\theta_N X}}$$

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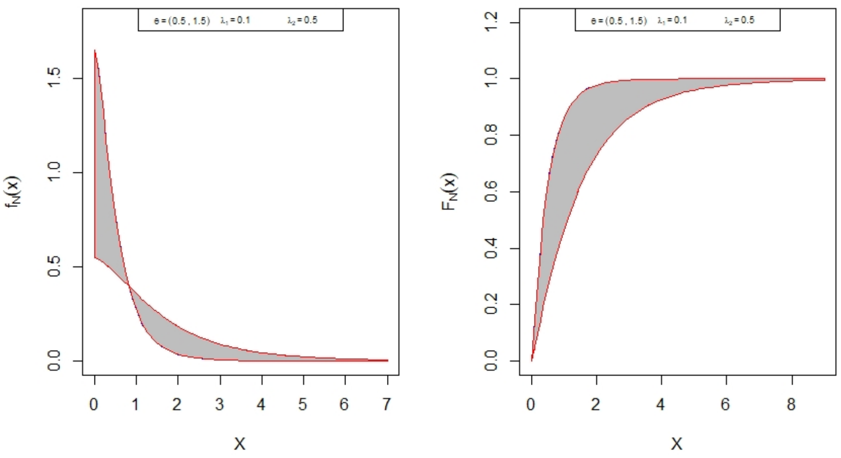

Figure 9.

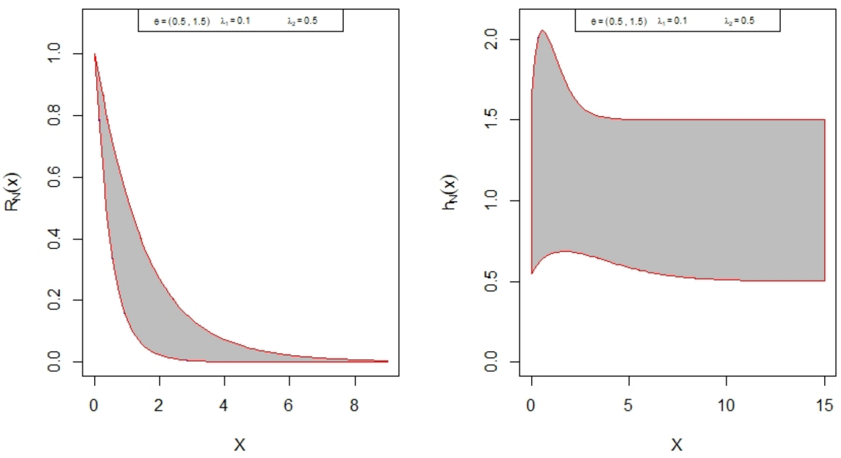

Figure 10.

The reliability function and hazard rate function graphs are plotted in Figure 10, for the model parameters, $\theta \in [0.5, 1.5], \lambda_1 = 0.1 \; and \; \lambda_2 = 0.5$

b. Quantile Function Of CTNED

The qth quantile x_q of the CTNED can be obtained from Eqn. (29) as

$$x_q = -\frac{1}{\theta_N} ln(y)$$

where,

$$y = -\frac{b}{3a} - \frac{2^{\frac{1}{3}}\eta_1}{3a\left(\eta_2 + \sqrt{4\eta_1^3 + \eta_2^2}\right)^{\frac{1}{3}}} + \frac{\left(\eta_2 + \sqrt{4\eta_1^3 + \eta_2^2}\right)^{\frac{1}{3}}}{3\left(2^{\frac{1}{3}}\right)a}$$

$\eta_1 = -b^2 + 3ac; \; \eta_2 = -2b^3 + 9abc - 27a^2d$
$a = \lambda_2; \;\; b = -\lambda_1 - 2\lambda_2$
$c = \lambda_1 + \lambda_2 - 1; \;\; d = 1 - q$

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2. **Cubic Transmuted Neutrosophic Weibull Distribution**
The CDF of Cubic Transmuted Neutrosophic Weibull Distribution (CTNWD) is:

$$F_N(X) = 1 + (\lambda_1 + \lambda_2 - 1)e^{-(\frac{X}{\alpha_N})^{\beta_N}} - (\lambda_1 + 2\lambda_2)e^{-2(\frac{X}{\alpha_N})^{\beta_N}} + \lambda_2 e^{-3(\frac{X}{\alpha_N})^{\beta_N}} \tag{31}$$

PDF is,

$$f_N(X) = \frac{\beta_N}{\alpha_N^{\beta_N}} X^{\beta_N - 1} e^{-3(\frac{X}{\alpha_N})^{\beta_N}} \left[(1 - \lambda_1 - \lambda_2)e^{2(\frac{X}{\alpha_N})^{\beta_N}} + 2(\lambda_1 + 2\lambda_2)e^{(\frac{X}{\alpha_N})^{\beta_N}} - 3\lambda_2\right] \tag{32}$$

Figure 11 provides the PDF and CDF of CTNWD for $\alpha \in [1, 1.5], \beta \in [1.5, 2], \lambda_1 \in 0.5 \; and \; \lambda_2 = -1$

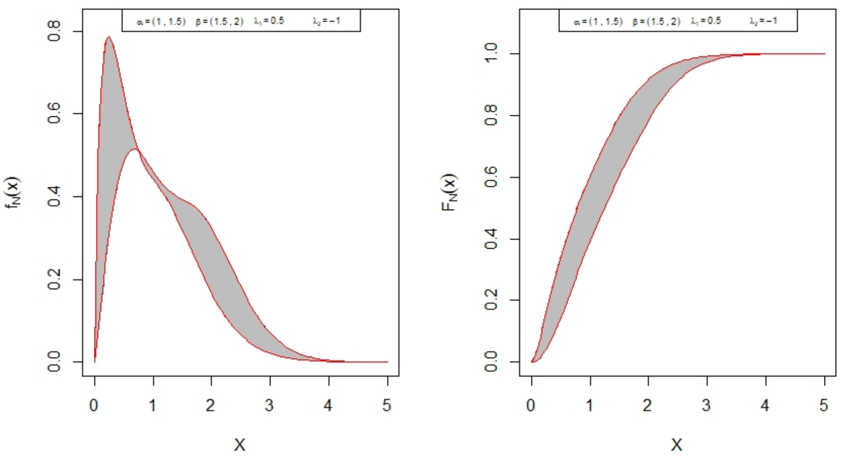

Figure 11.

a. Reliability Properties of CTNWD

- The reliability function,

$$R_N(X) = (1 - \lambda_1 - \lambda_2)e^{-(\frac{X}{\alpha_N})^{\beta_N}} + (\lambda_1 + 2\lambda_2)e^{-2(\frac{X}{\alpha_N})^{\beta_N}} - \lambda_2 e^{-3(\frac{X}{\alpha_N})^{\beta_N}}$$

- The hazard rate function,

$$h_N(X) = \frac{\frac{\beta_N}{\alpha_N^{\beta_N}} X^{\beta_N - 1} e^{-3(\frac{X}{\alpha_N})^{\beta_N}} \left[(1 - \lambda_1 - \lambda_2)e^{2(\frac{X}{\alpha_N})^{\beta_N}} + 2(\lambda_1 + 2\lambda_2)e^{(\frac{X}{\alpha_N})^{\beta_N}} - 3\lambda_2\right]}{(1 - \lambda_1 - \lambda_2)e^{-(\frac{X}{\alpha_N})^{\beta_N}} + (\lambda_1 + 2\lambda_2)e^{-2(\frac{X}{\alpha_N})^{\beta_N}} - \lambda_2 e^{-3(\frac{X}{\alpha_N})^{\beta_N}}}$$

Figure 12 illustrates the graphs of reliability and hazard rate functions for fixed values of parameters.

b. Quantile Function Of CTNWD
The qth quantile x_q of the CTNED can be obtained from Eqn. (31) as

$$x_q = \alpha_N(-ln(y))^{\frac{1}{\beta_N}}$$

where,

$$y = -\frac{b}{3a} - \frac{2^{\frac{1}{3}}\eta_1}{3a\left(\eta_2 + \sqrt{4\eta_1^3 + \eta_2^2}\right)^{\frac{1}{3}}} + \frac{\left(\eta_2 + \sqrt{4\eta_1^3 + \eta_2^2}\right)^{\frac{1}{3}}}{3\left(2^{\frac{1}{3}}\right)a}$$

$\eta_1 = -b^2 + 3ac; \; \eta_2 = -2b^3 + 9abc - 27a^2d$
$a = \lambda_2; \;\; b = -\lambda_1 - 2\lambda_2$
$c = \lambda_1 + \lambda_2 - 1; \;\; d = 1 - q$

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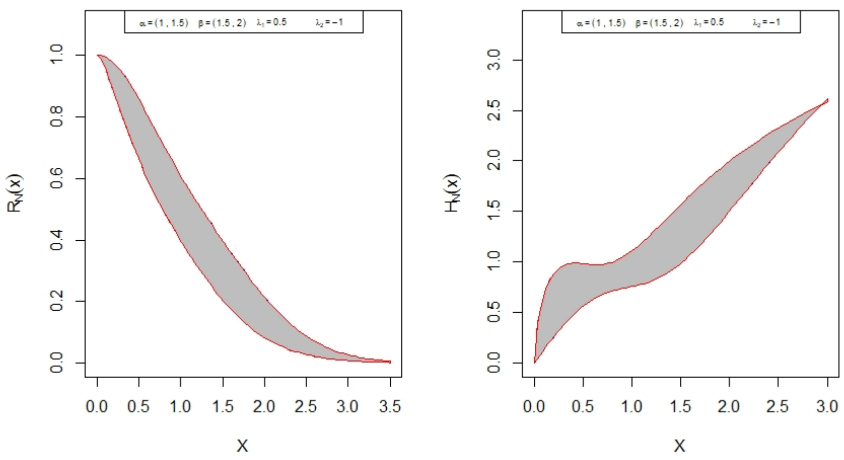

Figure 12.

3. **Cubic Transmuted Neutrosophic Rayleigh Distribution**
The CDF and PDF of Cubic Transmuted Neutrosophic Rayleigh Distribution (CTNRD) are given in the following, respectively,

$$F_N(X) = 1 + (\lambda_1 + \lambda_2 - 1)e^{-\left(\frac{X^2}{2\theta_N^2}\right)} - (\lambda_1 + 2\lambda_2)e^{-2\left(\frac{X^2}{2\theta_N^2}\right)} + \lambda_2 e^{-3\left(\frac{X^2}{2\theta_N^2}\right)} \tag{33}$$

and

$$f_N(X) = \frac{X}{\theta_N^2} e^{-3\left(\frac{X^2}{2\theta_N^2}\right)} \left[(1 - \lambda_1 - \lambda_2)e^{2\left(\frac{X^2}{2\theta_N^2}\right)} + 2(\lambda_1 + 2\lambda_2)e^{\left(\frac{X^2}{2\theta_N^2}\right)} - 3\lambda_2\right] \tag{34}$$

Figure 13 illustrates the shape of PDF and CDF of CTNRD for selected values of the parameters.

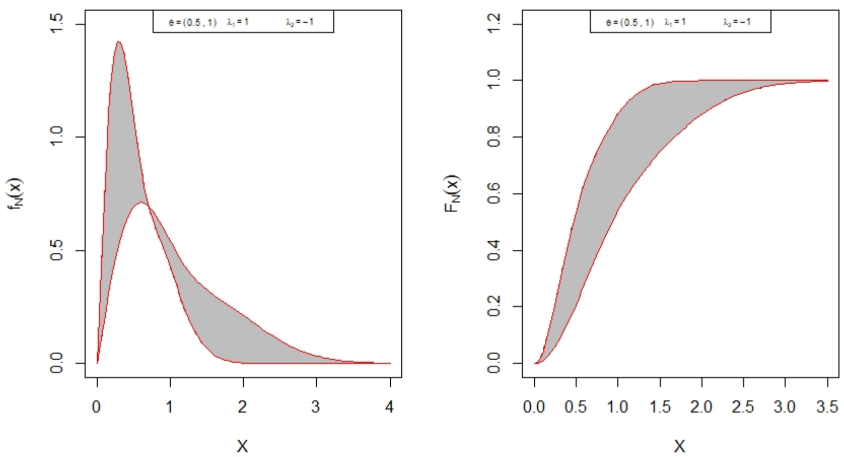

Figure 13.

a. Reliability Properties of CTNRD

- The reliability function,

$$R_N(X) = (1 - \lambda_1 - \lambda_2)e^{-\left(\frac{X^2}{2\theta_N^2}\right)} + (\lambda_1 + 2\lambda_2)e^{-2\left(\frac{X^2}{2\theta_N^2}\right)} - \lambda_2 e^{-3\left(\frac{X^2}{2\theta_N^2}\right)}$$

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- The hazard rate function,

$$h_N(X) = \frac{X}{\theta_N^2} \frac{\left[(1-\lambda_1-\lambda_2)e^{2\left(\frac{X^2}{2\theta_N^2}\right)} + 2(\lambda_1+2\lambda_2)e^{\left(\frac{X^2}{2\theta_N^2}\right)} - 3\lambda_2\right]}{\left[(1-\lambda_1-\lambda_2)e^{2\left(\frac{X^2}{2\theta_N^2}\right)} + (\lambda_1+2\lambda_2)e^{\left(\frac{X^2}{2\theta_N^2}\right)} - \lambda_2\right]}$$

. Figure 14 illustrates the behavior of the reliability and hazard rate function of a CTNRD.

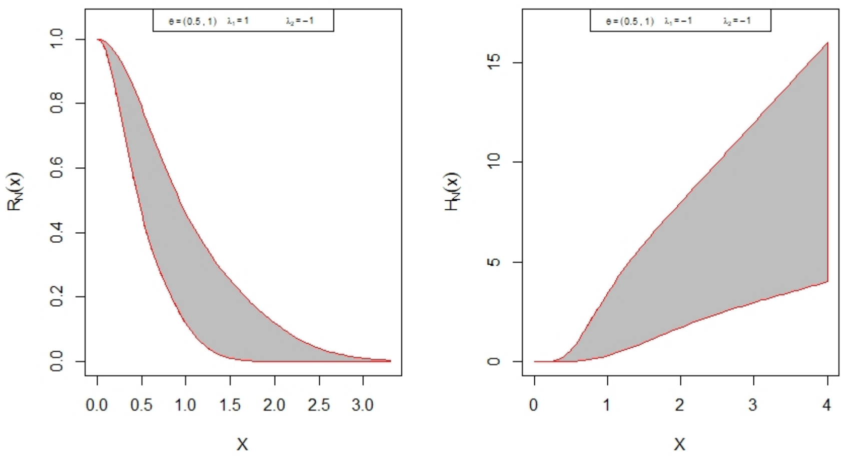

Figure 14.

b. Quantile Function Of CTNRD
The qth quantile x_q of the CTNED can be obtained from Eqn. (33) as

$$x_q = \theta_N\sqrt{-2ln(y)}$$

where,

$$y = -\frac{b}{3a} - \frac{2^{\frac{1}{3}}\eta_1}{3a\left(\eta_2+\sqrt{4\eta_1^3+\eta_2^2}\right)^{\frac{1}{3}}} + \frac{\left(\eta_2+\sqrt{4\eta_1^3+\eta_2^2}\right)^{\frac{1}{3}}}{3\left(2^{\frac{1}{3}}\right)a}$$

$\eta_1 = -b^2+3ac;\ \eta_2 = -2b^3+9abc-27a^2d$
$a = \lambda_2;\ \ b = -\lambda_1-2\lambda_2$
$c = \lambda_1+\lambda_2-1;\ \ \ d = 1-q$

4. **Cubic Transmuted Neutrosophic Pareto Distribution**
The CDF of Cubic Transmuted Pareto Distribution (CTNPD) can be obtained from Eqn. 27 as,

$$F_N(X) = \left[1-\left(\frac{\theta_N}{X}\right)^{\alpha_N}\right]\left[1+(\lambda_1+\lambda_2)\left(\frac{\theta_N}{X}\right)^{\alpha_N} - \lambda_2\left(\frac{\theta_N}{X}\right)^{2\alpha_N}\right] \quad (35)$$

and its corresponding PDF is;

$$f_N(X) = \frac{\alpha_N\theta_N^{\alpha_N}}{X^{\alpha_N+1}}\left[(1-\lambda_1-\lambda_2)+2(\lambda_1+2\lambda_2)\left(\frac{\theta_N}{X}\right)^{\alpha_N} - 3\lambda_2\left(\frac{\theta_N}{X}\right)^{2\alpha_N}\right] \quad (36)$$

Figure 15 presents the PDF and CDF of CTNPD for some selected model parameter values.

a. Reliability Properties of CTNPD

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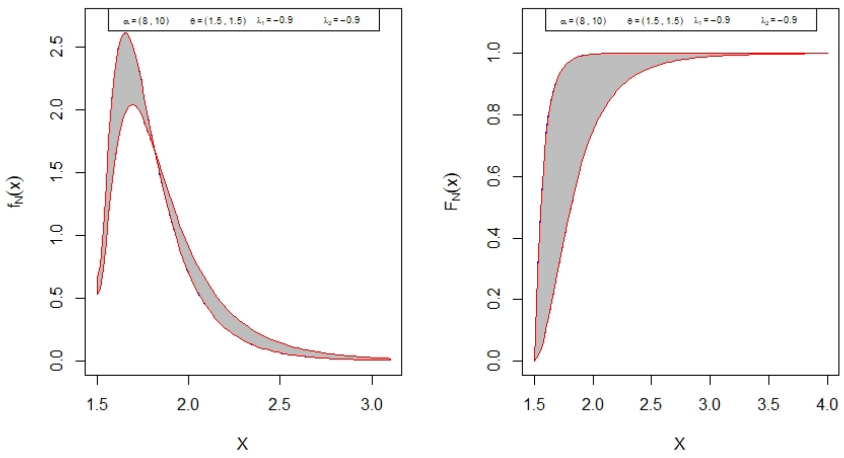

Figure 15.

- The reliability function,

$$R_N(X) = 1 - \left[1 - \left(\frac{\theta_N}{X}\right)^{\alpha_N}\right]\left[1 + (\lambda_1 + \lambda_2)\left(\frac{\theta_N}{X}\right)^{\alpha_N} - \lambda_2\left(\frac{\theta_N}{X}\right)^{2\alpha_N}\right]$$

- The hazard rate function,

$$h_N(X) = \frac{\frac{\alpha_N \theta_N^{\alpha_N}}{X^{\alpha_N+1}}\left[(1-\lambda_1-\lambda_2) + 2(\lambda_1+2\lambda_2)\left(\frac{\theta_N}{X}\right)^{\alpha_N} - 3\lambda_2\left(\frac{\theta_N}{X}\right)^{2\alpha_N}\right]}{1 - \left[1 - \left(\frac{\theta_N}{X}\right)^{\alpha_N}\right]\left[1 + (\lambda_1+\lambda_2)\left(\frac{\theta_N}{X}\right)^{\alpha_N} - \lambda_2\left(\frac{\theta_N}{X}\right)^{2\alpha_N}\right]}$$

The plots for reliability and hazard rate function are plotted in Figure 16.

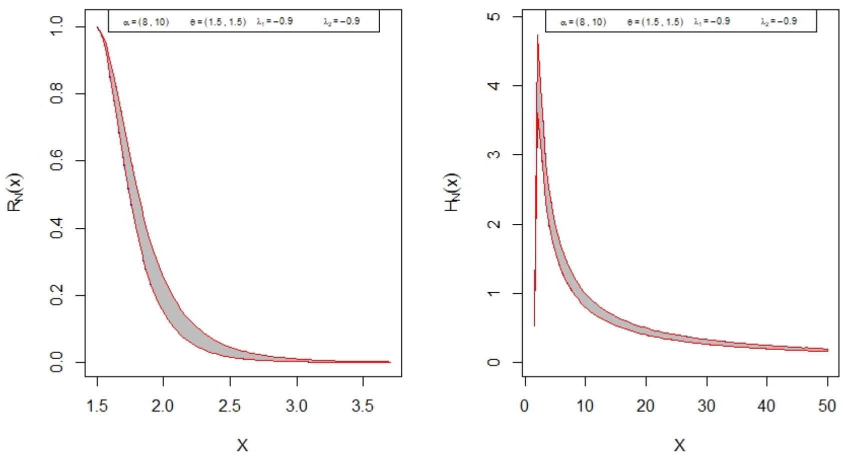

Figure 16.

b. Quantile Function Of CTNPD
The qth quantile x_q of the CTNED can be obtained from Eqn. (35) as

$$x_q = \theta_N e^{-\frac{1}{\alpha_N} ln(y)}$$

where,

$$y = -\frac{b}{3a} - \frac{2^{\frac{1}{3}}\eta_1}{3a\left(\eta_2 + \sqrt{4\eta_1^3 + \eta_2^2}\right)^{\frac{1}{3}}} + \frac{\left(\eta_2 + \sqrt{4\eta_1^3 + \eta_2^2}\right)^{\frac{1}{3}}}{3\left(2^{\frac{1}{3}}\right)a}$$

$\eta_1 = -b^2 + 3ac;\ \eta_2 = -2b^3 + 9abc - 27a^2d$
$a = \lambda_2;\ \ b = -\lambda_1 - 2\lambda_2$
$c = \lambda_1 + \lambda_2 - 1;\ \ \ d = 1 - q$

3 Applications

In order to gauge interest in the flexibility, sustainability, and applicability of the proposed neutrosophic distribution over an existing neutrosophic distribution is analyzed using a real-world data set in this section. Here, four actual data sets demonstrate the practical implementation of the proposed Neutrosophic distributions. For comparison purposes, Quadratic Transmuted Neutrosophic Distributions (QTND) and Neutrosophic Distributions (ND) are used. From the following model selection standards, log-likelihood (Loglik.), AIC(Akaike's Information criteria), BIC (Bayesian Information criteria) values, and K-S test statistic for the verification of the goodness of fit of the proposed neutrosophic distribution. The distribution with minimum AIC, BIC, and K-S statistic values and maximum log-likelihood and p-value of K-S statistic are deemed as a best-fit model.

3.1 Rain Repining Data:

In this section, The research relies on district-level survey data pertaining to agricultural crops, sourced primarily from the crop reporting services of the agriculture department in Punjab, Pakistan. Additionally, Rain-reducing data are gathered from the meteorological department in Punjab, Pakistan.[22] This data set utilizes for the illustration of QTNED.

The MLEs of the parameters along with goodness of fit measures from the considered real data set of competing distributions are present in Table 1. The results indicate that QTNED is more efficacious in exploring the characteristics of uncertainties related to climate variability in neutrosophic data compared to NED.

Table 1: **Estimation of Parameters and model adequacy measures of Rain Repining observations during wheat phenology**

Distribution	Estimates	LogLik.	AIC	BIC	KS	P-value
QTNED	$\hat{\theta}_N$=[1.8753,0.0499] $\hat{\lambda}_N$=[0.6837,-0.64775]	[0.1049,-71.8412]	[3.7902,147.6825]	[5.4566,149.3489]	[0.3203,0.0946]	[0.0611,0.9942]
NED	$\hat{\theta}_N$=[2.5,0.0379]	[-1.4231,-72.6131]	[4.8461,147.2261]	[5.6793,148.0593]	[0.3712,0.1717]	[0.0185,0.6369]

The results tracked down from above Table 1 confirm that the QTNED model is a better model compared to NED since it gives more efficient values for the goodness of fit measures and the highest K-S statistic p-value.

3.2 Population density of villages in USA:

In order to provide a practical example, we examined the population compactness of several villages in rural USA, considering data sourced from.[17] The data set comprises population information for 17 villages in the USA along with their associated neutrosophic data. The parameter estimates and related goodness of fit criteria values of the neutrosophic Weibull distribution (NWD) for processing complex data, as explored,[24] is contrasted with the model adequacy of the proposed QTNWD. The findings in Table 2 further indicate that QTNWD is better suited to model the data compared to the NWD.

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Table 2: **Estimation of Parameters and model adequacy measures of Population density of villages in USA**

Distribution	Estimates	LogLik.	AIC	BIC	KS	P-value
QTNWD	$\hat{\alpha}_N$ =[6.1026,6.1826] $\hat{\beta}_N$ =[6.5645,6.2496] $\hat{\lambda}_N$ =[0.7173,0.7328]	[-22.3237,-23.1144]	[50.6473,52.2288]	[53.1470,54.7284]	[0.1960,0.1983]	[0.4724,0.5156]
NWD	$\hat{\alpha}_N$ =[5.0565,6.9361] $\hat{\beta}_N$ =[0.0500,0.0520]	[-96.2151,-95.6476]	[124.5214,125.2397]	[126.1788,126.9061]	[0.6284,0.6222]	[$5.03\text{x}10^{-7}$,$3.84\text{x}10^{-6}$]

3.3 Remission periods of 128 cancer patients:

The data set pertains to the remission time in months for 128 cancer patients. Originally examined and reported[5] within the context of bladder cancer research. We employ the same data set to illustrate the QTNRD distribution, with the parameters of the developed QTNRD estimated based on the uncertainties associated with the remission periods of cancer patients. The results in Table 3 reveal that QTNRD is more effective in exploring the uncertainties of remission periods in cancer patients' neutrosophic data compared to NRD.

Table 3: **Estimation of Parameters and model adequacy measures of Remission periods of 128 cancer patients**

Distribution	Estimates	LogLik.	AIC	BIC	KS	P-value
QTNRD	$\hat{\theta}_N$ =[11.0574,11.3365] $\hat{\lambda}_N$ =[0.7949,0.8052]	[-463.1498,-466.6273]	[930.2996,937.2546]	[936.0037,942.9587]	[0.2805,0.2807]	[$3\text{x}10^{-9}$,$3\text{x}10^{-9}$]
NRD	$\hat{\theta}_N$ =[9.6432,9.8702]	[-486.1404,-490.4138]	[974.280,982.8275]	[977.6728,985.6796]	[0.3544,0.3542]	[$2\text{x}10^{-14}$,$2\text{x}10^{-14}$]

3.4 The Dioxin Data:

The annual estimation of dioxin absorption from the average diet during the period from 1998 to 2015 is documented in the 2017 annual report on environmental statistics published by the Ministry of Environment Japan.[23] Table 4 presents the neutrosophic maximum likelihood estimators and sufficiency indicators for the model QTNPD. The findings indicate that the QTNPD outperforms the NPD in handling the data.

Table 4: **Estimation of Parameters and model adequacy measures of dioxins consumption from total food samples collected with uncertainties.**

Distribution	Estimates	LogLik.	AIC	BIC	KS	P-value
QTNPD	$\hat{\theta}_N$ =[0.0200,0.6500] $\hat{\alpha}_N$ =[0.5031,1.9731] $\hat{\lambda}_N$ =[-0.7711,-0.7821]	[-11.1064,-13.0795]	[26.2128,30.1590]	[27.8792,31.8254]	[0.2316,0.2308]	[0.2762,0.3258]
NPD	$\hat{\theta}_N$ =[0.0200,0.6500] $\hat{\alpha}_N$ =[0.37164,1.4492]	[-13.0664,-15.09932]	[28.1328,32.1987]	[28.9660,33.0319]	[0.3097,0.2845]	[0.0603,0.1276]

4 Conclusion

In this article, a general family of transmuted neutrosophic probability distributions is introduced. This paper gives a special reference to the quadratic and cubic transmuted families of distributions, and we present some lifetime baseline transmuted neutrosophic distributions. In order to assess the performance of this new family of distributions, the quadratic transmuted neutrosophic distributions (QTND) and neutrosophic distributions have been focused. The efficacy of the QTND was evaluated through its application to four real-life datasets, aiming to compare its performance with the neutrosophic distributions. The findings suggested that the QTNDs exhibited superior behavior and demonstrated enhanced suitability for modeling the datasets when compared to the neutrosophic baseline distributions. In the forthcoming research, an analysis of the characteristics of the properties and estimation of the distributions will be undertaken.

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