On The 4-Cyclic Refined Neutrosophic Solutions of The Diophantine Equation $X^n = 1$ and m-Cyclic Refined Neutrosophic Modulo Integers

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Abstract

The ring of n-cyclic refined neutrosophic integers is a logical extension of the integer ring Z based on a special multiplication operation defined between the indeterminacy algebraic elements. In this paper, we provide a full description of the 4-cyclic refined neutrosophic integer roots of unity, where we prove that for odd values of n we get exactly two different solutions. For even values of n, we get exactly 15 different solutions. On the other hand, we characterize the m-cyclic refined neutrosophic modulo integers rings and present many of their algebraic properties based on neutrosophic homomorphisms and substructures.

Keywords: 4-cyclic refined neutrosophic integer; Diophantine equation; 4-cyclic refined neutrosophic solution; roots of unity; m-cyclic refined neutrosophic modulo integers ring.

1. Introduction

The ring of the n-cyclic refined neutrosophic integers was proposed for the first time in [9]. The authors combined algebraic rings with neutrosophic algebraic indeterminate elements in one algebraic structure by using a multiplication property very similar to the structure of the cyclic abelian group Z_n.

Later, this concept became a fertile material for studying generalizations of algebraic structures associated with it, in which we find several important results related to n-cyclic refined neutrosophic groups [11], n-cyclic refined neutrosophic spaces [10], and even complex numbers [14]. Von Shtawzen et.al [1-4, 12] studied the group of units in different types of n-cyclic refined neutrosophic rings, where they showed a close correlation between the classification of the group of units and the nonlinear Diophantine equations [5].

In [6-8], Sankari et.al have proved Von Shtawzen's conjectures and found all solutions for the related Diophantine equations. Laterally, Salman and others have presented an important result that clarifies how 3-cyclic refined neutrosophic integers can be represented using complex unity roots [13].

The 3-cyclic refined neutrosophic solutions of the Diophantine equation $X^n = 1$ were proposed in [15] by using the classification isomorphism of the 3-cyclic refined neutrosophic ring of integers.

This has prompted us to study the 4-cyclic refined neutrosophic integer solutions for the equation $X^n = 1$, where we show that it has two different solutions if n is odd, and 15 different solutions if n is even and to characterize the m-cyclic refined neutrosophic modulo integers rings and present many of their algebraic properties based on neutrosophic homomorphisms and substructures.

Definition [9]
If R is a ring, the corresponding n-cyclic refined neutrosophic ring is defined as
$R_n(I) = \{r_0 + r_1 I_1 + \cdots + r_n I_n; r_i \in R, I_i I_j = I_{i+j\,(mod\ n)}\}$.

Doi: https://doi.org/10.54216/JNFS.080205
Received: November 10, 2023 Revised: January 12, 2024 Accepted: April 19, 2024

Addition on $R_n(I)$ is defined as follows:
$(r_0 + r_1 I_1 + \cdots + r_n I_n)$+$(m_0 + m_1 I_1 + \cdots + m_n I_n)$= $(r_0 + m_0) + (r_1 + m_1)I_1 + \cdots + (r_n + m_n)I_n$.
Multiplication on $R_n(I)$ is defined as follows:
$(r_0 + r_1 I_1 + \cdots + r_n I_n)$.$(m_0 + m_1 I_1 + \cdots + m_n I_n)$= $(r_0 . m_0) + \left(\sum_{i+j\equiv 1(mod\ n)} r_i . m_j\right) I_1 + \cdots +$ $\left(\sum_{i+j\equiv n(mod\ n)} r_i . m_j\right) I_n$.

Remark:

The 4-cyclic refined neutrosophic roots of unity are exactly the solutions of the Diophantine equation $X^n = 1$ in the 4-cyclic refined neutrosophic ring of integers $Z_4(I)$.

2. Main Discussion

Definition:

Let $X = x_0 + \sum_{i=1}^{4} x_i I_i \in Z_4(I)$, then X is called a Diophantine 4-cyclic refined neutrosophic root of unity if and only if: $X^n = 1$ with $n \in \mathbb{N}$.

Remark:

According to [13], $X^n = 1$ if and only if:

$$\begin{cases} x_0^n = 1 & (1) \\ \left(\sum_{i=0}^{4} x_i\right)^n = 1 & (2) \\ \left(\sum_{i=0}^{4} (-1)^i x_i\right)^n = 1 & (3) \\ [(x_0 + x_4 - x_2) + i(x_1 - x_3)]^n = 1 & (4) \end{cases}$$

Main Results:

Equation (1) means that: $x_0 = 1$ if n is odd, and $x_0 \in \{-1.1\}$ if n is even.
Equation (2) means that: $\sum_{i=0}^{4} x_i = 1$ if n is odd, and $\sum_{i=0}^{4} x_i \in \{-1.1\}$ if n is even.
Equation (3) means that: $\left(\sum_{i=0}^{4} (-1)^i x_i\right)^n = 1$ if n is odd, and $\sum_{i=0}^{4} (-1)^i x_i \in \{-1.1\}$ if n is even.
Equation (4) is equivalent to: $|(x_0 + x_4 - x_2) + i(x_1 - x_3)| = 1$. hence: $(x_0 + x_4 - x_2)^2 + (x_1 - x_3)^2 = 1$

The possible solutions are:

$$\begin{cases} x_0 + x_4 - x_2 \in \{-1.1\} \\ x_1 - x_3 = 0 \end{cases} . \begin{cases} x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 \in \{-1.1\} \end{cases}$$

We discuss the possible cases:

For n is odd, we have:

Case (1):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

Thus: $x_2 = x_4 = x_3 = x_1 = 0$, and $X = 1$

Case (2):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = -1 \\ x_1 - x_3 = 0 \end{cases}$$

Thus: $x_4 = -1.\ x_2 = 1.\ x_3 = x_1 = 0$, and $X = 1 + I_2 - I_4$

Case (3):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

Thus: $x_4 = -\frac{1}{2}$ a contradiction.

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Case (4):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = -1 \end{cases}$$

Thus: $x_4 = -\frac{1}{2}$ a contradiction.

For n is even, we see:

Cases (1),(2),(3), and (4) are similar to the case of odd n.

The rest of cases are:

Case (5):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = -2 \\ x_0 + x_4 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

$x_4 = -\frac{1}{2}$ a contradiction.

Case (6):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -2 \\ - x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

$x_4 = -\frac{1}{2}$ a contradiction.

Case (7):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -2 \\ -x_1 + x_2 - x_3 + x_4 = -2 \\ x_0 + x_4 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

$x_1 = -\frac{1}{2}$ a contradiction.

Case (8):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ x_0 + x_4 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

Thus: $x_1 = x_3 = 0.\ x_4 = 2.\ x_2 = 0.$, and $X = -1 + 2I_4$.

Case (9):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

Thus: $x_4 = \frac{3}{2}$ a contradiction.

Case (10):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ x_0 + x_4 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

Thus: $x_4 = \frac{3}{2}$ a contradiction.

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Case (11):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = -2 \\ x_0 + x_4 - x_2 = -1 \\ x_1 - x_3 = 0 \end{cases}$$

Thus: $x_4 = \frac{-3}{2}$ a contradiction.

Case (12):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -2 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = -1 \\ x_1 - x_3 = 0 \end{cases}$$

Thus: $x_4 = \frac{-3}{2}$ a contradiction.

Case (13):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -2 \\ -x_1 + x_2 - x_3 + x_4 = -2 \\ x_0 + x_4 - x_2 = -1 \\ x_1 - x_3 = 0 \end{cases}$$

$x_1 = x_3 = x_2 = 0.\ x_4 = -2$, and $X = 1 - 2I_4$.

Case (14):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ x_0 + x_4 - x_2 = -1 \\ x_1 - x_3 = 0 \end{cases}$$

$x_1 = x_3 = 0 . x_2 = x_4 = 1$, and $X = -1 + I_2 + I_4$

Case (15):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = -1 \\ x_1 - x_3 = 0 \end{cases}$$

Thus $x_1 = \frac{1}{2}$ a contradiction.

Case (16):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = -2 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

$x_3 = 0. x_1 = 1. x_2 = 0.\ x_4 = -1$, and $X = 1 + I_1 - I_4$

Case (17):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -2 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

$x_3 = -1. x_1 = 0. x_2 = 0.\ x_4 = -1$, and $X = 1 - I_3 - I_4$

Case (18):

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$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -2 \\ -x_1 + x_2 - x_3 + x_4 = -2 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

Hence $x_4 = \frac{-3}{2}$ which is forbidden.

Case (19):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ x_0 + x_4 - x_2 = -1 \\ x_1 - x_3 = 0 \end{cases}$$

Hence $x_4 = \frac{1}{2}$ is a contradiction.

Case (20):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

We get $x_4 = \frac{3}{2}$, which is impossible.

Case (21):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

We get $x_1 = 1. x_3 = 0. x_2 = 0.\ x_4 = 1$, and $X = -1 + I_1 + I_4$

Case (22):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

We get $x_4 = 1. x_2 = 0. x_1 = 0.\ x_3 = -1$, and $X = -1 - I_3 + I_4$

Case (23):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = -2 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = -1 \end{cases}$$

We get: $x_3 = 1. x_1 = 0. x_2 = 0. x_4 = -1$, and $X = 1 + I_3 - I_4$

Case (24):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -2 \\ -\ x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = -1 \end{cases}$$

We get $x_1 = -1. x_3 = 0. x_4 = -1.\ x_2 = 0$, and $X = 1 - I_1 - I_4$

Case (25):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -2 \\ -x_1 + x_2 - x_3 + x_4 = -2 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = -1 \end{cases}$$

We get $x_4 = \frac{-3}{2}$ which is a contradiction.

Case (26):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = -1 \end{cases}$$

We get $x_4 = \frac{3}{2}$, which is a contradiction.

Case (27):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = -1 \end{cases}$$

We get $x_4 = 1. x_2 = 0. x_1 = 0.\ x_3 = 1$, and $X = -1 + I_3 + I_4$

Case (28):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 2 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = -1 \end{cases}$$

We get: $x_4 = 1. x_2 = 0. x_1 = -1.\ x_3 = 0$, and $X = -1 - I_1 + I_4$

Case (29):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 1 \\ x_1 - x_3 = 0 \end{cases}$$

We get: $x_1 = x_3 = 0\ . x_4 = 1. x_2 = -1$, and $X = -1 - I_2 + I_4$

Case (30):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

We get: $x_4 = \frac{1}{2}$, which is impossible.

Case (31):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = -1 \\ x_1 - x_3 = 0 \end{cases}$$

We get: $x_1 = x_2 = x_3 = x_4 = 0$, and $X = -1$

Doi: https://doi.org/10.54216/JNFS.080205
Received: November 10, 2023 Revised: January 12, 2024 Accepted: April 19, 2024

Case (32):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 - x_3 = -1 \end{cases}$$

We get: $x_4 = \frac{1}{2}$, which is a contradiction.

3. The General result:

For odd n: $X \in \{1.\ 1 + I_2 - I_4\}$

For even n: $X \in$

$$\left\{\begin{matrix} 1.\ 1 + I_2 - I_4 . -1 + 2I_4 . 1 - 2I_4 . -1 + I_2 + I_4 . 1 + I_1 - I_4 . 1 - I_3 - I_4 . -1 - I_3 + I_4 .\ -1 + I_1 + I_4 . \\ 1 + I_3 - I_4 . 1 - I_1 - I_4 . -1 + I_3 + I_4 . -1 - I_1 + I_4 . -1 - I_2 + I_4 . -1 \end{matrix}\right\}$$

m-cyclic refined neutrosophic modulo integers rings:

Definition:

Let $Z_m = \{0.1. \ldots . m - 1\}$ be the ring of integers modulo m, let n be fixed positive integer, the n-cyclic refined neutrosophic ring of integers modulo m is defined as follows:

$Z_m^{(n)}(I) = \{a_0 + a_1 I_1 + \cdots + a_n I_n\ ; a_i \in \mathbb{Z}_m\}$

Addition is defined as follows:

$+: Z_m^{(n)}(I) \times Z_m^{(n)}(I) \to Z_m^{(n)}(I)$, such that:

$(a_0 + \sum_{i=1}^{n} a_i I_i) + (b_0 + \sum_{i=1}^{n} b_i I_i) = (a_0 + b_0) + \sum_{i=1}^{n} (a_i + b_i) I_i$.

Multiplication is defined as follows:

$(a_0 + \sum_{i=1}^{n} a_i I_i) \times (b_0 + \sum_{i=1}^{n} b_i I_i) = a_0 b_0 + \sum_{i.j=1}^{n} a_i b_j I_i I_j + \sum_{i=1}^{n} a_0 b_i I_i + \sum_{i=1}^{n} b_0 a_i I_i$, where:

$I_i I_j = I_{i+j (mod\ n)}$

Example:

For $m = 2. n = 3$, we have:

$Z_2^{(3)}(I) = \{0.1. I_1 . I_2 . I_3 . I_1 + I_2 . I_1 + I_3 . I_2 + I_3\}$.

For example $(I_1 + I_2) \times (I_1 + I_3) = I_2 + I_1 + I_3 + I_2 = I_1 + I_3$.

Remark:

1] $(Z_m^{(n)}(I). +. \times)$ is commutative finite ring.

2] $|Z_m^{(n)}(I)| = m^{n+1}$.

Definition:

The 2-cyclic refined ring modulo m is:

$Z_m^{(2)}(I) = \{x_0 + x_1 I_1 + x_2 I_2\ ; x_i \in \mathbb{Z}_m\}$.

It is clear that $|Z_m^{(2)}(I)| = m^3$.

Theorem:

The mapping $f: Z_m^{(2)}(I) \to Z_m \times Z_m \times Z_m$ such that $f(x_0 + x_1 I_1 + x_2 I_2) = (x_0 . x_0 + x_1 + x_2 . x_0 - x_1 + x_2)$ is a ring homomorphism.

Proof:

For $x_0 + x_1 I_1 + x_2 I_2 = y_0 + y_1 I_1 + y_2 I_2$, then $x_i = y_i$ for all $0 \le i \le 2$, thus $(x_0 . x_0 + x_1 + x_2 . x_0 - x_1 + x_2) = (y_0 . y_0 + y_1 + y_2 . y_0 - y_1 + y_2)$ so that $f(x_0 + x_1 I_1 + x_2 I_2) = f(y_0 + y_1 I_1 + y_2 I_2)$.

For $X = x_0 + x_1 I_1 + x_2 I_2 . Y = y_0 + y_1 I_1 + y_2 I_2$, then:

$f(X + Y) = (x_0 + y_0 . \sum_{i=0}^{2} (x_i + y_i) . \sum_{i \neq 1} (x_i + y_i) - (x_1 + y_1)) = (x_0 . x_0 + x_1 + x_2 . x_0 - x_1 + x_2) + (y_0 . y_0 + y_1 + y_2 . y_0 - y_1 + y_2) = f(X) + f(Y)$.

$X \times Y = x_0 y_0 + I_1 (x_0 y_1 + x_1 y_0 + x_1 y_2 + x_2 y_1) + I_2 (x_0 y_2 + x_2 y_0 + x_1 y_1 + x_2 y_2)$,

$f(X \times Y) = \left(x_0 y_0 . \sum_{i.j=0}^{2} x_i y_j . x_0 y_0 - x_0 y_1 - x_1 y_0 - x_1 y_2 - x_2 y_1 + x_0 y_2 + x_2 y_0 + x_1 y_1 + x_2 y_2\right) = (x_0 . x_0 + x_1 + x_2 . x_0 - x_1 + x_2) \times (y_0 . y_0 + y_1 + y_2 . y_0 - y_1 + y_2) = f(X) f(Y)$.

Remark:

If $f(X) = (0.0.0)$. then: $\begin{cases} x_0 = 0 \\ x_1 + x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases}$

So that $x_1 = x_2 . 2x_1 = 0$, thus: $k_{er}(f) = \{x_1 I_1 + x_1 I_2 ;\ x_1 \in \mathbb{Z}_m . 2x_1 = 0\}$.

This implies that if $\gcd(m. 2) = 1$. then $k_{er}(f) = \{0\}$ and (f) will be injective.

On the other hand, if m is even, then:

Doi: https://doi.org/10.54216/JNFS.080205
Received: November 10, 2023 Revised: January 12, 2024 Accepted: April 19, 2024

$k_{er}(f) = \{\frac{m}{2}(I_1 + I_2), 0\}$.

Remark:

If m is odd, then for every $(x_0.x_1.x_2) \in \mathbb{Z}_m^3$, we have:

$X = x_0 + I_1(x_1 - x_2)(2)^{-1} + I_2(x_1 + x_2 - 2x_0)(2^{-1}) \in Z_m^{(2)}(I)$. such that:

$f(X) = \left(x_0.2^{-1}(2x_1 - 2x_0) + x_0.x_0 - 2^{-1}(x_1 - x_2) + 2^{-1}(x_1 + x_2 - 2x_0)\right) = (x_0.x_1.x_2)$, thus f will be a bijection.

This implies that $Z_m^{(2)}(I) \cong Z_m \times Z_m \times Z_m$ if m is odd.

Remark:

If m is even, then:

$Z_m^{(2)}(I)/\, k_{er}(f) \cong f(Z_m^{(2)}(I))$.

Theorem:

For even m, we have $f\left(Z_m^{(2)}(I)\right) = L$; where $L = \{(a, b, c); a \in Z_m, b - c \in \langle 2\rangle\}$.

Proof:

Let $X = x_0 + x_1 I_1 + x_2 I_2 \in Z_m^{(2)}(I)$, then:

$f(X) = (x_0.x_0 + x_1 + x_2.x_0 - x_1 + x_2) \in f\left(Z_m^{(2)}(I)\right)$, and $(x_0 + x_1 + x_2) - (x_0 - x_1 + x_2) = 2x_1 \in \langle 2\rangle$. thus $f(X) \in L$.

On the other hand, for $(a, b, c) \in L$, then:

$T = a + \frac{(b-c)}{2} I_1 + \left(\frac{b+c}{2} - a\right) I_2 \in Z_m^{(2)}(I)$, with $f(T) = (a.b.c)$. thus $f\left(Z_m^{(2)}(I)\right) = L$.

Result:

If m is even, then $Z_m^{(2)}(I)/\, Z_2 \cong L$.

Idempotents in $Z_m^{(2)}(I)$:

Definition:

Let $X = x_0 + x_1 I_1 + x_2 I_2 \in Z_m^{(2)}(I)$, then X is idempotent if $X^2 = X$.

Theorem:

Let $Z_m^{(2)}(I)$ be the 2-cyclic refined modulo m ring, then we have:

1] X is idempotent in $Z_m^{(2)}(I)$ if and only if: $x_0.x_0 + x_1 + x_2.x_0 - x_1 + x_2$ are idempotents in Z_m.

2] If m is odd, then the idempotent elements in $Z_m^{(2)}(I)$ are $\{1.1 + 2^{-1}I_1 + (2^{-1} - 1)I_2 . 1 + (m - 1)I_2 . 0. I_2 . 2^{-1}(m - 1)I_1 + 2^{-1}I_2 . 1 + 2^{-1}(m - 1)I_1 + (2^{-1} - 1)I_2 . 2^{-1}I_1 + 2^{-1}I_2\}$.

3] If m is even, then $X = x_0 + x_1 I_1 + x_2 I_2$ is idempotent in $Z_m^{(2)}(I)/\, k_{er}(f)$ if and only if:

$$\begin{cases} m|x_0(x_0 - 1) \\ m|(x_0 + x_1 + x_2)(x_0 + x_1 + x_2 - 1) \\ m|(x_0 - x_1 + x_2)(x_0 - x_1 + x_2 - 1) \end{cases}$$

Proof:

1]X is idempotent in $Z_m^{(2)}(I)$ if and only if $f(X)$ is idempotent in $f(Z_m^{(2)}(I))$, thus X is idempotent in $Z_m^{(2)}(I)$ if and only if $\begin{cases} x_0 \\ x_0 + x_1 + x_2 \\ x_0 + x_1 + x_2 \end{cases}$ are idempotents in Z_m.

2] For odd m, we have $Z_m^{(2)}(I) \cong Z_m \times Z_m \times Z_m$.

So that, we must find idempotents in $Z_m \times Z_m \times Z_m$ firstly the idempotents of $(Z_m)^3$are:

$\{(0.0.0). (1.1.1). (1.1.0). (1.0.0). (0.1.0). (0.0.1). (0.1.1). (1.0.1)\}$

Hence, the idempotents of $Z_m^{(2)}(I)$ are exactly the inverses of previous elements, which are:

$\{0.1.1 + 2^{-1}I_1 + (2^{-1} - 1)I_2. 1 + (m - 1)I_2 . I_2 . 2^{-1}(m - 1)I_1 + 2^{-1}I_2 . 1 + 2^{-1}(m - 1)I_1 + (2^{-1} - 1)I_2 . 2^{-1}I_1 + 2^{-1}I_2\}$.

3] For even m, if $a \in Z_m$, then a is idempotent if and only if $a^2 \equiv a(mod\ m)$, thus $m|a(a - 1)$which implies the proof.

Example:

For $m = 5. 2^{-1} = 3$, and the idempotents of $Z_5^{(2)}(I)$ are:

$\{0.1.1 + 3I_1 + 2I_2. 1 + 4I_2 . I_2 . 2I_1 + 3I_2 . 1 + 2I_1 + 2I_2 . 3I_1 + 3I_2\}$.

For $m = 12$. We have: $m|a(a - 1)$ if and only if: $a \in \{4.0.1\}$.

The idempotents of $f(Z_{12}^{(2)}(I))$ are:

$\{(0.0.0). (1.1.1). (4.4.4). (0.4.0). (0.0.4). (1.0.4). (1.4.0). (4.0.4). (4.4.0). (4.1.1). (0.1.1)\}$.

The idempotents of $Z_{12}^{(2)}(I)/\, k_{er}(f)$ are:

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Received: November 10, 2023 Revised: January 12, 2024 Accepted: April 19, 2024

$$\begin{cases} f^{-1}(0.0.0) = 0. f^{-1}(1.1.1) = 1. f^{-1}(4.4.4) = 4 \\ f^{-1}(0.4.0) = 2I_1 + 2I_2 . f^{-1}(0.0.4) = 10I_1 + 2I_2 . f^{-1}(1.0.4) = 1 + 10I_1 + I_2 \\ f^{-1}(1.4.0). f^{-1}(4.0.4). f^{-1}(4.4.0) \\ f^{-1}(4.1.1). f^{-1}(0.1.1). \end{cases}$$

Theorem:

For even values of m, the element $X = x_0 + x_1 I_1 + x_2 I_2$ is idempotent in $Z_m^{(2)}(I)$ if and only if:

$$\begin{cases} m | x_0(x_0 - 1) \\ m | x_1(2x_0 + 2x_2 - 1) \\ m | x_2(x_2 + 2x_0 - 1) + {x_1}^2 \end{cases}$$

Proof:

$X^2 = X$ is equivalent to: $\begin{cases} {x_0}^2 = x_0 \\ 2x_0 x_1 + 2x_1 x_2 = x_1 \\ {x_1}^2 + {x_2}^2 + 2x_0 x_2 = x_2 \end{cases}$

Thus: $\begin{cases} m | x_0(x_0 - 1) \\ m | x_1(2x_0 + 2x_2 - 1) \\ m | x_2(x_2 + 2x_0 - 1) + {x_1}^2 \end{cases}$

Example:

For $m = 12$, the possible idempotents are:

$f^{-1}(0.0.0) = 0. f^{-1}(1.1.1) = 1. f^{-1}(4.4.4) = 4.$

Definition:

An element $X = x_0 + x_1 I_1 + x_2 I_2 \in Z_m^{(2)}(I)$ is called 2-potent if and only if $X^2 = 0$.

This definition is equivalent to:

$$\begin{cases} {x_0}^2 \equiv 0 (mod\ m) \\ 2x_1(x_0 + x_2) \equiv 0 (mod\ m) \\ {x_1}^2 + {x_2}^2 + 2x_0 x_2 \equiv 0 (mod\ m) \end{cases}$$

Example:

For $Z_8^{(2)}(I)$, the 2-potent elements can be found under the following conditions:

$$\begin{cases} x_0 \in \{0.4\} \\ 2x_1(x_0 + x_2) = 8\, k_1 \ ;\ k_1 \in \mathbb{Z} \\ {x_1}^2 + {x_2}^2 + 2x_0 x_2 = 8\, k_2 \ ;\ k_2 \in \mathbb{Z} \end{cases} \qquad x_1 . x_2 \in Z_8 .$$

For $x_0 = 0$, we have: $\begin{cases} 2x_1 x_2 \equiv 0 (mod\ 8) \\ {x_1}^2 + {x_2}^2 \equiv 0 (mod\ 8) \end{cases}$

$\Longrightarrow (x_1 . x_2) \in \{(2.2).(0.4).(4.0).(6.6).(6.2).(2.6)\}$

Thus $X \in \{2I_1 + 2I_2 . 4I_2 . 4I_1 . 6I_1 + 6I_2 . 6I_1 + 2I_2 . 2I_1 + 6I_2\}$.

For $x_0 = 4$, we get: $\begin{cases} 2x_1 x_2 \equiv 0 (mod\ 8) \\ {x_1}^2 + {x_2}^2 \equiv 0 (mod\ 8) \end{cases}$

$\Longrightarrow (x_1 . x_2) \in \{(2.2).(0.4).(4.0).(6.6).(6.2).(2.6)\}$,

Thus $X \in \{4 + 2I_1 + 2I_2 . 4 + 4I_2 . 4 + 4I_1 . 4 + 6I_1 + 6I_2 . 4 + 6I_1 + 2I_2 . 4 + 2I_1 + 6I_2\}$.

Idempotents in 3 cyclic refined modulo m rings:

Remark:

If $X = x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3 \in Z_m^{(3)}(I)$, then X is idempotent if and only if $X^2 = X$, which is equivalent to:

$$\begin{cases} {x_0}^2 \equiv x_0 (mod\ m) \\ {x_2}^2 + 2x_0 x_1 + 2x_1 x_3 \equiv x_1 (mod\ m) \\ {x_1}^2 + 2x_0 x_2 + 2x_2 x_3 \equiv x_2 (mod\ m) \\ {x_3}^2 + 2x_0 x_3 + 2x_1 x_2 \equiv x_3 (mod\ m) \end{cases}$$

Example:

For $m = 3$, consider $Z_3^{(3)}(I)$. $X = x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3$ is idempotent, hence: $x_0 \in \{0.1\}$.

If $x_0 = 0$, we get: $\begin{cases} {x_2}^2 + 2x_1 x_3 \equiv x_1 (mod\ 3) \\ {x_1}^2 + 2x_2 x_3 \equiv x_2 (mod\ 3) \\ {x_3}^2 + 2x_1 x_2 \equiv x_3 (mod\ 3) \end{cases}$

For $x_1 = 0$, we get: $x_2 = x_3 = 0$. and $X = 0$.

Doi: https://doi.org/10.54216/JNFS.080205
Received: November 10, 2023 Revised: January 12, 2024 Accepted: April 19, 2024

For $x_1 = 1$, we get: $\begin{cases} x_2{}^2 + 2x_3 \equiv 1(mod\ 3) \\ 1 + 2x_2x_3 \equiv x_2(mod\ 3) \\ x_3{}^2 + 2x_2 \equiv x_3(mod\ 3) \end{cases}$

Which implies a contradiction.

For $x_1 = 2$, we get: $\begin{cases} x_2{}^2 + x_3 \equiv 2(mod\ 3) \\ 1 + 2x_2x_3 \equiv x_2(mod\ 3) \\ x_3{}^2 + x_2 \equiv x_3(mod\ 3) \end{cases}$

Which is impossible.

Now, if $x_0 = 1$, we get: $\begin{cases} x_2{}^2 + 2x_1 + 2x_1x_3 \equiv x_1(mod\ 3) \\ x_1{}^2 + 2x_2 + 2x_2x_3 \equiv x_2(mod\ 3) \\ x_3{}^2 + 2x_3 + 2x_1x_2 \equiv x_3(mod\ 3) \end{cases}$

For $x_1 = 1$, we get: $x_2 = 0\,.\,x_3 \in \{0.2\}$.

Thus: ;$X = 1.X = 1 + 2I_3$.

For $x_1 = 1$, we get: $\begin{cases} x_2{}^2 + 2x_3 \equiv 2(mod\ 3) \\ 1 + 2x_2 + 2x_2x_3 \equiv x_2(mod\ 3) \\ x_3{}^2 + 2x_3 + 2x_2x_3 \equiv x_3(mod\ 3) \end{cases}$

Which is impossible.

For $x_1 = 2$, we get: $\begin{cases} x_2{}^2 + x_3 \equiv 1(mod\ 3) \\ 2x_2 + 2x_2x_3 + 1 \equiv x_2(mod\ 3) \\ x_3{}^2 + 2x_3 + x_2 \equiv x_3(mod\ 3) \end{cases}$

Which is impossible.

2-potent element in $Z_m^{(3)}(I)$:

Remark:

$X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 \in Z_m^{(3)}(I)$ is 2-potent if and only if:

$$\begin{cases} x_0{}^2 \equiv 0\ (mod\ m) \\ x_2{}^2 + 2x_0x_1 + 2x_1x_3 \equiv 0\ (mod\ m) \\ x_1{}^2 + 2x_0x_2 + 2x_2x_3 \equiv 0\ (mod\ m) \\ x_3{}^2 + 2x_0x_3 + 2x_1x_2 \equiv 0\ (mod\ m) \end{cases}$$

Example:

For $m = 4$, let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3$ be a 2-potent element, then:

$x_0 \in \{0.2\}$ and: $\begin{cases} x_2{}^2 + 2x_1x_3 \equiv 0\ (mod\ 4) \\ x_1{}^2 + 2x_2x_3 \equiv 0\ (mod\ 4) \\ x_3{}^2 + 2x_1x_2 \equiv 0\ (mod\ 4) \end{cases}$

For $x_1 = 0$, we get: $x_2 \in \{0.2\}\,.\,x_3 \in \{0.2\}$.

For $x_1 = 2$, we get: $x_2 \in \{0.2\}\,.\,x_3 \in \{0.2\}$, so that:

$X \in \{0.2I_1.2I_2.2I_3.2I_1 + 2I_2.2I_1 + 2I_3.2I_2 + 2I_3.2.2 + 2I_1.2 + 2I_2.2 + 2I_3.2 + 2I_1 + 2I_2.2 + 2I_1 + 2I_3.2 + 2I_1 + 2I_2 + 2I_3.2 + 2I_2 + 2I_3.2I_1 + 2I_2 + 2I_3\}$.

4. Conclusion

In this paper, we computed all 4-cyclic refined neutrosophic Diophantine roots of unity, where we proved that only two solutions exist for the case of odd order (n), and 15 different solutions for the case of even order (n). In the future, we aim to extend our results for the general case of the n-cyclic refined neutrosophic ring of integers.

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Doi: https://doi.org/10.54216/JNFS.080205
Received: November 10, 2023 Revised: January 12, 2024 Accepted: April 19, 2024