Statistical Optimization of Industrial Processes for Sustainable Growth using Neutrosophic Maddala Distribution

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Abstract

The family of neutrosophic distributions has received considerable attention from the scientific community, due to the flexible parametric form of its probability density function, in modeling many physical phenomena with imprecise information. In this study, we consider a generalization of Singh Maddala distribution for handling fuzzy data sets. This study presents a new research endeavor: quantifying the lifespan of manufacturing enterprises using the Neutrosophic Singh Maddala Distribution (NSMD). This work significantly enhances the theoretical foundations by providing novel formulations for the moments and mode of the NSMD distribution. In addition, it expands the study beyond the traditional Maddala model by examining conventional statistical models. For estimating the unknown parameters, the maximum likelihood estimation has been used in neutrosophic framework. Characterizations are obtained in terms of neutrosophic measures. The assessment of model performance, carried out using the goodness of fit criterion, highlights the superiority of NSMD compared to other models. In the application section, a real data on carbon emission is provided for usefulness of the proposed model.

Keywords: Neutrosophic probability; imprecise density; interval estimation; risk analysis; simulation

1. Introduction

In recent decades, many continuous univariate distributions have been developed. Despite this improvement, datasets from a variety of domains, including dependability, engineering, environmental science, finance, and healthcare, frequently deviate from the established distributions [1]. As a result, there is a critical need for the creation of modified, extended, and generalized distributions, as well as their application to meet issues in these areas[2]. Transformations or the addition of one or more factors to the baseline distribution create modified, extended, and generalized distributions. These enhancements enable the newly generated distributions to provide better fits to the data than competing models [3]. The Singh Maddala distribution is a special case of the beta distribution[4-8]. Burr suggested 12 distributions, known as the Burr family, to handle cumulative frequency functions in frequency data analysis [9]. Burr distributions XII, III, and X are often used. Notably, the Burr-XII (BXII) distribution is extremely useful for modeling insurance data in the banking and business fields[10]. It is also widely used to model failure time data in reliability studies, survival analysis, and the development of acceptance sampling strategies. The hazard rate related to income was examined by Singh and Maddala in order to formulate their distribution [5]. Although

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declining failure rate (DFR) distributions are not usually expected in time-based variables, they pointed out that there is a strong theoretical justification for detecting DFR distributions in relation to income. They also noted that income may facilitate further earnings, implying that DFR distributions in variables related to income have a reasonable foundation. The opportunity to make more money may increase alongside one's current earnings. This concept corresponds to the standard Pareto distribution [11]. Consequently, a plethora of modified, extended, and generalized versions of the Burr distribution, featuring additional shape and scale parameters, have been documented in the literature. However, these distributions are primarily designed for analyzing precise data and may not be suitable for handling interval-valued data [12]. Smarandache pioneered neutrosophic set theory, which extends classical set theory to include uncertain, imprecise, and indeterminate information by assigning membership, non-membership, and indeterminacy degrees to each element [13], [14]. Neutrosophic sets and Logic plays a significant role in approximation theory [15]. This flexible framework can be utilized across areas such as decision-making processes, expert systems, medical diagnostics, pattern recognition, risk management and environmental studies [16]. Neutrosophic sets offer a way to interpret data effectively and improve reasoning and decision making in real world scenarios by embracing uncertainty [17-19]. They provide a method for dealing with imprecision and uncertainty offering insights into unpredictable events while enhancing the accuracy and dependability of systems and models in various fields. Smrandache proposed an extension of statistics to encompass data [20]. Neutrosophic Statistics means statistical analysis of population or sample that has indeterminate (imprecise, ambiguous, vague, incomplete, unknown) data. Neutrosophic statistics is a branch that focuses on analyzing uncertain, imprecise and ambiguous data to draw conclusions [21-23]. It achieves this through the application of set theory. By using sets to represent data uncertainty based on membership levels, non-membership levels and indeterminacy categories statisticians can effectively manage imprecise information systematically for modeling and evaluating uncertain details [24]. Neutrosophic statistics finds applications in fields where conventional statistical methods may fall short due to ambiguity or vagueness such as decision making, under uncertainty, medical diagnosis, risk evaluation and environmental research [25]. The combination of neutrosophic set theory with analysis has led to the development of probability distributions within the realm of neutrosophic statistics as mentioned in several references, in the literature.
The primary aim of this study is to develop a more adaptable distribution, termed NSMD, based on the Maddala model, with the intention of effectively managing fuzzy applications. The core motivation behind this endeavor is to scrutinize ambiguous data that conforms to the Maddala model, thereby enhancing our capacity to analyze and interpret such vague information. The sections of this study are structured as follows: Section 2 provides the definition of NSMD. Section 3 provides the statistical aspects, focusing on a probability density of the proposed model and its important functions. Section 4 describes the examination of estimating methods. Section 4 conducts a simulation analysis to define the parameters of the suggested distribution estimation methods. Section 5 explains the real application of the proposed model. Finally, the conclusion section summarizes the study's findings.

2. Proposed Model with Some Useful properties

In this section, the probability density function and its related important statistical properties of the proposed model are discussed in neutrosophic framework. The neutrosophic Singh-Maddala distribution, denoted as NSMD, is indeed a special case of the classical Maddala distribution, specifically designed for handling neutrosophic data. The proposed NSMD has the following the following mathematical expression with three parameters:

$$h(z_N) = (\alpha_N q z^{\alpha_N - 1})/(\beta_N{}^{\alpha_N}[1 + (z/\beta_N)^{\alpha_N}]^{1+q}); \quad z > 0, \alpha_N > 0,\ \beta_N > 0 \qquad (1)$$

where α_N is the shape parameter of the proposed distribution, β_N

is the scale parameter of the proposed distribution and q influences the right tail and corresponds to a parameter that only affects the right tail in the NSMD.
The Singh-Maddala distribution has been rediscovered independently numerous times across various loosely connected fields. As a result, it has acquired multiple names. Its initial consideration dates back to Burr (1942), where it emerges as the twelfth example within the solutions of a differential equation defining the Burr system of distributions. Shape of the probability density function is shown in Figure 1.

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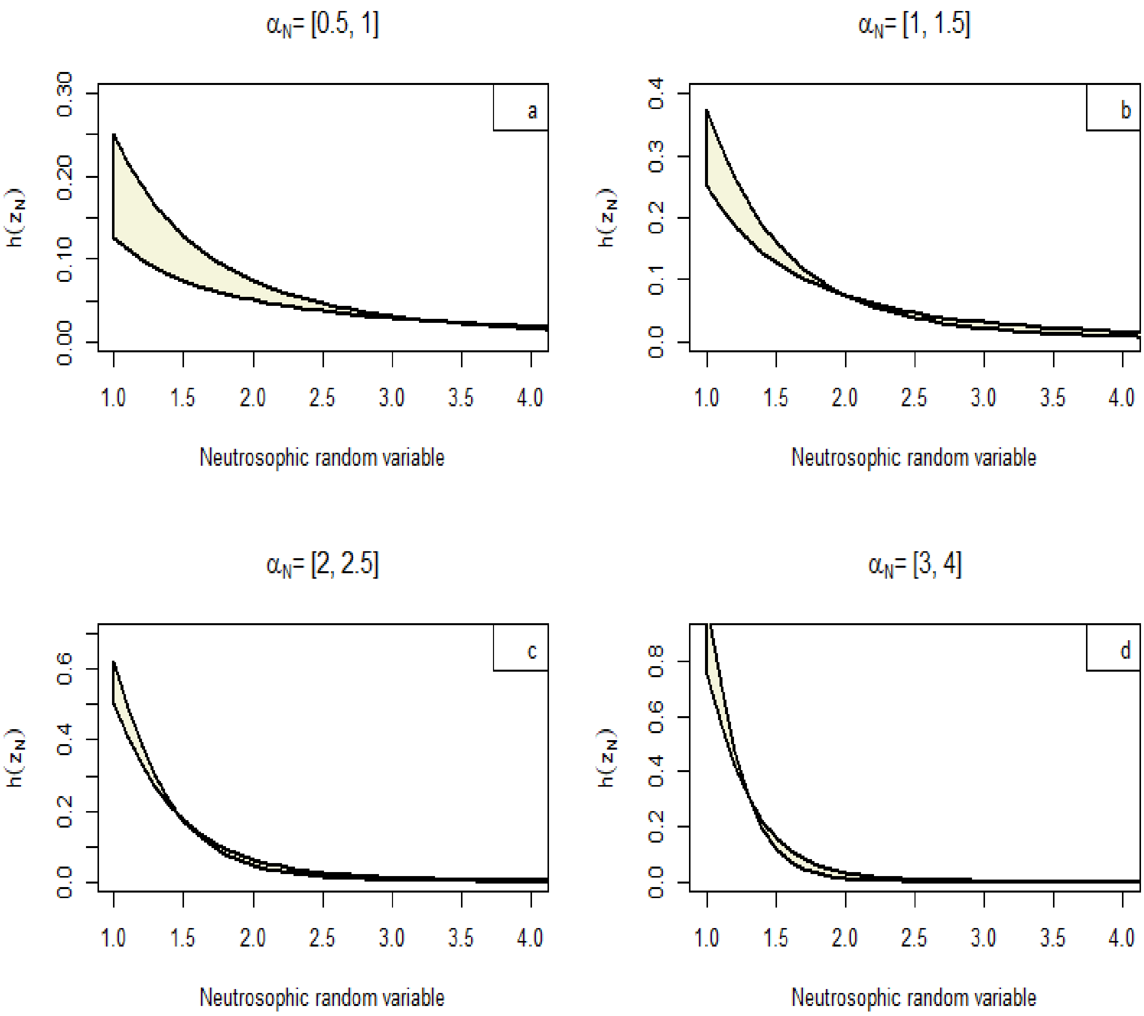

Figure 1: Density plots of the proposed model with different imprecise parameter values

Figure 1 shows the neutrosophic curves of the proposed model different neutrosophic parameter setting. Unlike classic neutrosophic probability density functions, which deal with crisp, precise data, NPDFs deal with data that contains various degrees of truth, untruth, and uncertainty. The parameters or data in a neutrosophic probability density function could be vague or imprecise. When typical statistical approaches fail due to insufficient or contradictory data, NPDFs are particularly useful. They provide a more sophisticated framework for simulating uncertainty, making it possible to depict intricate and erratic occurrences. Research on neurophilosophic statistics and probability is still underway, with applications in risk analysis, image processing, pattern recognition, and decision making. In light of the ongoing evolution of our understanding of uncertainty, NPDFs offer a potential approach to managing the inherent complexity of A distribution function is a fundamental idea in probability theory and statistics. It is also referred to as a cumulative distribution function (or CDF). It represents the likelihood that a random variable will have a value that is either less than or equal to a specific point. In essence, it is a summary of a random variable's probability distribution. By visualising the probability distribution across the spectrum of potential outcomes, the CDF plot helps reveal characteristics and behaviour of the random variable. In fields ranging from engineering and medicine to finance and economics, distribution functions are essential instruments for risk assessment, hypothesis testing, and decision-making.
CDF of the proposed based on (1) can be written as:

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$$F(z_N) = 1 - \left[1 + \left(\frac{z}{\beta_N}\right)^{\alpha_N}\right]^{-q}; \; z > 0, \alpha_N > 0, \; \beta_N > 0 \tag{2}$$

Neutrosophic Cumulative Distribution Function is an extension of the classical cumulative CDF in neutrosophic statistics, which handles with uncertainty characterized by incomplete, indeterminate, and inconsistent information.

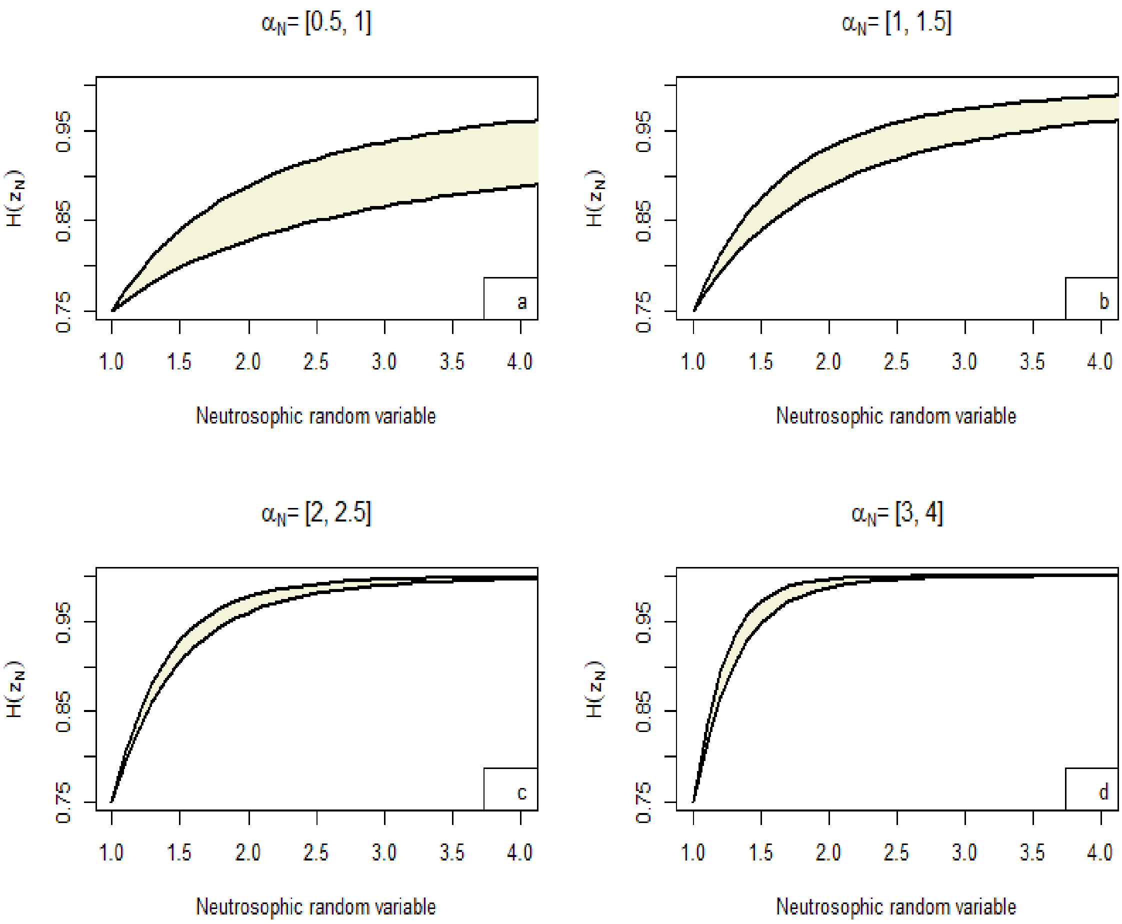

Figure 2: CDF plots of the proposed model with different imprecise parameter values

The neutrosophic CDF plots of the suggested model with imprecise values for the shape parameter and accurate constants for the other parameters are displayed in Figure 2. It is not the same as the traditional CDF. Neutrosophic CDF takes into consideration the inherent uncertainties in real-world situations, where data may be imprecise, vague, or ambiguous, whereas traditional CDF deal with clear, exact data. Parameters, observations, or the distribution itself may have different degrees of truth, indeterminacy, or falsehood within a neutrosophic framework. The way Neutrosophic CDF handles uncertainty is the primary difference. Classical CDF presumes a perfect understanding of the underlying probability distribution, but Neutrosophic CDF explicitly recognize and model uncertainty. Because of this, Neutrosophic CDF can be applied to situations where insufficient or inconsistent data prevent the application of classic statistical approaches. Neutrosophic CDF are useful in many areas where uncertainties need to be taken into consideration, such as image processing, risk analysis, pattern recognition, and decision making. They provide a way to modeling uncertain data that is more advanced and flexible, which makes them an effective tool for handling the complexities of real-world events.
Another important function related to any probability density function is survival function. The survival function of the proposed distribution is provided by:

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$$S(z_N) = \left[1 + \left(\frac{z}{\beta_N}\right)^{\alpha_N}\right]^{-q}; \; z > 0, \alpha_N > 0, \; \beta_N > 0 \quad (3)$$

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Survival function of the proposed distribution is shown in Figure 3

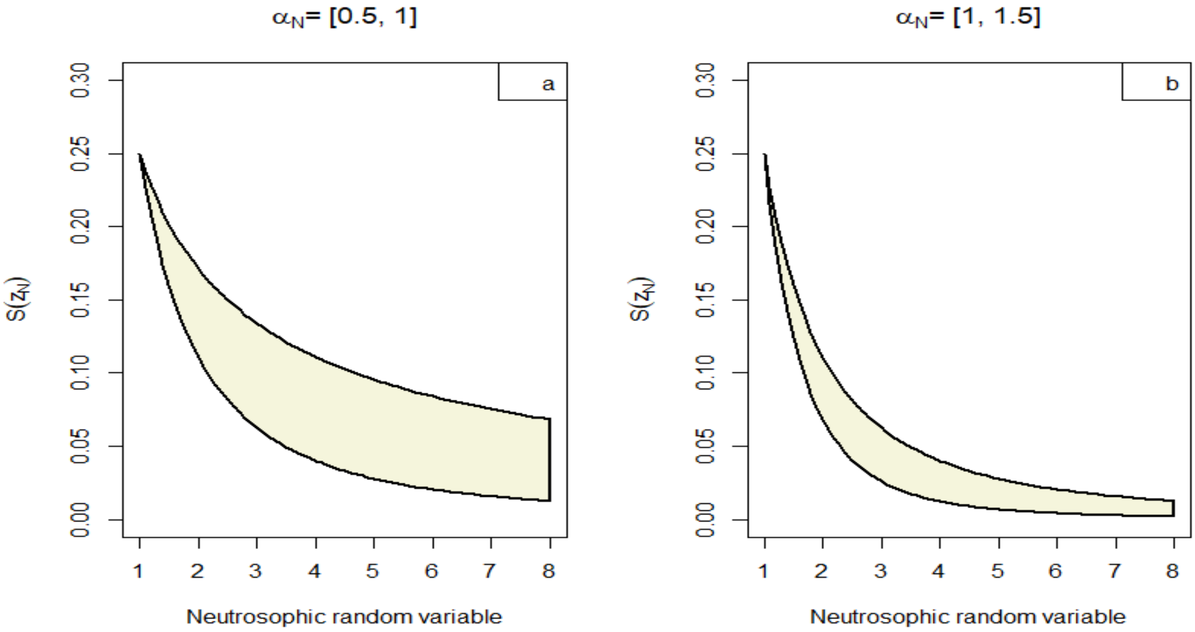

Figure 3: Survival function of the proposed distribution with imprecise shape parameter values

The survival function is a statistical concept that measures the probability of an event occurring at a specific time. It is crucial in survival analysis, a field that studies survival data in various fields like social sciences, engineering, health, and economics, to analyze time-to-event data and predict future events.

3. Some Statistical Properties

In this section, we study several key statistical characteristics of the proposed distribution within the neutrosophic framework. These properties aid in elucidating its distinctions from the classical model.

Theorem 1: Derive the expression for the rth central moment of the proposed model

Proof: By definition the rth moment of origin is defined as:

$$\mu_{rN}''(\mathcal{Z}) = \int_{-\infty}^{\infty} \mathcal{Z}^r \frac{\alpha_N q z^{\alpha_N - 1}}{{\beta_N}^{\alpha_N}\left[1+\left(\frac{z}{\beta_N}\right)^{\alpha_N}\right]^{1+q}} d\mathcal{Z}$$

$$= \left[\int_{0}^{\infty} \mathcal{Z}^r \frac{\alpha_l q z^{\alpha_l - 1}}{{\beta_l}^{\alpha_l}\left[1+\left(\frac{z}{\beta_l}\right)^{\alpha_l}\right]^{1+q}} d\mathcal{Z}, \int_{0}^{\infty} \mathcal{Z}^r \frac{\alpha_u q z^{\alpha_u - 1}}{{\beta_u}^{\alpha_u}\left[1+\left(\frac{z}{\beta_u}\right)^{\alpha_u}\right]^{1+q}} d\mathcal{Z}\right]$$

$$= \left[\frac{{\beta_l}^r \Gamma(1+{}^r/_{\alpha_l})\Gamma(q-{}^r/_{\alpha_l})}{\Gamma q}, \frac{{\beta_u}^r \Gamma(1+{}^r/_{\alpha_u})\Gamma(q-{}^r/_{\alpha_u})}{\Gamma q}\right]$$

This further can be written as:

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$$\mu''_{rN}(\mathcal{Z}) = \frac{{\beta_N}^k B\left(1+\frac{r}{\alpha_N},\ q-\frac{r}{\alpha_N}\right)}{B(1,q)} = \frac{{\beta_N}^k \Gamma(1+{}^{r}/_{\alpha_N})\Gamma(q-{}^{r}/_{\alpha_N})}{\Gamma q} \tag{4}$$

which is required result.

Theorem 2: If $\mathcal{Z}$ follows the $NSMD$ then $E(\mathcal{Z}) = \frac{\beta_N \Gamma(1+{}^{1}/_{\alpha_N})\Gamma(q-{}^{1}/_{\alpha_N})}{\Gamma q}$

Proof: By definition

$$E(\mathcal{Z}) = \int_{-\infty}^{\infty} \mathcal{Z} \frac{\alpha_N q z^{\alpha_N - 1}}{{\beta_N}^{\alpha_N}\left[1+\left(\frac{z}{\beta_N}\right)^{\alpha_N}\right]^{1+q}} d\mathcal{Z}$$

$$\left[\int_0^{\infty} \mathcal{Z} \frac{\alpha_l q z^{\alpha_l - 1}}{{\beta_l}^{\alpha_l}\left[1+\left(\frac{z}{\beta_l}\right)^{\alpha_l}\right]^{1+q}} d\mathcal{Z},\ \int_0^{\infty} \mathcal{Z} \frac{\alpha_u q z^{\alpha_u - 1}}{{\beta_u}^{\alpha_u}\left[1+\left(\frac{z}{\beta_u}\right)^{\alpha_u}\right]^{1+q}} d\mathcal{Z}\right] \tag{5}$$

Simplification of (5) provides:

$$E(\mathcal{Z}) = \frac{\beta_N \Gamma(1+{}^{1}/_{\alpha_N})\Gamma(q-{}^{1}/_{\alpha_N})}{\Gamma q} \tag{6}$$

Theorem 3: If $\mathcal{Z}$ follows the $SMND$ then show that variance of is finite positive quantity.

Proof: By definition

$$\mathcal{V}_N(\mathcal{Z}) = \mu''_{2N}(\mathcal{Z}) - [\mu''_{1N}(\mathcal{Z})]^2 \tag{7}$$

We know from (4)

$$\mu''_{2N}(\mathcal{Z}) = \frac{{\beta_N}^2 \Gamma(1+{}^{2}/_{\alpha_N})\Gamma(q-{}^{2}/_{\alpha_N})}{\Gamma q} \tag{8}$$

$$\mu''_{1N}(\mathcal{Z}) = \frac{\beta_N \Gamma(1+{}^{1}/_{\alpha_N})\Gamma(q-{}^{1}/_{\alpha_N})}{\Gamma q} \tag{9}$$

Putting (8) and (9) in (7) provides

$$\mathcal{V}_N(\mathcal{Z}) = \frac{{\beta_N}^2 \Gamma q \Gamma(1+{}^{2}/_{\alpha_N})\Gamma(q-{}^{2}/_{\alpha_N}) - (\Gamma(1+{}^{1}/_{\alpha_N})\Gamma(q-{}^{1}/_{\alpha_N}))^2}{\Gamma q^2} \tag{10}$$

which is required expression.

Theorem 4: Derive the expression for CV of the proposed model

Proof: The CV of the model is defined as:

$$CV = SD/mean \tag{11}$$

Putting values from (6) and (10) in (11) yields:

$$CV = \frac{\sqrt{\frac{{\beta_N}^2 \Gamma q \left[\Gamma(1+{}^{2}/_{\alpha_N})\right]\Gamma(q-{}^{2}/_{\alpha_N}) - \left(\left[\Gamma(1+{}^{1}/_{\alpha_N})\right]\Gamma(q-{}^{1}/_{\alpha_N})\right)^2}{\Gamma q^2}}}{\frac{\beta_N \left[\Gamma(1+{}^{1}/_{\alpha_N})\right]\left[\Gamma(q-{}^{1}/_{\alpha_N})\right]}{\Gamma q}} \tag{12}$$

Theorem 5: Derive the expression of shape coefficients

Proof: By definition shape coefficients are defined as:

$$\beta_{1N} = \frac{\mu'_{3N}}{(\mu'_{2N})^{3/2}} \tag{13}$$

$$\beta_{2N} = \frac{\mu'_{4N}}{{\mu'_{2N}}^2} \tag{14}$$

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Putting the expression from (4) and simplification yielded:

$$\beta_{1N} = \left[\frac{\lceil q^2\lambda_3 - 3\sqrt{q}\lambda_1\lambda_2 + 2\lambda_1^3}{\left(\sqrt{q}\lambda_2 - \lambda_1^2\right)^{3/2}}\right]^2 \tag{15}$$

$$\beta_{2N} = \frac{\lceil q^3\lambda_4 - 4\lceil q^2\lambda_3\lambda_1 + 6\lceil q\lambda_2{\lambda_1}^2 - 3\lambda_1^4}{\left(\sqrt{q}\lambda_2 - \lambda_1^2\right)^2} \tag{16}$$

where $\lambda_j = \left(\left\lceil q - {}^{j}/_{\alpha_N}\right)\left(\left\lceil 1 + {}^{j}/_{\alpha_N}\right)$

Theorem 6 Derive the expression for mode of the proposed model

Proof: By definition model value is define as:

$$F(m) = 1/2 \tag{17}$$

$$1 - \left[1 + \left(\frac{m}{\beta_N}\right)^{\alpha_N}\right]^{-q} = \frac{1}{2} \tag{18}$$

Simplification of (18) yielded:

$$m = \beta_N \left(\frac{\alpha_N - 1}{\alpha_N q + 1}\right)^{1/\alpha_N} ;\ \alpha_N > 1$$

which is required result.

4. Random Number Generation

In this section, we have discussed random generation using inverse CDF methodology. By applying the inverse CDF transformation, we can transform uniform random numbers into numbers that adhere to a specific distribution. This technique is particularly useful for generating random numbers that align with a desired distribution, making it valuable for statistical modeling and Monte Carlo simulations. While this method is a fundamental tool in producing accurate random numbers across various scientific and technical fields, it may not always be suitable for all distributions due to limitations on invertibility or computational challenges. The inverse CDF of the proposed model is given by:

$$u = 1 - \left[1 + (z/\beta_N)^{(\alpha_N)}\right]^{(-q)} \tag{19}$$

$$\mathrm{F}^{-1}\,(\mathrm{u}_\mathrm{N}) = \beta_\mathrm{N}\left\{(1 - \mathrm{u}_\mathrm{N})^{-1/q} - 1\right\}^{1/\alpha_\mathrm{N}} \tag{20}$$

where u_N follows the neutrosophic uniform distribution.

Generating neutrosophic random numbers involves integrating randomness with the degrees of truth, indeterminacy, and falsehood linked to each value. Various methods can be used to accomplish this, including modifying existing random number generation algorithms to accommodate neutrosophic uncertainty or developing new algorithms tailored for neutrosophic sets.

Now providing some different values of neutrosophic parameters in (20), we can generate random some from the proposed model. For example, if $\beta_N = [2,4]$, $\alpha_N = [1,1]$ and $q = [2,2]$

random number are provide in Table 1

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Table 1: Some random numbers from the proposed model

Random numbers				
[0.369, 0.739]	[2.236, 4.694]	[0.602, 1.203]	[3.847, 7.695]	[6.196, 12.394]
[0.047, 0.094]	[0.911, 1.823]	[4.097, 1.972]	[0.986, 1.426]	[0.713, 15.252]
[1.522, 1.410]	[1.059, 3.044]	[0.111, 2.119]	[4.319, 0.223]	[0.303, 8.638]
[0.439, 0.607]	[7.376, 0.879]	[4.017, 14.753]	[0.303,0.401]	[0.043, 0.478]
[0.439, 8.035]	[7.376, 3.217]	[4.017, 2.671]	[1.608, 2.671]	[1.335, 48.841]
[24.420, 2.817]	[1.408, 2.487]	[1.704, 3.409]	[0.961, 1.924]	[1.139, 2.279]
[0.372, 0.744]	[0.165, 0.331]	[8.400, 16.802]	[4.398, 3.192]	[1.596, 4.845]

In the scenario where uncertainty is present solely in the scale parameter of the proposed model, Table 1 displays the random numbers. Neutrosophic random number generation merges neutrosophic logic with random number generation principles. Neutrosophic random numbers in this context seek to represent ambiguity and imprecision in data or decision-making procedures. Incorporating neutrosophic logic into decision-making processes, risk analysis, fuzzy systems, and artificial intelligence can provide a more comprehensive framework for handling prevalent uncertainty and ambiguity. This approach is particularly valuable when traditional probabilistic methods may not fully capture the complexity of uncertain situations. By utilizing neutrosophic parameters, we can also calculate the statistical properties of the model with greater accuracy. In similar way we can calculate the statistical properties of the proposed model using the same neutrosophic parameters. Table 1 provides some essential statistical properties of the proposed model for neutrosophic parameters $\beta_N = [2,4]$, $\alpha_N = [1,1]$ and $q = [3,3]$

Table 2: Descriptive measures of NSMD

Statistical measures	Computed values
Mean	[1, 2]
Variance	[1, 4]
Mode	[0, 0]
CV	[1,1]

Table 2 displays inaccurate neutrosophic values derived from neutrosophic parameters. Descriptive statistics are crucial in understanding the characteristics of neutrosophic datasets, as they offer valuable insights. In this context, the mean, variance, mode, and coefficient of variation (CV) carry specific and detailed meanings that represent the fundamental uncertainty and indeterminacy present in neutrosophic data. The neutrosophic mean is a statistical measure that calculates the average value of a dataset by taking into account ambiguity, inconsistency, and indeterminacy. In addition to the numerical values, factors such as degrees of truth, indeterminacy, and falsehood associated with each data point are also considered in determining the balance point. The dispersion or spread of neutrosophic data around its mean is quantified by the neutrosophic variance. By considering the uncertainty and variability present in each data point, a quantification is offered for the acceptable range of values that can accommodate the imprecision and discrepancies found within the dataset.

5. Estimation Method

The maximum likelihood estimation (MLE) technique can be used to find the unknown values of the suggested model from sample data. MLE is a popular and reliable statistical estimation method known for its robustness and efficiency. It provides parameter estimates that are asymptotically normal, efficient, and consistent, ensuring high reliability as the sample size increases. MLE is a versatile tool that is beneficial for statistical analysis due to its ability to work with various models and data types.

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By optimizing the data, MLE can provide accurate parameter estimations and support in-depth analysis by creating confidence intervals and conducting hypothesis testing. The importance of the Maximum Likelihood Estimation (MLE) method lies in its ability to compare models using likelihood ratio tests, aiding in the selection of the most optimal model. With its emphasis on precision, adaptability, and extensive inferential capabilities, MLE plays a crucial role in statistical practice.

The loglikelihood function of the proposed model can be written as:

$$l(\alpha_N, \beta_N, q|x) = n\ln\alpha_N - n\alpha_N \ln\beta_N + n\ln q + (\alpha_N - 1)\sum_{i=1}^{n}\ln x_i - (q+1)\sum_{i=1}^{n}\ln\left[1 + \left(\frac{x_i}{\beta_N}\right)^{\alpha_N}\right] \quad (21)$$

Partial derivative of (21) with respect to unknow values are given by:

$$\frac{\partial l(\alpha_N, \beta_N, q|x)}{\partial \alpha_N} = \frac{n}{\alpha_N} - n\ln\beta_N + \sum_{i=1}^{n}\ln x_i - (q+1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta_N}\right)^{\alpha_N}\ln\left(\frac{x_i}{\beta_N}\right)}{\left[1+\left(\frac{x_i}{\beta_N}\right)^{\alpha_N}\right]} = 0 \quad (22)$$

$$\frac{\partial l(\alpha_N, \beta_N, q|x)}{\partial \beta_N} = -\frac{n\alpha_N}{\beta_N} + \alpha_N(q+1)\sum_{i=1}^{n}\frac{x_i^{\alpha_N}}{\beta_N^{\alpha_N+1}\left[1+\left(\frac{x_i}{\beta_N}\right)^{\alpha_N}\right]} = 0 \quad (23)$$

$$\frac{\partial l(\alpha_N, \beta_N, q|x)}{\partial q} = \frac{n}{q} - (\alpha_N + 1)\sum_{i=1}^{n}\ln\left[1 + \left(\frac{x_i}{\beta_N}\right)^{\alpha_N}\right] = 0 \quad (24)$$

Structures of (21), (22) and (23) are in the way that we cannot find analytical results so we need some numerial method to find optimum solution for unknown parameter that can maximize (20).

We have simulated random sample from the NSMD with parameters $\beta\_N = [1,3]$, $\alpha\_N = [1.5,1.5]$ and $q = [2,2]$, results of unknown values with mean square error (MSE) are shown in Table 3

Table 3: Estimated values of the proposed model with essential MSE

Sample size	MSE			Estimated values		
	α_N	β_N	q	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{q}$
25	[28.7, 31.86]	[48, 223.31]	[157, 348]	[1.39, 1.39]	[1.81,5.45	[2.58, 2.59]
50	[0.12, 0.12]	[15.6, 68.3]	[54, 108]]	[1.63, 1.63]	[0.78, 2.36]	[1.43, 1.44]
100	[0.05, 0.05]	[6.67,45.96]	[29, 43.1]	[1.52, 1.53]	[1.04, 3.13]	[2.14, 2.15]
150	[0.03, 0.03]	[4.0, 22.53]	[14.6, 23.5]	[1.54, 1.55]	[0.96, 2.89]	[1.92, 1.92]
200	[0.02,0.02]	[1.04, 7.56]	[4.46, 5.81]	[1.78, 1.78]	[0.73, 2.22]	[1.45, 1.45]
300	[0.02, 0.016]	[0.23, 2.11]	[0.97, 0.97]	[1.65, 1.70]	[0.84, 2.52]	[1.70, 1.70]

Table 3 demonstrates that the MSE of the parameter estimates for α_N, β_N, and q dramatically lowers with increasing sample size, from 25 to 300, showing better accuracy. The benefit of bigger sample sizes in obtaining more accurate estimates is highlighted by this decrease in MSE. As sample size increases, the predicted values for α_N, β_N, and q converge closer to their true values, which is consistent with the expected result of statistical analysis where larger samples yield more dependable data. This pattern emphasizes how crucial it is to use sample sizes that are big enough to improve the accuracy of parameter estimation in any statistical model.

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6. Real Application

The renewable energy sector is essential in combatting climate change and reducing greenhouse gas emissions. It is highlighted that the burning of fossil fuels for energy production is the primary contributor to greenhouse gas emissions, leading to climate change. Given the urgency to decrease these emissions and their detrimental effects, transitioning to renewable energy sources is imperative. Despite being a major oil-producing nation, the Kingdom of Saudi Arabia recognizes the importance of renewable energy for its environmental benefits. According to GlobalData, Saudi Arabia's renewable power capacity has increased at a compound annual growth rate of 82.4 percent from 0.02 GW to 3 GW from 2015 to 2023. By transitioning to renewable energy sources, Saudi Arabia can significantly decrease its carbon emissions and contribute to global efforts in combating climate change. The growing population has resulted in a rise in carbon emissions, as shown in Table 4 [26]. Saudi Arabia would need to reduce its emissions in order to meet the international standards.

Table 4: Carbon emission in Saudi Arabia (2012-2021)

Year	2012	2013	2014	2015	2016	2017	2018	2019	2020
Carbon emission	463.3	471.1	506.9	531.3	527.7	521.3	498.0	498.3	498.3

This indicator shows the extent to which human society and its supporting sectors are affected by the future changing climate conditions. The solution lies within the renewable energy industry and requires concerted efforts to promote sustainable solutions that can meet our energy demands without the harmful consequences connected with the burning of fossil fuels. Solar photovoltaic (PV) and wind power are significant contributors to the renewable energy sector. The quantification of carbon emissions in precise numerical terms may seem impractical, as the measurement of carbon emissions cannot be completely accurate due to a multitude of reasons. However, displaying the data in intervals instead of precise values, as shown in Table 5, can provide a more precise representation of the inherent uncertainty in these measures. The interval values can be acquired by implementing the technique outlined by [27]. Using neutrosophic density function analysis is crucial for analyzing these types of data sets. The model analysis we offer enables the study of data within specific ranges or intervals, taking into account the inherent uncertainty and unpredictability in carbon emission measurements. By integrating this analytical methodology, we can gain a deeper comprehension of the subtleties and variations in emissions data, resulting in more knowledgeable decision-making procedures and efficient mitigation tactics.

Table 5: Carbon emission with imprecise values

Carbon emission				
[460.23, 465.25]	[465.24, 480.24]	[500.31, 515.41]	[520.12, 535.24]	[522.12, 530.54]
[518.54, 525.24]	[480.12, 500.24]	[485.12, 505.14]		

Now the conventional model of the Singh Maddala distribution cannot be utilized here due to imprecise observations. We employed our proposed model to the data with imprecise observations and results are shown in Table 6

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Table 6: Basic characteristics of the carbon emissions data

Statistical characteristics	Computed values
$\widehat{\beta}_N$	[580.47, 582.81]
$\hat{q}$	[34.41, 42.51]
$\widehat{\alpha}_N$	[25.26, 28.27]
Estimated average	[602.98, 610.97]

The Table 6 provides estimated values for the parameters and statistical characteristics of the proposed model, particularly in the context of uncertainty and imprecise data. The spread of the distribution is managed by the scale parameter $\widehat{\beta}_N$. More dispersion of the data is indicated by a greater $\widehat{\beta}_N$ value. A reasonably stable estimate is suggested by the range [580.47, 582.81], which shows stability in the distribution's scale. The tail behaviour of the distribution is influenced by the shape parameter $\hat{q}$. A heavy-tailed distribution, indicated by the range [34.41, 42.51], means that extreme values—both high and low—have a higher probability than they would in a normal distribution. Although there is significant variability in this range, it stays inside the designated band that determines the distribution's form. Like $\hat{q}$, $\widehat{\alpha}_N$ affects the form and behaviour of the tail. The form attributes appear to be moderately variable, as indicated by the range [25.26, 28.27], which also influences the kurtosis and the probability of extreme events. An approximation of the distribution's mean, or central tendency, can be found in this range. The close range suggests that the average's location is highly certain. The data set modelled by this distribution appears to have a high mean, which may be a sign of high-value measurements within the data set, given that the values are more than 600.

7. Conclusions

This study shows that the Neutrosophic Singh Maddala model is a reliable and useful tool for estimating the life expectancy of manufacturing businesses. The study goes beyond the conventional Maddala model by improving the theoretical foundations with new formulations for the moments and mode of the SNMD distribution. The effectiveness of the SNMD is further validated by including comparisons with traditional models. The survival probability parameters and confidence ranges have been precisely computed through the application of maximum likelihood estimation, facilitating reliable comparisons. The actual data application highlights the usefulness of the suggested approach in practical situations, validating its applicability and efficacy. The study makes a substantial contribution to the analytical tools available for evaluating the likelihood of manufacturing firms surviving, providing a solid foundation for further study and useful applications. The suggested model for financial analysts or risk managers suggests that there could be substantial gains or losses, which calls for cautious risk management techniques. This distribution can be used as a model for income or wealth distributions, where extreme values and considerable variability are typical.

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