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FULL LENGTH ARTICLE

# Finding the HK Completions of Some Sequence Spaces

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## ABSTRACT

In this paper, we prove the following result: If the space $\phi$ (the finite sequences) is equipped with the norm which is naturally induced by a positive definite Hermitian and diagonally blockwise constructed matrix, then an HK completion exists. Also, we investigate our result through many illustrated examples.

**Keywords:** HK completion ▪ vector space ▪ normed space ▪ sequence

## 1. INTRODUCTION AND PRELIMINARIES

An HK space is a Hilbert space of sequences on which coordinate projections are all continuous.

An FK space is a linear topological space of sequences which is a locally convex Fréchet space with continuous coordinate projections.

A BK space is the special case of the foregoing in which the Fréchet space is a Banach space. An AD space is an FK space in which $\phi$ is dense.

Throughout this paper, $\omega$ will stand for the collection of all complex sequences, $\mathbb{C}$ for the complex numbers, and $\ell^2$ will denote the space of all members of $\omega$ which are square summable:

$$\ell^2 = \left\{ x \in \omega : \sum_{j=1}^{\infty} |x_j|^2 = \|x\|_2^2 < \infty \right\}. \tag{1}$$

For $n = 1, 2, \ldots$, let $e^n$ be the sequence defined by

$$e_k^n = \begin{cases} 1, & n = k, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Finally, let $M$ be the set of all infinite matrices $A = (a_{ij})$ which are Hermitian and positive definite. For $x = \sum_{k=1}^{n} x_k e^k \in \phi$ and $A \in M$, define the norm induced by $A$ as

$$\|x\|_A^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} \overline{x_j}. \tag{3}$$

This, of course, makes $(\phi, \|\cdot\|_A)$ a normed sequence space. In this paper, we present sufficient conditions for this space to have an HK completion.

### 1.1 The Continuity of Coordinate Projections

For $n = 1, 2, \ldots$, we define the $n$th coordinate projection $p_n$ as

$$p_n(x) = x_n, \qquad x \in \omega. \tag{4}$$

The HK definition requires that $p_n$ be continuous on $(\phi, \|\cdot\|_A)$ for each $n$. One needs the following criterion.

**Continuity Criterion** [1]. If $X$ is a linear topological space with topology determined by a family $P$ of seminorms, then the linear functional $f$ is continuous on $X$ if and only if there exist $s > 0$ and $q_1, q_2, \ldots, q_n$ selected from $P$ such that

$$|f(x)| \leq s \sum_{k=1}^{n} q_k(x), \qquad x \in X. \tag{5}$$

In the present setting, $P = \{\|\cdot\|_A\}$, and the following example shows a case where $p_1$ is not continuous on $(\phi, \|\cdot\|_A)$.

**Example.** For each $n$, let

$$E^n = \{x \in \omega : x_k = 0 \text{ for all } k > n\}.$$

Consider the matrix $A=(a_{ij})$, where

$$a_{ij}=\begin{cases}1, & i=j=1,\\ 5, & i=j>1,\\ -2, & |i-j|=1,\\ 0, & \text{elsewhere.}\end{cases} \tag{6}$$

$A$ is clearly Hermitian. As for positive definiteness, let $x\neq 0$ be an arbitrary element in $E^n$. We may assume that $x_n\neq 0$. Then

$$\begin{aligned}\|x\|_A^2&=x^*Ax\\ &=(\bar{x}_1-2\bar{x}_2,-2\bar{x}_1+5\bar{x}_2-2\bar{x}_3,\ldots,\\ &\qquad -2\bar{x}_{n-1}+5\bar{x}_n)x.\end{aligned} \tag{7}$$

Expanding and regrouping the terms gives

$$\begin{aligned}\|x\|_A^2&=|x_1|^2-2x_1\bar{x}_2-2\bar{x}_1x_2+5|x_2|^2\\ &\quad-2x_2\bar{x}_3-2\bar{x}_2x_3+5|x_3|^2-\cdots\\ &\quad+5|x_{n-1}|^2-2x_{n-1}\bar{x}_n-2\bar{x}_{n-1}x_n+5|x_n|^2\\ &=\sum_{k=1}^{n-1}\left(|x_k|^2-4\Re(x_k\bar{x}_{k+1})+4|x_{k+1}|^2\right)+|x_n|^2\\ &=\sum_{k=1}^{n-1}|x_k-2x_{k+1}|^2+|x_n|^2>0.\end{aligned} \tag{8}$$

Therefore, $A\in M$.

For $k\geq 2$, choose $x_k=x_{k-1}/2$ to obtain $x_n=x_1/2^{n-1}$. Hence

$$\|x\|_A^2=|x_n|^2=\frac{|x_1|^2}{2^{2n-2}},\qquad |x_1|^2=2^{2n-2}\|x\|_A^2. \tag{9}$$

This implies that $p_1$ is not continuous on $(\phi,\|\cdot\|_A)$.

We proceed to establish sufficient conditions for coordinate projections to be continuous.

## 2. MAIN DISCUSSION

**Lemma.** For the $n\times n$ positive definite matrix $A$, let

$$\lambda=\min\{\lambda_k:\lambda_k \text{ is an eigenvalue of } A\}.$$

It is then known that, for $x\in E^n$,

$$x^*Ax\geq\lambda\|x\|_2^2,\qquad \|x\|_2^2=\sum_{k=1}^{n}|x_k|^2. \tag{10}$$

*Proof.* Let $U$ be a unitary matrix which diagonalizes $A$, and let $x=Uy$ for some $y\in E^n$. Then

$$\begin{aligned}x^*Ax&=y^*U^*AUy=y^*Dy\\ &=\sum_{k=1}^{n}\lambda_k|y_k|^2\geq\lambda\sum_{k=1}^{n}|y_k|^2=\lambda\|x\|_2^2,\end{aligned} \tag{11}$$

where $D=\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$. □

**Theorem.** If $A\in M$ has the form

$$A=\operatorname{diag}(A_1,A_2,\ldots), \tag{12}$$

where, for each $k$, $A_k$ is an $S_k\times S_k$ matrix, then $p_n$ is continuous on $(\phi,\|\cdot\|_A)$ for each $n=1,2,\ldots$.

*Proof.* It is clear that, for each $k$, $A_k$ is Hermitian and positive definite. Define

$$\begin{aligned}A_k'&=\operatorname{diag}(0,\ldots,0,A_k,0,\ldots),\\ B_k&=\operatorname{diag}(A_1,A_2,\ldots,A_k,0,\ldots).\end{aligned} \tag{13}$$

Fix $n$, let

$$\lambda=\min\{\lambda_i:\lambda_i \text{ is an eigenvalue of } B_n\},$$

and let $\|x\|_2$ be as in Equation (10). Now,

$$\begin{aligned}|x_n|^2&\leq\|x\|_2^2\leq\frac{1}{\lambda}\|x\|_{B_n}^2\\ &=\frac{1}{\lambda}\sum_{k=1}^{n}\|x\|_{A_k'}^2\leq\frac{1}{\lambda}\sum_{k=1}^{\infty}\|x\|_{A_k'}^2\\ &=\frac{1}{\lambda}\|x\|_A^2.\end{aligned} \tag{14}$$

Thus, $p_n$ is continuous on $(\phi,\|\cdot\|_A)$. It is now our objective to show that this diagonal blockwise construction also solves the existence problem. The existing completions are to be shown unique and of the desired form. As for the uniqueness question, the next well-known lemma takes care of it. □

**Lemma.** Let $X$ and $Y$ be two BK spaces. If $S$ is a dense subspace of $X$ and of $Y$ such that $\|x\|_X=\|x\|_Y$ for all $x\in S$, then $X=Y$.

*Proof.* Let $x\in X$ be arbitrary. Choose a sequence $\{x^n\}\subseteq S$ with $x^n\to x$ in $X$. The sequence $\{x^n\}$ is then a Cauchy sequence in $X$. Thus, for every $\varepsilon>0$, there exists a positive integer $N$ for which

$$\|x^m-x^n\|_X<\varepsilon\qquad(m,n\geq N).$$

But this says that $\|x^m-x^n\|_Y<\varepsilon$ for all $m,n\geq N$. Therefore, $\{x^n\}$ is Cauchy in $Y$, hence converges to some $y\in Y$.

Now, $x^n\to x$ in $X$, so, for each $k$, $x_k^n\to x_k$ in $\mathbb{C}$. Also, $x^n\to y$ in $Y$, so, for each $k$, $x_k^n\to y_k$ in $\mathbb{C}$, implying that $x=y$. Thus, $X\subseteq Y$. Reversing the roles of $X$ and $Y$ gives $Y\subseteq X$, and hence $X=Y$. □

Luckily enough, inner product spaces transfer their inner products to their completions. The problem is that a completion of a specific type may not exist, and this is actually the existence problem being considered here.

**Proposition.** Let $(X,\langle\cdot,\cdot\rangle_X)$ be an inner product space and $K$ a completion of $X$. Then $K$ is an inner product space.

*Proof.* First of all, one has to note that a completion always exists. Take, for example, the closure of $X$ in its second dual under the canonical embedding. For $x,y\in K$, define

$$\langle x,y\rangle_K=\lim_{n\to\infty}\langle x^n,y^n\rangle_X, \tag{15}$$

where $\{x^n\},\{y^n\}\subseteq X$ are two sequences converging respectively to $x$ and $y$. This inner product is well defined. Indeed,

$$\begin{aligned}&|\langle x^m,y^m\rangle_X-\langle x^n,y^n\rangle_X|\\ &\quad\leq|\langle x^m,y^m-y^n\rangle_X|+|\langle x^m-x^n,y^n\rangle_X|\\ &\quad\leq\|x^m\|\,\|y^m-y^n\|+\|y^n\|\,\|x^m-x^n\|\longrightarrow 0.\end{aligned} \tag{16}$$

Thus, $\{\langle x^n,y^n\rangle_X\}$ is a Cauchy sequence in $\mathbb{C}$ and hence converges to a unique limit. □

As remarked earlier, turning $\phi$ into an inner product space does not necessarily force an FK completion to exist. The following example shows this claim.

**Example.** Suppose that $A\in M$ makes coordinate projections continuous on $(\phi,\|\cdot\|_A)$. For $x\in\phi$, let

$$\|x\|=\left(\|x\|_A^2+|f(x)|^2\right)^{1/2}, \tag{17}$$

where $f$ is a noncontinuous linear functional on $(\phi,\|\cdot\|_A)$; such an $f$ exists since $\phi$ is infinite dimensional [2]. We claim that $(\phi,\|\cdot\|)$ is an inner product space which is continuously embedded in $\omega$, but has no FK completion.

To prove this claim, we need the following lemma.

**Lemma.** Let $X$ be a linear metric space of sequences on which $p_n$ is continuous for all $n$. Then $X$ has an FK completion if and only if, for every Cauchy sequence $\{x^n\} \subset X$ which converges to zero pointwise, we have $x^n \to 0$ in $X$.

We can now prove our claim. Note first that the norm $\|\cdot\|$ is given by the inner product

$$\langle x,y\rangle = \langle x,y\rangle_A + f(x)\overline{f(y)}. \tag{18}$$

Define the matrix $B=(b_{nk})$ by

$$b_{nk} = a_{nk} + f(e^n)\overline{f(e^k)}. \tag{19}$$

Thus, $\|\cdot\| = \|\cdot\|_B$. Moreover, $(\phi, \|\cdot\|_B)$ is continuously embedded in $\omega$. Fix $n$ and let $x \in \phi$ be arbitrary. By Equation (5), for some $s>0$,

$$|x_n| \leq s\|x\|_A \leq s\|x\|_B. \tag{20}$$

However, $(\phi, \|\cdot\|_B)$ has no FK completion. The set

$$\{x \in \phi : f(x) = 1\}$$

is dense in $(\phi, \|\cdot\|_A)$. Therefore, there exists a sequence $\{x^n\} \subset \phi$ with $f(x^n)=1$ for all $n$ and $\|x^n\|_A \to 0$. Finally, $\{x^n\}$ is Cauchy in $(\phi, \|\cdot\|_B)$ since

$$\begin{aligned}\|x^m - x^n\|_B &= \left(\|x^m - x^n\|_A^2 + |f(x^m - x^n)|^2\right)^{1/2}\\ &= \|x^m - x^n\|_A.\end{aligned} \tag{21}$$

Also, $x^n \to 0$ in $\omega$, but $\|x^n\|_B \to 1 \neq 0$. Our claim now follows from the lemma.

Turning to the existence problem, it is essential to recall the following remarks:

(a) A finite-dimensional inner product space is a Hilbert space [4].

(b) If $(H_n)$ is a sequence of Hilbert spaces, then the direct sum

$$\bigoplus_{n=1}^{\infty} H_n \tag{22}$$

is the Hilbert space $H$ of all sequences $x=(x^n)$, where $x^n \in H_n$, such that $\{\|x^n\|_{H_n}\} \in \ell^2$ [5].

Addition, scalar multiplication, and the inner product are defined on $H$ as follows. For $x=(x^n)$, $y=(y^n) \in H$, and $\alpha \in \mathbb{C}$, define

$$\begin{aligned} x+y &= (x^n + y^n),\\ \alpha x &= (\alpha x^n),\\ \langle x,y\rangle_H &= \sum_{n=1}^{\infty} \langle x^n, y^n\rangle_{H_n}. \end{aligned} \tag{23}$$

Consider the matrix $A$ in Equation (12). For each $n$, let

$$r_n = \sum_{k=1}^{n} S_k, \qquad r_0 = 0, \tag{24}$$

and define

$$\phi_n = \{x \in \phi : x_k = 0 \text{ whenever } k \notin (r_{n-1}, r_n]\}, \tag{25}$$

$$A'_n = \operatorname{diag}(\underbrace{0,\ldots,0}_{r_{n-1}\text{ entries}}, A_n, 0, 0, \ldots), \tag{26}$$

$$H_n = (\phi_n, \|\cdot\|_{A'_n}). \tag{27}$$

By the preceding remarks, $H = \bigoplus_{n=1}^{\infty} H_n$ is a Hilbert space of sequences and, by its very construction, $H$ has AD. With all this at hand, the preceding theorem now implies the main result.

**Theorem (main result).** Suppose that $A$ is an infinite matrix which is Hermitian and positive definite. If $A$ has the form

$$A = \operatorname{diag}(A_1, A_2, \ldots),$$

where, for each $n$, $A_n$ is an $S_n \times S_n$ matrix, then the normed sequence space $(\phi, \|\cdot\|_A)$ has an HK completion which has the AD property.

*Proof.* Done already. □

## 3. CONCLUSION

In this paper, we proved the following result: If the space $\phi$ (the finite sequences) is equipped with the norm which is naturally induced by a positive definite Hermitian and diagonally blockwise constructed matrix, then an HK completion exists. Also, we investigated our result through many illustrated examples.

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