



On the Non-Commutative Logical Rings As Novel Extensions of Neutrosophic Rings

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Abstract

This paper uses some logical algebraic elements to extend any ring into a non-commutative ring containing the original ring with many generalized substructures and special elements. On the other hand, we study the substructures of non-commutative logical rings such as AH-homomorphisms and AH-ideals with many examples that explain their algebraic validity. Also, we discuss the possibility of solving a linear Diophantine equation with two variables in the non-commutative logical ring of integers, where we present an easy algorithm to solve this kind of generalized Diophantine equation.

Keywords: neutrosophic ring; non-commutative; AH-ideal; AH-homomorphism.

1. Introduction

Algebraic ring theory is one of modern algebra's most broad and important branches of mathematics. Ring extension plays a major role in modern algebraic theories and in solving algebraic equations.

It was customary to build ring expansions through algebraic elements and the roots of polynomials until Smarandache et al. [17] introduced the idea of using logical elements defined by specific algebraic properties in constructing new expansions and defining Neutrosophic algebraic structures of all kinds, including vector spaces, integers, modules, and even matrices [7-10, 18-19]. These new ideas have been widely used by many researchers in expanding algebraic structures through plithogenic sets, which have been widely used in expanding rings, matrices, and even distance-preserving algebraic functions [1-6, 11-13].

On the other hand, some expansions related to real numbers that depend on fuzzy sets have been studied, and their algebraic properties and geometric constructions have been studied in many recently published research papers [14-16].

As a continuation of the extensive research efforts made previously, we have created a new expansion of rings based on algebraic elements of a logical nature, where we present the concept of non-commutative logical extension of a ring, with many elementary properties of this new generalization. Also, we discuss the solvability of linear Diophantine equations in two variables built over those rings in the integer case.

2. Main Discussion

Definition:

Let R be a ring, the corresponding non-commutative logical ring is defined as follows:

$$R_N = \{a + bN_1 + cN_2 \ ; \ N_1^2 = N_1, N_2^2 = N_2, N_1N_2 = N_1, N_2N_1 = N_2, a, b, c \in R\}.$$

We call it the non-commutative logical ring of type (1), and we denote it by NCR_1 .

Definition:

Addition: $R_N \times R_N \rightarrow R_N$ such that:

$$(a + bN_1 + cN_2) + (a' + b'N_1 + c'N_2) = (a + a') + (b + b')N_1 + (c + c')N_2.$$

Multiplication: $R_N \times R_N \rightarrow R_N$ such that:

$$(a + bN_1 + cN_2)(a' + b'N_1 + c'N_2) = aa' + N_1(ab' + ba' + bb' + bc') + N_2(ac' + ca' + cc' + cb').$$

Remark:

$(R_N, \times)$ is not commutative, that is because for $a \in R$, $(aN_1)(aN_2) = a^2N_1 \neq (aN_2)(aN_1) = a^2N_2$.

Remark:

If R has a unity (1), then R_N has as a unity.

Theorem:

$(R_N, +, \times)$ is a ring.

Proof:

Let $A = a_0 + a_1 N_1 + a_2 N_2, B = b_0 + b_1 N_1 + b_2 N_2, C = c_0 + c_1 N_1 + c_2 N_2 \in R_N$, we have:

$$A + B = B + A, A + 0 = 0 + A = A, A + (-A) = (-A) + A = 0.$$

Also, $(A \times B) \times C = [a_0b_0 + N_1(a_0b_1 + a_1b_0 + a_1b_1 + a_1b_2) + N_2(a_0b_2 + a_2b_0 + a_2b_2 + a_2b_1)] \times (c_0 + c_1N_1 + c_2N_2) = a_0b_0c_0 + a_0b_0c_1N_1 + a_0b_0c_2N_2 + N_1(a_0b_1c_0 + a_1b_0c_0 + a_1b_1c_0 + a_1b_2c_0) + N_1(a_0b_1c_1 + a_1b_0c_1 + a_1b_1c_1 + a_1b_2c_1) + N_1(a_0b_1c_2 + a_1b_0c_2 + a_1b_1c_2 + a_1b_2c_2) + N_2(a_0b_2c_0 + a_2b_0c_0 + a_2b_2c_0 + a_2b_1c_0) + N_2(a_0b_2c_1 + a_2b_0c_1 + a_2b_2c_1 + a_2b_1c_1) + N_2(a_0b_2c_2 + a_2b_0c_2 + a_2b_2c_2 + a_2b_1c_2) = (a_0b_0c_0) + N_1(a_0b_0c_1 + a_0b_1c_0 + a_1b_0c_0 + a_1b_1c_0 + a_1b_2c_0 + a_0b_1c_1 + a_1b_0c_1 + a_1b_1c_1 + a_1b_2c_1 + a_0b_1c_2 + a_1b_0c_2 + a_1b_1c_2 + a_1b_2c_2) + N_2(a_0b_0c_2 + a_0b_2c_0 + a_2b_0c_0 + a_2b_2c_0 + a_2b_1c_0 + a_0b_2c_1 + a_2b_0c_1 + a_2b_2c_1 + a_2b_1c_1 + a_0b_2c_2 + a_2b_0c_2 + a_2b_2c_2 + a_2b_1c_2) = A \times (B \times C).$

$$A \times (B + C) = (a_0 + a_1N_1 + a_2N_2) \times [(b_0 + c_0) + (b_1 + c_1)N_1 + (b_2 + c_2)N_2] = a_0(b_0 + c_0) + N_1[a_0(b_1 + c_1) + a_1(b_0 + c_0) + a_1(b_1 + c_1) + a_1(b_2 + c_2)] + N_2[a_2(b_0 + c_0) + a_0(b_2 + c_2) + a_2(b_1 + c_1) + a_2(b_2 + c_2)] = [a_0b_0 + N_1(a_0b_1 + a_1b_0 + a_1b_1 + a_1b_2) + N_2(a_2b_0 + a_0b_2 + a_2b_1 + a_2b_2)] + [a_0c_0 + N_1(a_0c_1 + a_1c_0 + a_1c_1 + a_1c_2) + N_2(a_2c_0 + a_0c_2 + a_2c_1 + a_2c_2)] = (A \times B) + (A \times C).$$

Remark:

It is clear that $R \subset R_N$.

Example:

Take $R = Z_3 = \{0.1.2\}$. $R_N = \{0.1.2.N_1.N_2.1 + N_1.1 + N_2.2 + N_1.2 + N_2.1 + N_1 + N_2.2 + N_1 + N_2.2 + 2N_1 + 2N_2.N_1 + N_2.1 + 2N_1.1 + 2N_2.1 + 2N_1 + N_2.1 + N_1 + 2N_2..2N_1 + 2N_2..2N_1 + N_2..N_1 + 2N_2.2N_1.2N_2.1 + 2N_1 + 2N_2.2 + 2N_1 + N_2.2 + N_1 + 2N_2.2 + 2N_1.2 + 2N_2\}$.

For example:

$$(1 + N_1)(1 + N_2) = 2 + N_2 + 2N_1 + N_1 = 2 + N_2.$$
$$(2 + N_2)(1 + N_1) = 2 + 2N_1 + N_2 + N_2 = 2 + 2N_1 + 2N_2.$$

Remark:

If R is finite with $|R| = n$. then $|R_N| = n^3$.

Example:

Take $\bar{R} = Z_2 = \{0,1\}$. then $R_N = \{0,1,N_1,N_2,1+N_1,1+N_2,N_1+N_2,1+N_1+N_2\}$.

Definition:

If R is a field, then R_N is called the non-commutative logical field, which is not a field in the ordinary meaning.

It is only a ring.

Definition:

Let $A = a_0 + a_1N_1 + a_2N_2 \in R_N$, then it is called a unity if and only if there exists $B = b_0 + b_1N_1 + b_2N_2 \in R_N$ such that $AB = BA = 1$.

Under the condition that R_N has a unity (1).

Theorem:

Let R be a commutative ring with unity, R_N be its corresponding $N \subset R_1$, then $A = a_0 + a_1N_1 + a_2N_2 \in R_N$ is a unit if and only if: $a_0, a_0^2 + a_0a_2 + a_1a_2$ are units in R .

Proof:

Assume that there exists $B = b_0 + b_1 N_1 + b_2 N_2 \in R_N$ such that $AB = BA = 1$. This is equivalent to:

$$\begin{cases} a_0 b_0 = 1 \\ a_0 b_1 + a_1 b_0 + a_1 b_1 + a_1 b_2 = 0 \\ a_0 b_2 + a_2 b_0 + a_2 b_2 + a_0 b_1 = 0 \end{cases}$$

Which is equivalent to:

$$\begin{cases} b_0 = \frac{1}{a_0} & (a_0 \text{ is a unit in } R) \\ b_1(a_0 + a_1) + b_2a_1 = -\frac{a_1}{a_0} \\ b_1a_0 + b_2(a_0 + a_2) = -\frac{a_2}{a_0} \end{cases}$$

$$\Delta = \begin{vmatrix} a_0 + a_1 & a_1 \\ a_0 & a_0 + a_2 \end{vmatrix} = (a_0 + a_1)(a_0 + a_2) - a_0 a_1 = a_0^2 + a_0 a_2 + a_1 a_2$$

To get a solution, Δ should be invertible in R .

$$\Delta_{b_1} = \begin{vmatrix} -\frac{a_1}{a_0} & a_1 \\ -\frac{a_2}{a_0} & a_0 + a_2 \end{vmatrix} = \frac{a_1 a_2}{a_0} - \frac{a_1(a_0 + a_2)}{a_0} = \frac{a_1 a_2 - a_1(a_0 + a_2)}{a_0} = \frac{a_1(a_2 - a_0 - a_2)}{a_0} = -a_1$$

$$\Delta_{b_2} = \begin{vmatrix} a_0 + a_1 & -\frac{a_1}{a_0} \\ a_0 & -\frac{a_2}{a_0} \end{vmatrix} = -\frac{a_2(a_0 + a_1)}{a_0} + \frac{a_1 a_0}{a_0} = -\frac{a_2(a_0 + a_1)}{a_0} + a_1$$

$$\text{Thus: } b_1 = \frac{\Delta_{b_1}}{\Delta} = \frac{-a_1}{(a_0 + a_1)(a_0 + a_2) - a_0 a_1}$$

$$b_2 = \frac{\Delta_{b_2}}{\Delta} = \frac{-\frac{a_2(a_0 + a_1)}{a_0} + a_1}{(a_0 + a_1)(a_0 + a_2) - a_0 a_1}$$

$$\text{Thus: } \begin{cases} b_1 = \frac{-a_1}{a_0^2 + a_0 a_2 + a_1 a_2} \\ b_2 = \frac{-a_2 a_0 - a_2 a_1 + a_1 a_0}{a_0(a_0^2 + a_0 a_2 + a_1 a_2)} \end{cases}$$

$$\text{And } A^{-1} = \frac{1}{a_0} + N_1 \left(\frac{-a_1}{a_0^2 + a_0 a_2 + a_1 a_2} \right) + N_2 \left(\frac{a_1 a_0 - a_2 a_0 - a_2 a_1}{a_0(a_0^2 + a_0 a_2 + a_1 a_2)} \right).$$

Remark:

If R is a field, then A is invertible if and only if: $a_0 \neq 0, a_0^2 + a_0 a_2 + a_1 a_2 \neq 0$.

Open problem:

What is the classification of the group of units of R_N ?

What is the relationship between $U(R_N)$ and $U(R)$?

Definition:

The element $A \in R_N$ is called idempotent if and only if $A^2 = A$.

It is called 2-potent if and only if $A^2 = 0$.

Theorem:

Let $A = a_0 + a_1 N_1 + a_2 N_2 \in R_N$, where R is commutative ring, then:

1] A is idempotent in R_N if and only if:

a) a_0 is idempotent in R .

b) $(a_1, a_1 + a_2 + 2a_0), (a_2, a_1 + a_2 + 2a_0)$ are duplets in R , with $a_1 + a_2 + 2a_0$ acts as an identity.

2] A is 2-potent in R_N if and only if:

a) a_0 is 2-potent in R .

b) a_1, a_2 are zero divisors in R with: $a_1(a_1 + a_2 + 2a_0) = a_2(a_1 + a_2 + 2a_0) = 0$.

Proof:

1] $A^2 = A$ is equivalent to:

$$\begin{cases} a_0^2 = a_0 \\ a_1^2 + 2a_0 a_1 + a_2 a_1 = a_1 \\ a_2^2 + 2a_0 a_2 + a_1 a_2 = a_2 \end{cases}$$

$$\text{Thus: } \begin{cases} a_0^2 = a_0 \\ a_1(a_1 + a_2 + 2a_0) = a_1 \\ a_2(a_1 + a_2 + 2a_0) = a_2 \end{cases}$$

Which is equivalent to $\boxed{a}$ and $\boxed{b}$.

$$2] A^2 = 0 \text{ is equivalent to: } \begin{cases} a_0^2 = 0 \\ a_1(a_1 + 2a_0 + a_2) = 0 \\ a_2(a_1 + 2a_0 + a_2) = 0 \end{cases}$$

Which is equivalent to $\boxed{a}$ and $\boxed{b}$.

Definition:

Let R_N be the NCR_1 , then $K = k_0 + k_1 N_1 + k_2 N_2 = \{r_0 + r_1 N_1 + r_2 N_2 ; r_i \in k_i\}$ is called AH-ideal of R_N if k_i are ideal in R .

If $k_0 = k_1 = k_2$, then K is called AHS-ideal.

Example:

Take $R = \mathbb{Z}$. $R_N = \{a + bN_1 + cN_2 ; a, b, c \in \mathbb{Z}\}$. $k_0 = \langle 2 \rangle, k_1 = \langle 3 \rangle, k_2 = \langle 6 \rangle$, we have:

$K = k_0 + k_1 N_1 + k_2 N_2 = \{2x_0 + 3x_1 N_1 + 6x_2 N_2 ; x_i \in \mathbb{Z}\}$ is an AH-ideal of R_N .

Also, $k_0 + k_0 N_1 + k_0 N_2 = \{2x_0 + 2x_1 N_1 + 2x_2 N_2 ; x_i \in \mathbb{Z}\}$ is an AHS-ideal.

Theorem:

Let $K = k_0 + k_1N_1 + k_2N_2$, $R = R_0 + R_1N_1 + R_2N_2$ be two AH-ideals of R_N , then: $K \cap R$ is an AH-ideal.

If K, R are two AHS-ideals, then $K \cap R$ is an AHS-ideal.

Proof:

$K \cap R = (K_0 \cap R_0) + (K_1 \cap R_1)N_1 + (K_2 \cap R_2)N_2$, where $\{K_i \cap R_i\}$ is an ideal of R .

So that $K \cap R$ is an AH-ideal:

On the other hand, if K, R are two AHS-ideals, then:

$K \cap R = (K_0 \cap R_0) + (K_1 \cap R_1)N_1 + (K_2 \cap R_2)N_2$; $K_0 \cap R_0 = K_1 \cap R_1 = K_2 \cap R_2$, thus $K \cap R$ is an AHS-ideal.

Definition:

Let R, T be two rings, and $f_i: R \rightarrow T$; $0 \leq i \leq 2$ are three homomorphisms, then $f = f_0 + f_1N_1 + f_2N_2: R_N \rightarrow T_N$ such that:

$f(a + bN_1 + cN_2) = f_0(a) + f_1(b)N_1 + f_2(c)N_2$ is called an AH-homomorphism.

If $f_0 = f_1 = f_2$, then f is called AHS-homomorphism.

Remark:

If f_0, f_1, f_2 are isomorphisms, then f is called AHS-isomorphism.

Definition:

Let $f = f_0 + f_1N_1 + f_2N_2, g = g_0 + g_1N_1 + g_2N_2: R_N \rightarrow T_N$ be two AH-homomorphisms, we define:

$(f + g): R_N \rightarrow T_N$ such that:

$(f + g)(a + bN_1 + cN_2) = (f_0 + g_0)(a) + (f_1 + g_1)(b)N_1 + (f_2 + g_2)(c)N_2$.

$(f \times g): R_N \rightarrow T_N$ such that:

$(f \times g)(a + bN_1 + cN_2) = (f_0 \cdot g_0)(a) + (f_1 \cdot g_1)(b)N_1 + (f_2 \cdot g_2)(c)N_2$.

The set of all AH-homomorphisms between R_N and T_N is denoted by $AHH(R_N, T_N)$.

Definition:

Let $f = f_0 + f_1N_1 + f_2N_2: R_N \rightarrow T_N, g = g_0 + g_1N_1 + g_2N_2: T_N \rightarrow S_N$ be two AH-homomorphisms, we define:

$g \circ f: R_N \rightarrow S_N$; $g \circ f(a) = g \circ f(a + bN_1 + cN_2) = (g_0 \circ f_0) + (g_1 \circ f_1)N_1 + (g_2 \circ f_2)N_2$.

Definition:

Let $f = f_0 + f_1N_1 + f_2N_2: R_N \rightarrow T_N$ be an AH-homomorphism, then:

$AH_ker(f) = ker(f_0) + ker(f_1)N_1 + ker(f_2)N_2$.

$AH_Im(f) = Im(f_0) + Im(f_1)N_1 + Im(f_2)N_2$.

Theorem:

Let $f = f_0 + f_1N_1 + f_2N_2: R_N \rightarrow T_N$ be an AH-homomorphism, then:

1] $f(0) = 0, f(1) = 1 + N_1 + N_2$; $1 \in \mathbb{R}$.

2] $AH_ker(f)$ is an AH-ideal of R_N .

3] $AH_Im(f)$ is an AH-subring of T_N .

4] If K is an AH-ideal of R_N , then $f(K)$ is an AH-ideal of T_N .

Proof:

1] $f(0) = f_0(0) + f_1(0)N_1 + f_2(0)N_2 = 0 + 0 \cdot N_1 + 0 \cdot N_2 = 0$.

$f(1) = f_0(1) + f_1(1)N_1 + f_2(1)N_2 = 1 + N_1 + N_2$.

2] we have: $ker(f_0), ker(f_1), ker(f_2)$ are ideals of R , thus $AH_ker(f) = ker(f_0) + ker(f_1)N_1 + ker(f_2)N_2$ is an AH-ideal of R_N .

3] we have: $Im(f_0), Im(f_1), Im(f_2)$ are subrings of T , hence: $AH_Im(f)$ is an AH-subring of T_N .

4] Assume that $K = k_0 + k_1N_1 + k_2N_2$; k_i are ideals of R for $0 \leq i \leq 2$, then: $f(K) = f_0(k_0) + f_1(k_1)N_1 + f_2(k_2)N_2$ is an AH-ideal of T_N , that is because $f_i(k_i)$ is an ideal of T .

Theorem:

$(AHH(R_N, T_N), +, \times)$ is a ring.

Proof:

Suppose that $f = f_0 + f_1N_1 + f_2N_2, g = g_0 + g_1N_1 + g_2N_2, h = h_0 + h_1N_1 + h_2N_2$ be three arbitrary elements of $AHH(R_N, T_N)$, then:

$f + (g + h) = (f_0 + g_0 + h_0) + (f_1 + h_1 + g_1)N_1 + (f_2 + h_2 + g_2)N_2 = (f_0 + g_0) + h_0 + [(f_1 + h_1) + g_1]N_1 + [(f_2 + h_2) + g_2]N_2 = (f + g) + h$.

$f \times (g \times h) = (f_0g_0h_0) + (f_1g_1h_1)N_1 + (f_2g_2h_2)N_2 = (f_0g_0)h_0 + [(f_1g_1)h_1]N_1 + [(f_2g_2)h_2]N_2 = (f \times g) \times h$.

$f + 0 = 0 + f = f, f + (-f) = (-f) + f = 0, f + g = g + f$.

On the other hand, we have:

$f \times (g + h) = f_0(g_0 + h_0) + [f_1(g_1 + h_1)]N_1 + [f_2(g_2 + h_2)]N_2 = f_0g_0 + f_0h_0 + [f_1g_1 + f_1h_1]N_1 + [f_2g_2 + f_2h_2]N_2 = (f_0g_0 + f_1g_1N_1 + f_2g_2N_2) + (f_0h_0 + f_1h_1N_1 + f_2h_2N_2) = (f \times g) + (f \times h)$.

So that, our proof is complete.

Theorem:

Let $f : R_N \rightarrow T_N$, $g : T_N \rightarrow S_N$ be two AH-homomorphisms, then $g \circ f$ is an AH-homomorphism.

Proof:

$g \circ f = (g_0 \circ f_0) + (g_1 \circ f_1)N_1 + (g_2 \circ f_2)N_2$ and $g_i \circ f_i$ is a ring homomorphism between R and S , thus $g \circ f$ is an AH-homomorphism between R_N and S_N .

Definition:

Let R_N, T_N be two ($N \subset R_1$), then:

$$R_N \times T_N = \{(x, y); x \in R_N, y \in T_N\}.$$

Theorem:

$R_N \times T_N = (R \times T)_N$ is a ring.

Proof:

$$R_N \times T_N = \{(x, y); x \in R_N, y \in T_N\}; \begin{cases} x = x_0 + x_1N_1 + x_2N_2 \\ y = y_0 + y_1N_1 + y_2N_2 \end{cases} \quad \text{with } x_i \in R, y_i \in T; 0 \leq i \leq 2.$$

There for, $R_N \times T_N = \{(x_0, y_0) + (x_1, y_1)N_1 + (x_2, y_2)N_2\} = (R \times T)_N$ which is the non-commutative logical extension of $R \times T$.

Special types of NCR_1 :

$\mathbb{Z}_N = \{a + bN_1 + cN_2; a, b, c \in \mathbb{Z}\}$ is called the integer NCR_1 .

$\mathbb{Q}_N = \{a + bN_1 + cN_2; a, b, c \in \mathbb{Q}\}$ is called the rational NCR_1 .

$\mathbb{R}_N = \{a + bN_1 + cN_2; a, b, c \in \mathbb{R}\}$ is called the real NCR_1 .

$\mathbb{C}_N = \{a + bN_1 + cN_2; a, b, c \in \mathbb{C}\}$ is called the complex NCR_1 .

$\mathbb{W}_N = \{a + bN_1 + cN_2; a, b, c \in F_f\}$ is called the weak fuzzy complex NCR_1 .

$\mathbb{S}_N = \{a + bN_1 + cN_2; a, b, c \in \mathbb{S}\}$ is called the split-complex NCR_1 .

Where $\begin{cases} F_f = \{x + yJ; x, y \in \mathbb{R}, J^2 = t \in]0, 1[\} \\ \mathbb{S} = \{x + yJ; x, y \in \mathbb{R}, J^2 = 1 \} \end{cases}$.

$\mathbb{P}_N = \{a + bN_1 + cN_2; a, b, c \in n - sp_R\}$ is called the symbolic n-plithogenic NCR_1 .

$R_{NN} = \{a + bN_1 + cN_2; a, b, c \in R_n(I)\}$ is called the n-refined neutrosophic NCR_1 .

For definitions of n-refined neutrosophic rings and symbolic n-plithogenic rings.

3. Diophantine equations in two variables:**Definition:**

Let $A = a_0 + a_1N_1 + a_2N_2$, $B = b_0 + b_1N_1 + b_2N_2$, $C = c_0 + c_1N_1 + c_2N_2$ be three elements in

$$\mathbb{Z}_n = \{x + yN_1 + zN_2; x, y, z \in \mathbb{Z}, N_1^2 = N_1, N_2^2 = N_2, N_1N_2 = N_1, N_2N_1 = N_2\}.$$

We define the right non-commutative linear Diophantine equation in two variables

$X = x_0 + x_1N_1 + x_2N_2$, $Y = y_0 + y_1N_1 + y_2N_2$ as follows:

$A \cdot X + B \cdot Y = C$, we denote it by NCD_R .

Definition:

The left non-commutative linear Diophantine equation in two variables as follows:

$X \cdot A + Y \cdot B = C$, we denote it by NCD_L .

Definition:

The right - left non-commutative linear Diophantine equation in two variables as follows:

$A \cdot X + Y \cdot B = C$, we denote it by $NCD_{(R,L)}$.

The discussion of NCD_R :

$A \cdot X + B \cdot Y = C$ is equivalent to:

$$(a_0x_0 + b_0y_0) + N_1[a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + b_0y_1 + b_1y_0 + b_1y_1 + b_1y_2] + N_2[a_0x_2 + a_2x_0 + a_2x_1 + a_2x_2 + b_0y_2 + b_2y_0 + b_2y_1 + b_2y_2] = c_0 + c_1N_1 + c_2N_2.$$

This is equivalent to:

$$\begin{cases} a_0x_0 + b_0y_0 = c_0 & (1) \\ a_0x_1 + b_0y_1 + a_1(x_0 + x_1 + x_2) + b_1(y_0 + y_1 + y_2) = c_1 & (2) \\ a_0x_2 + b_0y_2 + a_2(x_0 + x_1 + x_2) + b_2(y_0 + y_1 + y_2) = c_2 & (3) \end{cases}$$

Equation (1) is solvable if and only if $\gcd(a_0, b_0) \mid c_0$.

Assume that (1) is solvable, and (k_0, l_0) is a solution.

By adding (1), (2), (3) we get:

$$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2 \quad (3)$$

Assume that (3) is solvable i.e. $\gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) \mid c_0 + c_1 + c_2$ and (k_1, l_1) is a solution of (3).

Then, we get from (2):

$$a_0x_1 + b_0y_1 + a_1k_1 + b_1l_1 = c_1, \text{ thus:}$$

$$a_0x_1 + b_0y_1 = c_1 - a_1k_1 - b_1l_1$$

It is solvable if and only if:

$$\gcd(a_0, \dots) | c_1 - a_1k_1 - b_1l_1.$$

If (II) is solvable, with (k_2, l_2) as a solution, then we get:

$$\begin{cases} x_0 = k_0 \\ y_0 = l_0 \\ x_0 + x_1 + x_2 = k_1 \\ y_0 + y_1 + y_2 = l_1 \\ x_1 = k_2 \\ y_1 = l_2 \end{cases} \text{ . thus } \begin{cases} x_0 = k_0 \\ y_0 = l_0 \\ x_2 = k_1 - k_2 - k_0 \\ y_2 = l_1 - l_0 - l_2 \\ x_1 = k_2 \\ y_1 = l_2 \end{cases}$$

Example:

Consider the following NCD_R :

$$(1 + 2N_1 + N_2)X + (3 + N_1 + 2N_2)Y = 7 + 12N_1 + 11N_2.$$

$$\text{Equation (1) is } x_0 + 3y_0 = 7$$

$$\text{Equation (2) is } 4(x_0 + x_1 + x_2) + 6(y_0 + y_1 + y_2) = 30$$

$$\text{Which is equivalent to: } 2(x_0 + x_1 + x_2) + 3(y_0 + y_1 + y_2) = 15.$$

Take $(k_0, l_0) = (1, 2)$ a solution of (1).

Take $(k_1, l_1) = (0, 5)$ a solution of (2).

$$\text{Equation (3) is: } x_1 + 3y_1 = 12 - 0 - (5) = 7.$$

Take $(k_2, l_2) = (4, 3)$ a solution of (3).

$$\text{Thus } \begin{cases} x_0 = 1, y_0 = 2 \\ x_1 = 1, y_1 = 2 \\ x_2 = -2 \\ y_2 = 1 \end{cases}$$

So that $X = 1 + N_1 - 2N_2$, $Y = 2 + 2N_1 + N_2$, is a solution of the original equation.

The discussion of NCD_L :

$X \cdot A + Y \cdot B = C$, is equivalent to:

$$(a_0x_0 + b_0y_0) + N_1[x_0a_1 + x_1a_0 + x_1a_1 + x_1a_2 + y_0b_1 + y_1b_0 + y_1b_1 + y_1b_2] + N_2[x_0a_2 + x_2a_0 + x_2a_1 + y_0b_2 + y_2b_0 + y_2b_1] = c_0 + c_1N_1 + c_2N_2.$$

There for:

$$\begin{cases} a_0x_0 + b_0y_0 = c_0 & (1) \\ a_1x_0 + (a_0 + a_1 + a_2)x_1 + b_1y_0 + y_1(b_0 + b_1 + b_2) = c_1 & (2) \\ a_2x_0 + (a_0 + a_1 + a_2)x_2 + b_2y_0 + (b_0 + b_1 + b_2)y_2 = c_2 & (3) \end{cases}$$

Assume that (1) is solvable, i.e $\gcd(a_0, b_0) | c_0$, and (k_0, l_0) is a solution of (1).

Then (2) will be:

$$(a_0 + a_1 + a_2)x_1 + (b_0 + b_1 + b_2)y_1 = c_1 - a_1k_0 - b_1l_0 \quad \text{it is solvable if and only if: } \gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) | c_1 - a_1k_0 - b_1l_0$$

Assume that it is solvable with (k_1, l_1) as a solution.

(2) will be:

$$(a_0 + a_1 + a_2)x_2 + (b_0 + b_1 + b_2)y_2 = c_2 - a_2k_0 - b_2l_0$$

is solvable if and only if: $\gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) | c_2 - a_2k_0 - b_2l_0$

Assume that it is solvable with (k_2, l_2) as a solution.

$$\text{Hence: } \begin{cases} x_0 = k_0, y_0 = l_0 \\ x_1 = k_1, y_1 = l_1 \\ x_2 = k_2, y_2 = l_2 \end{cases}$$

Example:

Consider the following NCD_L :

$$X \cdot (1 + 2N_1 + N_2) + Y \cdot (3 - N_1 + N_2) = 13 + 2N_1 + 2N_2.$$

$$\text{Equation (1) is: } x_0 + 3y_0 = 13, \text{ it is solvable, we can take } (k_0, l_0) = (1, 4).$$

$$\text{Equation (I) is: } 4x_1 + 3y_1 = 2 - (2)(1) + (1)(4) = 4, \text{ it is solvable, we can take } (k_1, l_1) = (1, 0).$$

$$\text{Equation (II) is: } 4x_2 + 3y_2 = 2 - (1)(1) - (1)(4) = -3, \text{ it is solvable, we can take } (k_2, l_2) = (0, -1).$$

Thus: $(X, Y) = (1 + N_1, 4 - N_2)$ is a solution of the original equation.

The discussion of $NCD_{(R,L)}$:

$A \cdot X + Y \cdot B = C$, is equivalent to:

$$(a_0x_0 + b_0y_0) + N_1[a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + y_0b_1 + y_1b_0 + y_1b_1 + y_1b_2] + N_2[a_0x_2 + a_2x_0 + a_2x_1 + a_2x_2 + y_0b_2 + y_2b_0 + y_2b_1] = c_0 + c_1N_1 + c_2N_2.$$

It is equivalent to:

$$\begin{cases} a_0x_0 + b_0y_0 = c_0 & (1) \\ a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + y_0b_1 + y_1(b_0 + b_1 + b_2) = c_1 & (2) \\ a_0x_2 + a_2x_0 + a_2x_2 + a_2x_1 + y_0b_2 + y_2(b_0 + b_1 + b_2) = c_2 & (3) \end{cases}$$

Equation (1) is solvable if and only if $\gcd(a_0, b_0) \mid c_0$.

Assume that it is solvable with (k_0, l_0) as a solution.

By adding (1),(2),(3), we get:

$$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$$

Assume that it is solvable, i.e $\gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) \mid c_0 + c_1 + c_2$ with (k_1, l_1) as a solution.

From equation (2), we can write:

$$a_0x_1 + (b_0 + b_1 + b_2)y_1 = c_1 - b_1l_0 - a_1k_1$$

Assume that it is solvable, i.e. $\gcd(a_0, b_0 + b_1 + b_2) \mid c_1 - b_1l_0 - a_1k_1$, with (k_2, l_2) as a solution.

There for, we get:

$$\begin{cases} x_0 = k_0, y_0 = l_0 \\ k_2 = x_1, l_2 = y_1 \\ x_0 + x_1 + x_2 = k_1, y_0 + y_1 + y_2 = l_1 \end{cases}$$

$$\text{Hence: } \begin{cases} x_0 = k_0, y_0 = l_0 \\ x_1 = k_2, y_1 = l_2 \\ x_2 = k_1 - k_0 - k_2 \\ y_2 = l_1 - l_0 - l_2 \end{cases}$$

Example:

Consider the following $NCD_{(R,L)}$:

$$(1 + N_1 + N_2)X + Y \cdot (3 + N_1 + N_2) = 10 + 6N_1,$$

According to our discussion, we can see that:

Equation (1) is: $x_0 + 3y_0 = 10$, take the solution: $(k_0, l_0) = (1, 3)$.

Equation (I) is: $3(x_0 + x_1 + x_2) + 5(y_0 + y_1 + y_2) = 16$ take the solution: $(k_1, l_1) = (2, 2)$.

Equation (II) is: $x_1 + 5y_1 = 6 - (1)(3) - (1)(2) = 1$, take the solution: $(k_2, l_2) = (1, 0)$.

$$x_0 = 1, y_0 = 3$$

Thus: $x_1 = 1, y_1 = 0$

$$x_2 = 0, y_2 = -1$$

And $(X, Y) = (1 + N_1, 3 - N_2)$ is a solution of the original equation.

4. Conclusion

In this paper, we used some logical algebraic elements to extend any ring into a non-commutative ring contains the original ring with many generalized substructures and special elements. On the other hand, we studied the substructures of non-commutative logical rings such as AH-homomorphisms and AH-ideals with many examples that explain their algebraic validity. Also, we discussed the possibility of solving linear Diophantine equation with two variables in the non-commutative logical ring of integers, where we present an easy algorithm to solve this kind of generalized Diophantine equations.

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