New approach to bisemiring via the q-neutrosophic cubic vague subbisemiring

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Abstract

We introduce the notion of q-neutrosophic cubic vague subbisemiring (q-NSCVSBS) and level set of q-NSCVSBS of a bisemiring. The q-NSCVSBS is a new concept of subbisemirings of bisemirings. Let Ξ be a neutrosophic vague subset of Λ. Then $\mho = ([\mathfrak{T}^-_\Xi, \mathfrak{T}^+_\Xi], [\mathfrak{I}^-_\Xi, \mathfrak{I}^+_\Xi], [\mathfrak{F}^-_\Xi, \mathfrak{F}^+_\Xi])$ is a q-NSCVSBS of Λ if and only if all non empty level set $\mho^{(\varrho_1,\varrho_2,s)}$ is also a SBS of Λ for every $\varrho_1, \varrho_2, s \in [0,1]$. Let Ξ be the q-NSCVSBS of Λ and Υ be the strongest cubic q-neutrosophic vague relation of Λ. Then Ξ is a q-NSCVSBS of $\Lambda \times \Lambda$. Let Ξ be the q-NSCVSBS of Λ, show that pseudo cubic q-neutrosophic vague coset $(\varsigma\Xi)^p$ is also a q-NSCVSBS of Λ, for all $\varsigma \in \Lambda$. Let $\Xi_1, \Xi_2, ..., \Xi_n$ be the any family of $q - NSCVSBS^s$ of $\Lambda_1, \Lambda_2, ..., \Lambda_n$ respectively, then $\Xi_1 \times \Xi_2 \times ... \times \Xi_n$ is also a q-NSCVSBS of $\Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$. The homomorphic image of every q-NSCVSBS is also a q-NSCVSBS. The homomorphic pre-image of every q-NSCVSBS is also a q-NSCVSBS.

Keywords: Subbisemiring; cubic neutrosophic subbisemiring; vague bisemiring; homomorphism.

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1 Introduction

Mathematics has developed theories to address uncertainty and fuzziness as a result of the limitations of classical mathematics, such as fuzzy set (FS)[1] and vague set (VS).[2] The FS introduced by Zadeh[1] is best suited to uncertain or vague situations. FS has been used to develop many hybrid fuzzy models in recent years. The notion of intuitionistic fuzzy sets (IFS) was first proposed in 1983 by Attanasov.[3] In 1999, Smarandache[4] proposed the neutrosophic set (NSS). The NSS estimates the truth, indeterminacy, and falsity degree of each proposition. In Smarandache,[5] the theory of IFSs was further generalized and expanded to include the neutrosophic model. The various structures and their applications were discussed by many researchers[7]-.[10] In 2004, Sen et al.[11] proposed a concept of bisemiring to develop semirings further. Biswas[12] introduced vague groups, vague cuts, and vague normal groups to vague algebra. A semiring $(S, +, \cdot)$ is a non-empty set in which $(S, +)$ and $(S, \cdot)$ are semigroups such that "$\cdot$" is distributive over "$+$".[13] Ahsan et al.[6] introduced fuzzy semirings in 1993. According to Sen and colleagues,[14] bisemirings were introduced in 2001. The FS theory had, however, generalized many algebraic concepts. Vandiver has extensively examined fuzzy algebraic structures of semirings.[15] Semirings have been extensively studied by Golan[13] and Glazek.[16] In,[23] he introduced a vague soft hyperring and a vague soft hyperideal. A concept of bisemiring was introduced to the field of semirings by Sen et al.[11] in 2004. Bisemiring homomorphisms and bisemirings are congruent studied by Hussain et al.[26] Furthermore, Hussain et al.[14,26] describe a semiring algebraic structure and a congruence relation between homomorphisms and n-semirings. Recently, Faisal et al. discussed more aggregating operators based on fuzzy approach. We discuss the notion of q-neutosophic cubic vague subbisemiring (q-NSCVSBS) and level sets. The q-NSCVSBS is a new concept of subbisemiring. Following is an outline of the preliminary definitions and results presented in Section 2. The concept of a q-NSCVSBS is introduced in Section 3.

2 Preliminaries

For our future studies, we will quickly review some fundamental terms in this section.

Definition 2.1. Let $\mathscr{X}$ and Q_1 denotes non-empty sets. The mapping $\Lambda : \mathscr{X} \circledast Q_1 \rightarrow [0, 1]$ is called a Q_1 fuzzy set in $\mathscr{X}$.

Definition 2.2. Let $(\mathbb{M}, +, \cdot)$ be semiring. The Q fuzzy set $\Lambda : \mathbb{M} \circledast Q \rightarrow [0, 1]$ is called a Q fuzzy subsemiring of $\mathbb{M}$ if $\Lambda_{Z_1}(\vartheta + \eta, \varsigma) \geq \min\{\Lambda_{Z_1}(\vartheta, \varsigma), \Lambda_{Z_1}(\eta, \varsigma)\}$, $\Lambda_{Z_1}(\vartheta \cdot \eta, \varsigma) \geq \min\{\Lambda_{Z_1}(\vartheta, \varsigma), \Lambda_{Z_1}(\eta, \varsigma)\}$ for all $\vartheta, \eta \in \mathbb{M}$ and $\varsigma \in Q$.

Definition 2.3. [4] A neutrosophic set (NSS) Ξ in a universal set $\mathbb{U}$ is $\Xi = \{(\flat, \mathfrak{T}_\Xi(\flat), \mathfrak{I}_\Xi(\flat), \mathfrak{F}_\Xi(\flat)) : \flat \in \mathbb{U}\}$, where $\mathfrak{T}_\Xi, \mathfrak{I}_\Xi, \mathfrak{F}_\Xi : \mathbb{U} \rightarrow [0, 1]$ denotes the truth, indeterminacy and the falsity membership function, respectively. For $\langle \mathfrak{T}_\Xi, \mathfrak{I}_\Xi, \mathfrak{F}_\Xi \rangle$ is used for the NSS $\Xi = \{(\flat, \mathfrak{T}_\Xi(\flat), \mathfrak{I}_\Xi(\flat), \mathfrak{F}_\Xi(\flat)) : \flat \in \mathbb{U}\}$.

Definition 2.4. [4] Let $\Xi = \langle \mathfrak{T}_\Xi, \mathfrak{I}_\Xi, \mathfrak{F}_\Xi \rangle$ and $\Gamma = \langle \mathfrak{T}_\Gamma, \mathfrak{I}_\Gamma, \mathfrak{F}_\Gamma \rangle$ be the two NSS of $\mathbb{U}$. Then

1. $\Xi \bigwedge \Gamma = \{(\flat, \min\{\mathfrak{T}_\Xi(\flat), \mathfrak{T}_\Gamma(\flat)\}, \min\{\mathfrak{I}_\Xi(\flat), \mathfrak{I}_\Gamma(\flat)\}, \max\{\mathfrak{F}_\Xi(\flat), \mathfrak{F}_\Gamma(\flat)\}) : \flat \in \mathbb{U}\}$,
2. $\Xi \bigvee \Gamma = \{(\flat, \max\{\mathfrak{T}_\Xi(\flat), \mathfrak{T}_\Gamma(\flat)\}, \max\{\mathfrak{I}_\Xi(\flat), \mathfrak{I}_\Gamma(\flat)\}, \min\{\mathfrak{F}_\Xi(\flat), \mathfrak{F}_\Gamma(\flat)\}) : \flat \in \mathbb{U}\}$.

Definition 2.5. [4] For any NSS $\Xi = \langle \mathfrak{T}_\Xi, \mathfrak{I}_\Xi, \mathfrak{F}_\Xi \rangle$ of $\mathbb{U}$, we defined a (ϱ, s)-cut of as the crisp subset $\{\flat \in \mathbb{U} : \mathfrak{T}_\Xi(\flat) \succeq \varrho, \mathfrak{I}_\Xi(\flat) \succeq \varrho, \mathfrak{F}_\Xi(\flat) \preceq s\}$ of $\mathbb{U}$.

Definition 2.6. [4] Let Ξ and Γ be two neutrosophic subsets of S. The Cartesian product of Ξ and Γ is defined as $\Xi \times \Gamma = \{((\flat, \hbar), \mathfrak{T}_{\Xi\times\Gamma}(\flat, \hbar), \mathfrak{I}_{\Xi\times\Gamma}(\flat, \hbar), \mathfrak{F}_{\Xi\times\Gamma}(\flat, \hbar)) : \flat, \hbar \in S\}$, where $\mathfrak{T}_{\Xi\times\Gamma}(\flat, \hbar) = \min\{\mathfrak{T}_\Xi(\flat), \mathfrak{T}_\Gamma(\hbar)\}$, $\mathfrak{I}_{\Xi\times\Gamma}(\flat, \hbar) = \frac{\mathfrak{I}_\Xi(\flat)+\mathfrak{I}_\Gamma(\hbar)}{2}$ and $\mathfrak{F}_{\Xi\times\Gamma}(\flat, \hbar) = \max\{\mathfrak{F}_\Xi(\flat), \mathfrak{F}_\Gamma(\hbar)\}$.

Definition 2.7. [12] A vague set (VS) $\Xi = (\mathfrak{T}_\Xi,\ \mathfrak{F}_\Xi)$ of Λ is said to be vague semiring if

$$\begin{cases} \mathfrak{T}_\Xi(\tau_1 + \tau_2) \succeq \min\{\mathfrak{T}_\Xi(\tau_1), \mathfrak{T}_\Xi(\tau_2)\} \\ \mathfrak{T}_\Xi(\tau_1 \cdot \tau_2) \succeq \min\{\mathfrak{T}_\Xi(\tau_1), \mathfrak{T}_\Xi(\tau_2)\} \end{cases}$$

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and

$$\begin{cases} 1-\mathfrak{F}_{\Xi}(\tau_1+\tau_2) \succeq \min\{1-\mathfrak{F}_{\Xi}(\tau_1), 1-\mathfrak{F}_{\Xi}(\tau_2)\} \\ 1-\mathfrak{F}_{\Xi}(\tau_1\cdot\tau_2) \succeq \min\{1-\mathfrak{F}_{\Xi}(\tau_1), 1-\mathfrak{F}_{\Xi}(\tau_2)\} \end{cases}.$$

for all $\tau_1, \tau_2 \in \Lambda$.

Definition 2.8. [12] A VS Ξ in $\mathbb{U}$. Then

1. A VS $\Xi = (\mathfrak{T}_{\Xi}, \mathfrak{F}_{\Xi})$, where $\mathfrak{T}_{\Xi} : \mathbb{U} \to [0,1], \mathfrak{F}_{\Xi} : \mathbb{U} \to [0,1]$ are mappings such that $\mathfrak{T}_{\Xi}(\flat) + \mathfrak{F}_{\Xi}(\flat) \preceq 1$, for all $\flat \in \mathbb{U}$ where $\mathfrak{T}_{\Xi}$ and $\mathfrak{F}_{\Xi}$ are called true and false membership function, respectively.
2. The interval $[\mathfrak{T}_{\Xi}(\flat), 1-\mathfrak{F}_{\Xi}(\flat)]$ is called the vague value of $\flat$ in Ξ and it is denoted by $V_{\Xi}(\flat)$, i.e., $V_{\Xi}(\flat) = [\mathfrak{T}_{\Xi}(\flat), 1-\mathfrak{F}_{\Xi}(\flat)]$.

Definition 2.9. [12] Let Ξ and Γ be the two VSs of $\mathbb{U}$. Then

1. Ξ is contained in Γ as $\Xi \subseteq \Gamma$ if and only if $V_{\Xi}(\flat) \preceq V_{\Gamma}(\flat)$, i.e. $\mathfrak{T}_{\Xi}(\flat) \preceq \mathfrak{T}_{\Gamma}(\flat)$ and $1-\mathfrak{F}_{\Xi}(\flat) \preceq 1-\mathfrak{F}_{\Gamma}(\flat)$ for all $\flat \in \mathbb{U}$,
2. the union of Ξ and Γ as $N = \Xi \bigvee \Gamma, \mathfrak{T}_N = \max\{\mathfrak{T}_{\Xi}, \mathfrak{T}_{\Gamma}\}$ and $1-\mathfrak{F}_N = \max\{1-\mathfrak{F}_{\Xi}, 1-\mathfrak{F}_{\Gamma}\} = 1-\min\{\mathfrak{F}_{\Xi}, \mathfrak{F}_{\Gamma}\}$,
3. the intersection of Ξ and Γ as $N = \Xi \bigwedge \Gamma, \mathfrak{T}_N = \min\{\mathfrak{T}_{\Xi}, \mathfrak{T}_{\Gamma}\}$ and $1-\mathfrak{F}_N = \min\{1-\mathfrak{F}_{\Xi}, 1-\mathfrak{F}_{\Gamma}\} = 1-\max\{\mathfrak{F}_{\Xi}, \mathfrak{F}_{\Gamma}\}$.

Definition 2.10. [12] Let Ξ be a VS of $\mathbb{U}$. Then

1. $\mathfrak{T}_{\Xi}(\flat) = 0$ and $\mathfrak{F}_{\Xi}(\flat) = 1$ is called zero VS of $\mathbb{U}$,
2. $\mathfrak{T}_{\Xi}(\flat) = 1$ and $\mathfrak{F}_{\Xi}(\flat) = 0$ is called unit VS of $\mathbb{U}$.

for all $\flat \in U$.

Definition 2.11. [12] Let Ξ be a VS of $\mathbb{U}$ with true membership function $\mathfrak{T}_{\Xi}$ and false membership function $\mathfrak{F}_{\Xi}$. For $\alpha, \omega \in [0,1]$ with $\alpha \preceq \omega$, the (α,ω)- cut or vague cut of a VS Ξ is the crisp subset of $\mathbb{U}$ is given by $\Xi_{(\alpha,\omega)} = \{\flat \in \mathbb{U} : V_{\Xi}(\flat) \succeq [\alpha,\omega]\}$. That is, $\Xi_{(\alpha,\omega)} = \{\flat \in \mathbb{U} : \mathfrak{T}_{\Xi}(\flat) \succeq \alpha,\ \ 1-\mathfrak{F}_{\Xi}(\flat) \succeq \omega]\}$.

Definition 2.12. [12] Let Ξ and Γ be any two VSs in $\mathbb{U}$. Then

1. $\Xi \bigwedge \Gamma = \big\{(\flat, \min\{\mathfrak{T}_{\Xi}(\flat), \mathfrak{T}_{\Gamma}(\flat)\}, \min\{1-\mathfrak{F}_{\Xi}(\flat), 1-\mathfrak{F}_{\Gamma}(\flat)\}) : \flat \in \mathbb{U}\big\}$,
2. $\Xi \bigvee \Gamma = \big\{(\flat, \max\{\mathfrak{T}_{\Xi}(\flat), \mathfrak{T}_{\Gamma}(\flat)\}, \max\{1-\mathfrak{F}_{\Xi}(\flat), 1-\mathfrak{F}_{\Gamma}(\flat)\}) : \flat \in \mathbb{U}\big\}$,
3. $\Box\Xi = \big\{(\flat, \mathfrak{T}_{\Xi}(\flat), 1-\mathfrak{T}_{\Xi}(\flat)) : \flat \in \mathbb{U}\big\}$,
4. $\Diamond\Xi = \big\{(\flat, 1-\mathfrak{F}_{\Xi}(\flat), \mathfrak{F}_{\Xi}(\flat)) : \flat \in \mathbb{U}\big\}$.

Definition 2.13. [11] A bisemiring $(\Lambda, \triangle, \circledcirc, \boxtimes)$ is an algebraic structure in which $(\Lambda, \triangle, \circledcirc)$ and $(\Lambda, \circledcirc, \boxtimes)$ are semirings in which $(\Lambda, \triangle)$, $(\Lambda, \circledcirc)$ and $(\Lambda, \boxtimes)$ are semigroups such that (a) $\varsigma \circledcirc (\hbar \triangle \varsigma) = (\flat \circledcirc \hbar) \triangle (\flat \circledcirc \varsigma)$, (b) $(\hbar \triangle \varsigma) \circledcirc \flat = (\hbar \circledcirc \flat) \triangle (\varsigma \circledcirc \flat)$, (c) $\flat \boxtimes (\hbar \circledcirc \varsigma) = (\flat \boxtimes \hbar) \circledcirc (\flat \boxtimes \varsigma)$ and (d) $(\hbar \circledcirc \varsigma) \boxtimes \flat = (\hbar \boxtimes \flat) \circledcirc (\varsigma \boxtimes \flat)$ for all $\flat, \hbar, \varsigma \in \Lambda$.

Definition 2.14. .[14] A non-empty subset Ξ of a bisemiring $(\Lambda, \triangle, \circledcirc, \boxtimes)$ is a subbisemiring (SBS) if and only if $\flat \triangle \hbar \in \Xi, \flat \circledcirc \hbar \in \Xi$ and $\flat \boxtimes \hbar \in \Xi$ for all $\flat, \hbar \in \Xi$.

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3 q-neutrosophic cubic vague subbisemirings

Definition 3.1. A q-neutrosophic cubic VS Ξ of Λ is called a q-NSCVSBS of Λ if

$$\begin{cases} \begin{pmatrix} \mathfrak{T}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\}, \\ 1-\mathfrak{F}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{1-\mathfrak{F}_\Xi(\flat,\varsigma),1-\mathfrak{F}_\Xi(\hbar,\varsigma)\} \end{pmatrix} \\ \begin{pmatrix} \mathfrak{T}_\Xi(\flat\heartsuit_2\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\}, \\ 1-\mathfrak{F}_\Xi(\flat\heartsuit_2\hbar,\varsigma) \succeq \min\{1-\mathfrak{F}_\Xi(\flat,\varsigma),1-\mathfrak{F}_\Xi(\hbar,\varsigma)\} \end{pmatrix} \\ \begin{pmatrix} \mathfrak{T}_\Xi(\flat\heartsuit_3\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\}, \\ 1-\mathfrak{F}_\Xi(\flat\heartsuit_3\hbar,\varsigma) \succeq \min\{1-\mathfrak{F}_\Xi(\flat,\varsigma),1-\mathfrak{F}_\Xi(\hbar,\varsigma)\} \end{pmatrix} \end{cases}$$

$$\begin{cases} \begin{pmatrix} \mathfrak{I}^u_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^u_\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Xi(\hbar,\varsigma)}{2}, \\ \mathfrak{I}^l_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^l_\Xi(\flat,\varsigma)-\mathfrak{I}^l_\Xi(\hbar,\varsigma)}{2} \end{pmatrix} \\ OR \\ \begin{pmatrix} \mathfrak{I}^u_\Xi(\flat\heartsuit_2\hbar,\varsigma) \succeq \frac{\mathfrak{I}^u_\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Xi(\hbar,\varsigma)}{2}, \\ \mathfrak{I}^l_\Xi(\flat\heartsuit_2\hbar,\varsigma) \succeq \frac{\mathfrak{I}^l_\Xi(\flat,\varsigma)-\mathfrak{I}^l_\Xi(\hbar,\varsigma)}{2} \end{pmatrix} \\ OR \\ \begin{pmatrix} \mathfrak{I}^u_\Xi(\flat\heartsuit_3\hbar,\varsigma) \succeq \frac{\mathfrak{I}^u_\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Xi(\hbar,\varsigma)}{2}, \\ \mathfrak{I}^l_\Xi(\flat\heartsuit_3\hbar,\varsigma) \succeq \frac{\mathfrak{I}^l_\Xi(\flat,\varsigma)-\mathfrak{I}^l_\Xi(\hbar,\varsigma)}{2} \end{pmatrix} \end{cases}$$

$$\begin{cases} \begin{pmatrix} \mathfrak{F}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\}, \\ 1-\mathfrak{T}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{1-\mathfrak{T}_\Xi(\flat,\varsigma),1-\mathfrak{T}_\Xi(\hbar,\varsigma)\} \end{pmatrix} \\ \begin{pmatrix} \mathfrak{F}_\Xi(\flat\heartsuit_2\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\}, \\ 1-\mathfrak{T}_\Xi(\flat\heartsuit_2\hbar,\varsigma) \preceq \max\{1-\mathfrak{T}_\Xi(\flat,\varsigma),1-\mathfrak{T}_\Xi(\hbar,\varsigma)\} \end{pmatrix} \\ \begin{pmatrix} \mathfrak{F}_\Xi(\flat\heartsuit_3\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\}, \\ 1-\mathfrak{T}_\Xi(\flat\heartsuit_3\hbar,\varsigma) \preceq \max\{1-\mathfrak{T}_\Xi(\flat,\varsigma),1-\mathfrak{T}_\Xi(\hbar,\varsigma)\} \end{pmatrix} \end{cases}$$

$$\begin{cases} \begin{pmatrix} \mathfrak{T}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\}, \end{pmatrix} \\ \begin{pmatrix} \mathfrak{T}_\Xi(\flat\heartsuit_2\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\}, \end{pmatrix} \\ \begin{pmatrix} \mathfrak{T}_\Xi(\flat\heartsuit_3\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\}, \end{pmatrix} \end{cases}$$

$$\begin{cases} \begin{pmatrix} \mathfrak{I}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}_\Xi(\flat,\varsigma)-\mathfrak{I}_\Xi(\hbar,\varsigma)}{2} \end{pmatrix} \\ OR \\ \begin{pmatrix} \mathfrak{I}_\Xi(\flat\heartsuit_2\hbar,\varsigma) \succeq \frac{\mathfrak{I}_\Xi(\flat,\varsigma)-\mathfrak{I}_\Xi(\hbar,\varsigma)}{2} \end{pmatrix} \\ OR \\ \begin{pmatrix} \mathfrak{I}_\Xi(\flat\heartsuit_3\hbar,\varsigma) \succeq \frac{\mathfrak{I}_\Xi(\flat,\varsigma)-\mathfrak{I}_\Xi(\hbar,\varsigma)}{2} \end{pmatrix} \end{cases}$$

$$\begin{cases} \begin{pmatrix} \mathfrak{F}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\}, \end{pmatrix} \\ \begin{pmatrix} \mathfrak{F}_\Xi(\flat\heartsuit_2\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\}, \end{pmatrix} \\ \begin{pmatrix} \mathfrak{F}_\Xi(\flat\heartsuit_3\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\}, \end{pmatrix} \end{cases}$$

for all $\flat,\hbar \in \Lambda$ and and $\varsigma \in Q \subseteq \Lambda$.

Example 3.2. Let $\Lambda = \{\wp_1,\wp_2,\wp_3,\wp_4\}$ be the bisemiring.

$\heartsuit_1$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_1$	$\wp_1$	$\wp_1$	$\wp_1$	$\wp_1$
$\wp_2$	$\wp_1$	$\wp_2$	$\wp_1$	$\wp_2$
$\wp_3$	$\wp_1$	$\wp_1$	$\wp_3$	$\wp_3$
$\wp_4$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$

$\heartsuit_2$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_1$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_2$	$\wp_2$	$\wp_2$	$\wp_4$	$\wp_4$
$\wp_3$	$\wp_3$	$\wp_4$	$\wp_3$	$\wp_4$
$\wp_4$	$\wp_4$	$\wp_4$	$\wp_4$	$\wp_4$

$\heartsuit_3$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_1$	$\wp_1$	$\wp_1$	$\wp_1$	$\wp_1$
$\wp_2$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_3$	$\wp_4$	$\wp_4$	$\wp_4$	$\wp_4$
$\wp_4$	$\wp_4$	$\wp_4$	$\wp_4$	$\wp_4$

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	$[\mathfrak{T}_\Xi(\omega), 1-\mathfrak{F}_\Xi(\omega)]$	$[\mathfrak{I}^l_\Xi(\omega), e\mathfrak{I}^u_\Xi(\omega)]$	$[\mathfrak{F}_\Xi(\omega), 1-\mathfrak{T}_\Xi(\omega)]$
$\omega = \wp_1$	$[0.87, 0.92]$	$[0.76, 0.81]$	$[0.08, 0.13]$
$\omega = \wp_2$	$[0.77, 0.82]$	$[0.66, 0.71]$	$[0.18, 0.23]$
$\omega = \wp_3$	$[0.57, 0.62]$	$[0.46, 0.51]$	$[0.38, 0.43]$
$\omega = \wp_4$	$[0.72, 0.77]$	$[0.51, 0.56]$	$[0.23, 0.28]$

	$\mathfrak{T}_\Xi(\omega)$	$\mathfrak{I}_\Xi(\omega)$	$\mathfrak{F}_\Xi(\omega)$
$\omega = \wp_1$	0.74	0.84	0.39
$\omega = \wp_2$	0.64	0.79	0.44
$\omega = \wp_3$	0.49	0.64	0.64
$\omega = \wp_4$	0.54	0.74	0.54

Clearly, Ξ is a q-NSCVSBS of Λ.

Theorem 3.3. *The intersection of a family of $q-NSCVSBS^s$ of Λ is a q-NSCVSBS of Λ.*

Proof. Let $\{\mho_i : i \in I\}$ be the collection of $q-NSCVSBS^s$ of Λ and $\Xi = \bigwedge_{i\in I} \mho_i$.

Let $\flat, \hbar$ in Λ. Then

$$\begin{aligned}\mathfrak{T}_\Xi(\flat\heartsuit_1\hbar, \varsigma) &= \inf_{i\in I}\ \mathfrak{T}_{\mho_i}(\flat\heartsuit_1\hbar, \varsigma)\\ &\succeq \inf_{i\in I}\ \min\{\mathfrak{T}_{\mho_i}(\flat, \varsigma), \mathfrak{T}_{\mho_i}(\hbar, \varsigma)\}\\ &= \min\left\{\inf_{i\in I}\ \mathfrak{T}_{\mho_i}(\flat, \varsigma), \inf_{i\in I}\ \mathfrak{T}_{\mho_i}(\hbar, \varsigma)\right\}\\ &= \min\{\mathfrak{T}_\Xi(\flat, \varsigma), \mathfrak{T}_\Xi(\hbar, \varsigma)\}.\end{aligned}$$

$$\begin{aligned}1-\mathfrak{F}_\Xi(\flat\heartsuit_1\hbar, \varsigma) &= \inf_{i\in I}\ 1-\mathfrak{F}_{\mho_i}(\flat\heartsuit_1\hbar, \varsigma)\\ &\succeq \inf_{i\in I}\ \min\{1-\mathfrak{F}_{\mho_i}(\flat, \varsigma), 1-\mathfrak{F}_{\mho_i}(\hbar, \varsigma)\}\\ &= \min\left\{\inf_{i\in I}\ 1-\mathfrak{F}_{\mho_i}(\flat, \varsigma), \inf_{i\in I}\ 1-\mathfrak{F}_{\mho_i}(\hbar, \varsigma)\right\}\\ &= \min\{1-\mathfrak{F}_\Xi(\flat, \varsigma), 1-\mathfrak{F}_\Xi(\hbar, \varsigma)\}.\end{aligned}$$

Thus, $\mho^{\mathfrak{T}}_\Xi(\flat\heartsuit_1\hbar, \varsigma) \succeq \min\{\mho_\Xi(\flat, \varsigma), \mho_\Xi(\hbar, \varsigma)\}$. Similarly, $\mho^{\mathfrak{T}}_\Xi(\flat\heartsuit_2\hbar, \varsigma) \succeq \min\{\mho_\Xi(\flat, \varsigma), \mho_\Xi(\hbar, \varsigma)\}$ and $\mho^{\mathfrak{T}}_\Xi(\flat\heartsuit_3\hbar, \varsigma) \succeq \min\{\mho_\Xi(\flat, \varsigma), \mho_\Xi(\hbar, \varsigma)\}$. Now,

$$\begin{aligned}\mathfrak{I}^l_\Xi(\flat\heartsuit_1\hbar, \varsigma) &= \inf_{i\in I}\ \mathfrak{I}^l_{\mho_i}(\flat\heartsuit_1\hbar, \varsigma)\\ &\succeq \inf_{i\in I}\ \frac{\mathfrak{I}^l_{\mho_i}(\flat, \varsigma)+\mathfrak{I}^l_{\mho_i}(\hbar, \varsigma)}{2}\\ &= \frac{\inf_{i\in I}\ \mathfrak{I}^l_{\mho_i}(\flat, \varsigma)+\inf_{i\in I}\ \mathfrak{I}^l_{\mho_i}(\hbar, \varsigma)}{2}\\ &= \frac{\mathfrak{I}^l_\Xi(\flat, \varsigma)+\mathfrak{I}^l_\Xi(\hbar, \varsigma)}{2}.\end{aligned}$$

$$\begin{aligned}\mathfrak{I}^u_\Xi(\flat\heartsuit_1\hbar, \varsigma) &= \inf_{i\in I}\ \mathfrak{I}^u_{\mho_i}(\flat\heartsuit_1\hbar, \varsigma)\\ &\succeq \inf_{i\in I}\ \frac{\mathfrak{I}^u_{\mho_i}(\flat, \varsigma)+\mathfrak{I}^u_{\mho_i}(\hbar, \varsigma)}{2}\\ &= \frac{\inf_{i\in I}\ \mathfrak{I}^u_{\mho_i}(\flat, \varsigma)+\inf_{i\in I}\ \mathfrak{I}^u_{\mho_i}(\hbar, \varsigma)}{2}\\ &= \frac{\mathfrak{I}^u_\Xi(\flat, \varsigma)+\mathfrak{I}^u_\Xi(\hbar, \varsigma)}{2}.\end{aligned}$$

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Thus, $\mho_{\Xi}^{\mathfrak{I}}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$. Similarly, $\mho_{\Xi}^{\mathfrak{I}}(\flat\heartsuit_2\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$ and $\mho_{\Xi}^{\mathfrak{I}}(\flat\heartsuit_3\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$.
Now,

$$\begin{aligned}
\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) &= \sup_{i\in I}\ \mathfrak{F}_{\mho_i}(\flat\heartsuit_1\hbar,\varsigma)\\
&\preceq \sup_{i\in I}\ \max\{\mathfrak{F}_{\mho_i}(\flat,\varsigma),\mathfrak{F}_{\mho_i}(\hbar,\varsigma)\}\\
&= \max\left\{\sup_{i\in I}\ \mathfrak{F}_{\mho_i}(\flat,\varsigma),\sup_{i\in I}\ \mathfrak{F}_{\mho_i}(\hbar,\varsigma)\right\}\\
&= \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}.
\end{aligned}$$

$$\begin{aligned}
1-\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) &= \sup_{i\in I}\ 1-\mathfrak{T}_{\mho_i}(\flat\heartsuit_1\hbar,\varsigma)\\
&\preceq \sup_{i\in I}\ \max\{1-\mathfrak{T}_{\mho_i}(\flat,\varsigma),1-\mathfrak{T}_{\mho_i}(\hbar,\varsigma)\}\\
&= \max\left\{\sup_{i\in I}\ 1-\mathfrak{T}_{\mho_i}(\flat,\varsigma),\sup_{i\in I}\ 1-\mathfrak{T}_{\mho_i}(\hbar,\varsigma)\right\}\\
&= \max\{1-\mathfrak{T}_{\Xi}(\flat,\varsigma),1-\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}.
\end{aligned}$$

Thus, $\mho_{\Xi}^{\mathfrak{F}}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$. Similarly, $\mho_{\Xi}^{\mathfrak{F}}(\flat\heartsuit_2\hbar,\varsigma) \preceq \max\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$ and $\mho_{\Xi}^{\mathfrak{F}}(\flat\heartsuit_3\hbar,\varsigma) \preceq \max\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$.
Now,

$$\begin{aligned}
\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) &= \inf_{i\in I}\ \mathfrak{T}_{\mho_i}(\flat\heartsuit_1\hbar,\varsigma)\\
&\succeq \inf_{i\in I}\ \min\{\mathfrak{T}_{\mho_i}(\flat,\varsigma),\mathfrak{T}_{\mho_i}(\hbar,\varsigma)\}\\
&= \min\left\{\inf_{i\in I}\ \mathfrak{T}_{\mho_i}(\flat,\varsigma),\inf_{i\in I}\ \mathfrak{T}_{\mho_i}(\hbar,\varsigma)\right\}\\
&= \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}.
\end{aligned}$$

Thus, $\mho_{\Xi}^{\mathfrak{T}}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$. Similarly, $\mho_{\Xi}^{\mathfrak{T}}(\flat\heartsuit_2\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$ and $\mho_{\Xi}^{\mathfrak{T}}(\flat\heartsuit_3\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$. Now,

$$\begin{aligned}
\mathfrak{I}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) &= \inf_{i\in I}\ \mathfrak{I}_{\mho_i}(\flat\heartsuit_1\hbar,\varsigma)\\
&\succeq \inf_{i\in I}\ \frac{\mathfrak{I}_{\mho_i}(\flat,\varsigma)+\mathfrak{I}_{\mho_i}(\hbar,\varsigma)}{2}\\
&= \frac{\inf_{i\in I}\ \mathfrak{I}_{\mho_i}(\flat,\varsigma)+\inf_{i\in I}\ \mathfrak{I}_{\mho_i}(\hbar,\varsigma)}{2}\\
&= \frac{\mathfrak{I}_{\Xi}(\flat,\varsigma)+\mathfrak{I}_{\Xi}(\hbar,\varsigma)}{2}.
\end{aligned}$$

Thus, $\mho_{\Xi}^{\mathfrak{I}}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$. Similarly, $\mho_{\Xi}^{\mathfrak{I}}(\flat\heartsuit_2\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$ and $\mho_{\Xi}^{\mathfrak{I}}(\flat\heartsuit_3\hbar,\varsigma) \succeq \min\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$.
Now,

$$\begin{aligned}
\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) &= \sup_{i\in I}\ \mathfrak{F}_{\mho_i}(\flat\heartsuit_1\hbar,\varsigma)\\
&\preceq \sup_{i\in I}\ \max\{\mathfrak{F}_{\mho_i}(\flat,\varsigma),\mathfrak{F}_{\mho_i}(\hbar,\varsigma)\}\\
&= \max\left\{\sup_{i\in I}\ \mathfrak{F}_{\mho_i}(\flat,\varsigma),\sup_{i\in I}\ \mathfrak{F}_{\mho_i}(\hbar,\varsigma)\right\}\\
&= \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}.
\end{aligned}$$

Thus, $\mho_{\Xi}^{\mathfrak{F}}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$. Similarly, $\mho_{\Xi}^{\mathfrak{F}}(\flat\heartsuit_2\hbar,\varsigma) \preceq \max\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$ and $\mho_{\Xi}^{\mathfrak{F}}(\flat\heartsuit_3\hbar,\varsigma) \preceq \max\{\mho_{\Xi}(\flat,\varsigma),\mho_{\Xi}(\hbar,\varsigma)\}$. Hence, Ξ is a q-NSCVSBS of Λ.

Theorem 3.4. *If Ξ and Γ are the $q-NSCVSBS^s$ of Λ_1 and Λ_2 respectively, then $\Xi\times\Gamma$ is a q-NSCVSBS of $\Lambda_1\times\Lambda_2$.*

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Proof. Let Ξ and Γ be the $q-NSCVSBS^s$ of Λ_1 and Λ_2 respectively. Let $(\flat_1,\flat_2)\in\Lambda_1$ and $(\hbar_1,\hbar_2)\in\Lambda_2$. Then $(\flat_1,\hbar_1),((\flat_2,\hbar_2))$ belong to $\Lambda_1\times\Lambda_2$. Now

$$\begin{aligned}\mathfrak{T}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= \mathfrak{T}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &=\min\{\mathfrak{T}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma),\mathfrak{T}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)\}\\ &\succeq\min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2,\varsigma)\},\min\{\mathfrak{T}_{\Gamma}(\hbar_1,\varsigma),\mathfrak{T}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &=\min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Gamma}(\hbar_1,\varsigma)\},\min\{\mathfrak{T}_{\Xi}(\flat_2,\varsigma),\mathfrak{T}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &=\min\{\mathfrak{T}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma),\mathfrak{T}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\}.\end{aligned}$$

$$\begin{aligned}1-\mathfrak{F}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= 1-\mathfrak{F}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &=\min\{1-\mathfrak{F}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma),1-\mathfrak{F}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)\}\\ &\succeq\min\{\min\{1-\mathfrak{F}_{\Xi}(\flat_1,\varsigma),1-\mathfrak{F}_{\Xi}(\flat_2,\varsigma)\},\min\{1-\mathfrak{F}_{\Gamma}(\hbar_1,\varsigma),1-\mathfrak{F}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &=\min\{\min\{1-\mathfrak{F}_{\Xi}(\flat_1,\varsigma),1-\mathfrak{F}_{\Gamma}(\hbar_1,\varsigma)\},\min\{1-\mathfrak{F}_{\Xi}(\flat_2,\varsigma),1-\mathfrak{F}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &=\min\{1-\mathfrak{F}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma),1-\mathfrak{F}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\}.\end{aligned}$$

Thus, $\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat\heartsuit_1\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat\heartsuit_2\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat\heartsuit_3\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$. Now,

$$\begin{aligned}\mathfrak{I}^{l}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= \mathfrak{I}^{l}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &=\frac{\mathfrak{I}^{l}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma)+\mathfrak{I}^{l}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)}{2}\\ &\succeq\frac{1}{2}\left[\frac{\mathfrak{I}^{l}_{\Xi}(\flat_1,\varsigma)+\mathfrak{I}^{l}_{\Xi}(\flat_2,\varsigma)}{2}+\frac{\mathfrak{I}^{l}_{\Gamma}(\hbar_1,\varsigma)+\mathfrak{I}^{l}_{\Gamma}(\hbar_2,\varsigma)}{2}\right]\\ &=\frac{1}{2}\left[\frac{\mathfrak{I}^{l}_{\Xi}(\flat_1,\varsigma)+\mathfrak{I}^{l}_{\Gamma}(\hbar_1,\varsigma)}{2}+\frac{\mathfrak{I}^{l}_{\Xi}(\flat_2,\varsigma)+\mathfrak{I}^{l}_{\Gamma}(\hbar_2,\varsigma)}{2}\right]\\ &=\frac{1}{2}\Big[\mathfrak{I}^{l}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mathfrak{I}^{l}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big].\end{aligned}$$

$$\begin{aligned}\mathfrak{I}^{u}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= \mathfrak{I}^{u}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &=\frac{\mathfrak{I}^{u}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma)+\mathfrak{I}^{u}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)}{2}\\ &\succeq\frac{1}{2}\left[\frac{\mathfrak{I}^{u}_{\Xi}(\flat_1,\varsigma)+\mathfrak{I}^{u}_{\Xi}(\flat_2,\varsigma)}{2}+\frac{\mathfrak{I}^{u}_{\Gamma}(\hbar_1,\varsigma)+\mathfrak{I}^{u}_{\Gamma}(\hbar_2,\varsigma)}{2}\right]\\ &=\frac{1}{2}\left[\frac{\mathfrak{I}^{u}_{\Xi}(\flat_1,\varsigma)+\mathfrak{I}^{u}_{\Gamma}(\hbar_1,\varsigma)}{2}+\frac{\mathfrak{I}^{u}_{\Xi}(\flat_2,\varsigma)+\mathfrak{I}^{u}_{\Gamma}(\hbar_2,\varsigma)}{2}\right]\\ &=\frac{1}{2}\Big[\mathfrak{I}^{u}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mathfrak{I}^{u}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big].\end{aligned}$$

Thus, $\mho^{\mathfrak{I}}_{\Xi\times\Gamma}(\flat\heartsuit_1\hbar,\varsigma)\succeq\frac{1}{2}\Big[\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big]$. Similarly, $\mho^{\mathfrak{I}}_{\Xi\times\Gamma}(\flat\heartsuit_2\hbar,\varsigma)\succeq\frac{1}{2}\Big[\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big]$ and $\mho^{\mathfrak{I}}_{\Xi\times\Gamma}(\flat\heartsuit_3\hbar,\varsigma)\succeq\frac{1}{2}\Big[\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big]$.
Now

$$\begin{aligned}\mathfrak{F}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= \mathfrak{F}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &=\max\{\mathfrak{F}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma),\mathfrak{F}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)\}\\ &\preceq\max\{\max\{\mathfrak{F}_{\Xi}(\flat_1,\varsigma),\mathfrak{F}_{\Xi}(\flat_2,\varsigma)\},\max\{\mathfrak{F}_{\Gamma}(\hbar_1,\varsigma),\mathfrak{F}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &=\max\{\max\{\mathfrak{F}_{\Xi}(\flat_1,\varsigma),\mathfrak{F}_{\Gamma}(\hbar_1,\varsigma)\},\max\{\mathfrak{F}_{\Xi}(\flat_2,\varsigma),\mathfrak{F}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &=\max\{\mathfrak{F}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma),\mathfrak{F}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\}.\end{aligned}$$

$$\begin{aligned}1-\mathfrak{T}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= 1-\mathfrak{T}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &=\max\{1-\mathfrak{T}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma),1-\mathfrak{T}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)\}\\ &\preceq\max\{\max\{1-\mathfrak{T}_{\Xi}(\flat_1,\varsigma),1-\mathfrak{T}_{\Xi}(\flat_2,\varsigma)\},\max\{1-\mathfrak{T}_{\Gamma}(\hbar_1,\varsigma),1-\mathfrak{T}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &=\max\{\max\{1-\mathfrak{T}_{\Xi}(\flat_1,\varsigma),1-\mathfrak{T}_{\Gamma}(\hbar_1,\varsigma)\},\max\{1-\mathfrak{T}_{\Xi}(\flat_2,\varsigma),1-\mathfrak{T}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &=\max\{1-\mathfrak{T}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma),1-\mathfrak{T}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\}.\end{aligned}$$

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Thus, $\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat\heartsuit_1\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat\heartsuit_2\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat\heartsuit_3\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$.
Now

$$\begin{aligned}\mathfrak{T}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= \mathfrak{T}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &= \min\{\mathfrak{T}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma),\mathfrak{T}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)\}\\ &\succeq \min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2,\varsigma)\},\min\{\mathfrak{T}_{\Gamma}(\hbar_1,\varsigma),\mathfrak{T}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &= \min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Gamma}(\hbar_1,\varsigma)\},\min\{\mathfrak{T}_{\Xi}(\flat_2,\varsigma),\mathfrak{T}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &= \min\{\mathfrak{T}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma),\mathfrak{T}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\}.\end{aligned}$$

Thus, $\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat\heartsuit_1\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat\heartsuit_2\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat\heartsuit_3\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$. Now,

$$\begin{aligned}\mathfrak{I}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= \mathfrak{I}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &= \frac{\mathfrak{I}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma)+\mathfrak{I}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)}{2}\\ &\succeq \frac{1}{2}\left[\frac{\mathfrak{I}_{\Xi}(\flat_1,\varsigma)+\mathfrak{I}_{\Xi}(\flat_2,\varsigma)}{2}+\frac{\mathfrak{I}_{\Gamma}(\hbar_1,\varsigma)+\mathfrak{I}_{\Gamma}(\hbar_2,\varsigma)}{2}\right]\\ &= \frac{1}{2}\left[\frac{\mathfrak{I}_{\Xi}(\flat_1,\varsigma)+\mathfrak{I}_{\Gamma}(\hbar_1,\varsigma)}{2}+\frac{\mathfrak{I}_{\Xi}(\flat_2,\varsigma)+\mathfrak{I}_{\Gamma}(\hbar_2,\varsigma)}{2}\right]\\ &= \frac{1}{2}\Big[\mathfrak{I}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mathfrak{I}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big].\end{aligned}$$

Thus, $\mho^{\mathfrak{I}}_{\Xi\times\Gamma}(\flat\heartsuit_1\hbar,\varsigma)\succeq\frac{1}{2}\Big[\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big]$. Similarly, $\mho^{\mathfrak{I}}_{\Xi\times\Gamma}(\flat\heartsuit_2\hbar,\varsigma)\succeq\frac{1}{2}\Big[\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big]$ and $\mho^{\mathfrak{I}}_{\Xi\times\Gamma}(\flat\heartsuit_3\hbar,\varsigma)\succeq\frac{1}{2}\Big[\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma)+\mho^{\mathfrak{I}}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\Big]$. Now

$$\begin{aligned}\mathfrak{F}_{\Xi\times\Gamma}[((\flat_1,\hbar_1),\varsigma)\heartsuit_1((\flat_2,\hbar_2),\varsigma)] &= \mathfrak{F}_{\Xi\times\Gamma}(\flat_1\heartsuit_1\flat_2,\hbar_1\heartsuit_1\hbar_2,\varsigma)\\ &= \max\{\mathfrak{F}_{\Xi}(\flat_1\heartsuit_1\flat_2,\varsigma),\mathfrak{F}_{\Gamma}(\hbar_1\heartsuit_1\hbar_2,\varsigma)\}\\ &\preceq \max\{\max\{\mathfrak{F}_{\Xi}(\flat_1,\varsigma),\mathfrak{F}_{\Xi}(\flat_2,\varsigma)\},\max\{\mathfrak{F}_{\Gamma}(\hbar_1,\varsigma),\mathfrak{F}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &= \max\{\max\{\mathfrak{F}_{\Xi}(\flat_1,\varsigma),\mathfrak{F}_{\Gamma}(\hbar_1,\varsigma)\},\max\{\mathfrak{F}_{\Xi}(\flat_2,\varsigma),\mathfrak{F}_{\Gamma}(\hbar_2,\varsigma)\}\}\\ &= \max\{\mathfrak{F}_{\Xi\times\Gamma}((\flat_1,\hbar_1),\varsigma),\mathfrak{F}_{\Xi\times\Gamma}((\flat_2,\hbar_2),\varsigma)\}.\end{aligned}$$

Thus, $\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat\heartsuit_1\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat\heartsuit_2\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat\heartsuit_3\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Xi\times\Gamma}(\hbar,\varsigma)\}$. Hence, $\Xi\times\Gamma$ is a q-NSCVSBS of Λ.

Corollary 3.5. *If $\Xi_1,\Xi_2,...,\Xi_n$ are the families of $q-NSCVSBS^s$ of $\Lambda_1,\Lambda_2,...,\Lambda_n$ respectively, then $\Xi_1\times\Xi_2\times...\times\Xi_n$ is a q-NSCVSBS of $\Lambda_1\times\Lambda_2\times...\times\Lambda_n$.*

Definition 3.6. Let Ξ be a q-neutrosophic VS in Λ, the strongest q-neutrosophic cubic vague relation (Sq-NSCVR) on Λ is defined as

$$\begin{cases}\mho^{\mathfrak{T}}_{\Upsilon}(\flat,\hbar,\varsigma)=\min\{\mho^{\mathfrak{T}}_{\Xi}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Xi}(\hbar,\varsigma)\}\\ \mho^{\mathfrak{I}}_{\Upsilon}(\flat,\hbar,\varsigma)=\frac{\mho^{\mathfrak{I}}_{\Xi}(\flat,\varsigma)+\mho^{\mathfrak{I}}_{\Xi}(\hbar,\varsigma)}{2}\\ \mho^{\mathfrak{F}}_{\Upsilon}(\flat,\hbar,\varsigma)=\max\{\mho^{\mathfrak{F}}_{\Xi}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Xi}(\hbar,\varsigma)\}\end{cases}.$$

Theorem 3.7. *Let Ξ be the q-NSCVSBS of Λ and Υ be the SqNSVR of Λ. Then Ξ is a q-NSCVSBS of Λ if and only if Υ is a q-NSCVSBS of $\Lambda\times\Lambda$.*

Proof. Let Ξ be the q-NSCVSBS of Λ and Υ be the SqNSVR of Λ. Then for any $\flat=((\flat_1,\flat_2),\varsigma)$ and

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$\hbar = ((\hbar_1, \hbar_2), \varsigma)$ are in $\Lambda \times \Lambda$. Now,

$$\begin{aligned}
\mathfrak{T}_\Upsilon(\flat \heartsuit_1 \hbar, \varsigma) &= \mathfrak{T}_\Upsilon[(((\flat_1, \flat_2), \varsigma) \heartsuit_1 ((\hbar_1, \hbar_2), \varsigma)] \\
&= \mathfrak{T}_\Upsilon(\flat_1 \heartsuit_1 \hbar_1, \flat_2 \heartsuit_1 \hbar_2, \varsigma) \\
&= \min\{\mathfrak{T}_\Xi(\flat_1 \heartsuit_1 \hbar_1, \varsigma), \mathfrak{T}_\Xi(\flat_2 \heartsuit_1 \hbar_2, \varsigma)\} \\
&\succeq \min\{\min\{\mathfrak{T}_\Xi(\flat_1, \varsigma), \mathfrak{T}_\Xi(\hbar_1, \varsigma)\}, \min\{\mathfrak{T}_\Xi(\flat_2, \varsigma), \mathfrak{T}_\Xi(\hbar_2, \varsigma)\}\} \\
&= \min\{\min\{\mathfrak{T}_\Xi(\flat_1, \varsigma), \mathfrak{T}_\Xi(\flat_2, \varsigma)\}, \min\{\mathfrak{T}_\Xi(\hbar_1, \varsigma), \mathfrak{T}_\Xi(\hbar_2, \varsigma)\}\} \\
&= \min\{\mathfrak{T}_\Upsilon((\flat_1, \flat_2), \varsigma), \mathfrak{T}_\Upsilon((\hbar_1, \hbar_2), \varsigma)\} \\
&= \min\{\mathfrak{T}_\Upsilon(\flat, \varsigma), \mathfrak{T}_\Upsilon(\hbar, \varsigma)\}.
\end{aligned}$$

$$\begin{aligned}
1 - \mathfrak{F}_\Upsilon(\flat \heartsuit_1 \hbar, \varsigma) &= 1 - \mathfrak{F}_\Upsilon[(((\flat_1, \flat_2), \varsigma) \heartsuit_1 ((\hbar_1, \hbar_2), \varsigma)] \\
&= 1 - \mathfrak{F}_\Upsilon(\flat_1 \heartsuit_1 \hbar_1, \flat_2 \heartsuit_1 \hbar_2, \varsigma) \\
&= \min\{1 - \mathfrak{F}_\Xi(\flat_1 \heartsuit_1 \hbar_1, \varsigma), 1 - \mathfrak{F}_\Xi(\flat_2 \heartsuit_1 \hbar_2, \varsigma)\} \\
&\succeq \min\{\min\{1 - \mathfrak{F}_\Xi(\flat_1, \varsigma), 1 - \mathfrak{F}_\Xi(\hbar_1, \varsigma)\}, \min\{1 - \mathfrak{F}_\Xi(\flat_2, \varsigma), 1 - \mathfrak{F}_\Xi(\hbar_2, \varsigma)\}\} \\
&= \min\{\min\{1 - \mathfrak{F}_\Xi(\flat_1, \varsigma), 1 - \mathfrak{F}_\Xi(\flat_2, \varsigma)\}, \min\{1 - \mathfrak{F}_\Xi(\hbar_1, \varsigma), 1 - \mathfrak{F}_\Xi(\hbar_2, \varsigma)\}\} \\
&= \min\{1 - \mathfrak{F}_\Upsilon((\flat_1, \flat_2), \varsigma), 1 - \mathfrak{F}_\Upsilon((\hbar_1, \hbar_2), \varsigma)\} \\
&= \min\{1 - \mathfrak{F}_\Upsilon(\flat, \varsigma), 1 - \mathfrak{F}_\Upsilon(\hbar, \varsigma)\}.
\end{aligned}$$

Thus, $\mho^{\mathfrak{T}}_\Upsilon(\flat \heartsuit_1 \hbar, \varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat, \varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar, \varsigma)\}$. Similarly, $\mho^{\mathfrak{T}}_\Upsilon(\flat \heartsuit_2 \hbar, \varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat, \varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar, \varsigma)\}$ and $\mho^{\mathfrak{T}}_\Upsilon(\flat \heartsuit_3 \hbar, \varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat, \varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar, \varsigma)\}$. Now,

$$\begin{aligned}
\mathfrak{I}^l_\Upsilon(\flat \heartsuit_1 \hbar, \varsigma) &= \mathfrak{I}^l_\Upsilon[(((\flat_1, \flat_2), \varsigma) \heartsuit_1 ((\hbar_1, \hbar_2), \varsigma)] \\
&= \mathfrak{I}^l_\Upsilon(\flat_1 \heartsuit_1 \hbar_1, \flat_2 \heartsuit_1 \hbar_2, \varsigma) \\
&= \frac{\mathfrak{I}^l_\Xi(\flat_1 \heartsuit_1 \hbar_1, \varsigma) + \mathfrak{I}^l_\Xi(\flat_2 \heartsuit_1 \hbar_2, \varsigma)}{2} \\
&\succeq \frac{1}{2}\left[\frac{\mathfrak{I}^l_\Xi(\flat_1, \varsigma) + \mathfrak{I}^l_\Xi(\hbar_1, \varsigma)}{2} + \frac{\mathfrak{I}^l_\Xi(\flat_2, \varsigma) + \mathfrak{I}^l_\Xi(\hbar_2, \varsigma)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\mathfrak{I}^l_\Xi(\flat_1, \varsigma) + \mathfrak{I}^l_\Xi(\flat_2, \varsigma)}{2} + \frac{\mathfrak{I}^l_\Xi(\hbar_1, \varsigma) + \mathfrak{I}^l_\Xi(\hbar_2, \varsigma)}{2}\right] \\
&= \frac{\mathfrak{I}^l_\Upsilon((\flat_1, \flat_2), \varsigma) + \mathfrak{I}^l_\Upsilon((\hbar_1, \hbar_2), \varsigma)}{2} \\
&= \frac{\mathfrak{I}^l_\Upsilon(\flat, \varsigma) + \mathfrak{I}^l_\Upsilon(\hbar, \varsigma)}{2}.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{I}^u_\Upsilon(\flat \heartsuit_1 \hbar, \varsigma) &= \mathfrak{I}^u_\Upsilon[(((\flat_1, \flat_2), \varsigma) \heartsuit_1 ((\hbar_1, \hbar_2), \varsigma)] \\
&= \mathfrak{I}^u_\Upsilon(\flat_1 \heartsuit_1 \hbar_1, \flat_2 \heartsuit_1 \hbar_2, \varsigma) \\
&= \frac{\mathfrak{I}^u_\Xi(\flat_1 \heartsuit_1 \hbar_1, \varsigma) + \mathfrak{I}^u_\Xi(\flat_2 \heartsuit_1 \hbar_2, \varsigma)}{2} \\
&\succeq \frac{1}{2}\left[\frac{\mathfrak{I}^u_\Xi(\flat_1, \varsigma) + \mathfrak{I}^u_\Xi(\hbar_1, \varsigma)}{2} + \frac{\mathfrak{I}^u_\Xi(\flat_2, \varsigma) + \mathfrak{I}^u_\Xi(\hbar_2, \varsigma)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\mathfrak{I}^u_\Xi(\flat_1, \varsigma) + \mathfrak{I}^u_\Xi(\flat_2, \varsigma)}{2} + \frac{\mathfrak{I}^u_\Xi(\hbar_1, \varsigma) + \mathfrak{I}^u_\Xi(\hbar_2, \varsigma)}{2}\right] \\
&= \frac{\mathfrak{I}^u_\Upsilon((\flat_1, \flat_2), \varsigma) + \mathfrak{I}^u_\Upsilon((\hbar_1, \hbar_2), \varsigma)}{2} \\
&= \frac{\mathfrak{I}^u_\Upsilon(\flat, \varsigma) + \mathfrak{I}^u_\Upsilon(\hbar, \varsigma)}{2}.
\end{aligned}$$

Thus, $\mho^{\mathfrak{I}}_\Upsilon(\flat \heartsuit_1 \hbar, \varsigma) \succeq \frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$. Similarly, $\mho^{\mathfrak{I}}_\Upsilon(\flat \heartsuit_2 \hbar, \varsigma) \succeq \frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$ and $\mho^{\mathfrak{I}}_\Upsilon(\flat \heartsuit_3 \hbar, \varsigma) \succeq \frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$.

Similarly, $\mho^{\mathfrak{F}}_\Upsilon(\flat \heartsuit_1 \hbar, \varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat, \varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar, \varsigma)\}$, $\mho^{\mathfrak{F}}_\Upsilon(\flat \heartsuit_2 \hbar, \varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat, \varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar, \varsigma)\}$ and $\mho^{\mathfrak{F}}_\Upsilon(\flat \heartsuit_3 \hbar, \varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat, \varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar, \varsigma)\}$.

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Now,

$$\begin{aligned}
\mathfrak{T}_{\Upsilon}(\flat\heartsuit_1\hbar,\varsigma) &= \mathfrak{T}_{\Upsilon}[(((\flat_1,\flat_2),\varsigma)\heartsuit_1((\hbar_1,\hbar_2),\varsigma)]\\
&= \mathfrak{T}_{\Upsilon}(\flat_1\heartsuit_1\hbar_1,\flat_2\heartsuit_1\hbar_2,\varsigma)\\
&= \min\{\mathfrak{T}_{\Xi}(\flat_1\heartsuit_1\hbar_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2\heartsuit_1\hbar_2,\varsigma)\}\\
&\succeq \min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\hbar_1,\varsigma)\},\min\{\mathfrak{T}_{\Xi}(\flat_2,\varsigma),\mathfrak{T}_{\Xi}(\hbar_2,\varsigma)\}\}\\
&= \min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2,\varsigma)\},\min\{\mathfrak{T}_{\Xi}(\hbar_1,\varsigma),\mathfrak{T}_{\Xi}(\hbar_2,\varsigma)\}\}\\
&= \min\{\mathfrak{T}_{\Upsilon}((\flat_1,\flat_2),\varsigma),\mathfrak{T}_{\Upsilon}((\hbar_1,\hbar_2),\varsigma)\}\\
&= \min\{\mathfrak{T}_{\Upsilon}(\flat,\varsigma),\mathfrak{T}_{\Upsilon}(\hbar,\varsigma)\}.
\end{aligned}$$

Thus, $\mho^{\mathfrak{T}}_{\Upsilon}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_{\Upsilon}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Upsilon}(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{T}}_{\Upsilon}(\flat\heartsuit_2\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_{\Upsilon}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Upsilon}(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_{\Upsilon}(\flat\heartsuit_3\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_{\Upsilon}(\flat,\varsigma),\mho^{\mathfrak{T}}_{\Upsilon}(\hbar,\varsigma)\}$.

Now,

$$\begin{aligned}
\mathfrak{I}_{\Upsilon}(\flat\heartsuit_1\hbar,\varsigma) &= \mathfrak{I}_{\Upsilon}[(((\flat_1,\flat_2),\varsigma)\heartsuit_1((\hbar_1,\hbar_2),\varsigma)]\\
&= \mathfrak{I}_{\Upsilon}(\flat_1\heartsuit_1\hbar_1,\flat_2\heartsuit_1\hbar_2,\varsigma)\\
&= \frac{\mathfrak{I}_{\Xi}(\flat_1\heartsuit_1\hbar_1,\varsigma)+\mathfrak{I}_{\Xi}(\flat_2\heartsuit_1\hbar_2,\varsigma)}{2}\\
&\succeq \frac{1}{2}\left[\frac{\mathfrak{I}_{\Xi}(\flat_1,\varsigma)+\mathfrak{I}_{\Xi}(\hbar_1,\varsigma)}{2}+\frac{\mathfrak{I}_{\Xi}(\flat_2,\varsigma)+\mathfrak{I}_{\Xi}(\hbar_2,\varsigma)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\mathfrak{I}_{\Xi}(\flat_1,\varsigma)+\mathfrak{I}_{\Xi}(\flat_2,\varsigma)}{2}+\frac{\mathfrak{I}_{\Xi}(\hbar_1,\varsigma)+\mathfrak{I}_{\Xi}(\hbar_2,\varsigma)}{2}\right]\\
&= \frac{\mathfrak{I}_{\Upsilon}((\flat_1,\flat_2),\varsigma)+\mathfrak{I}_{\Upsilon}((\hbar_1,\hbar_2),\varsigma)}{2}\\
&= \frac{\mathfrak{I}_{\Upsilon}(\flat,\varsigma)+\mathfrak{I}_{\Upsilon}(\hbar,\varsigma)}{2}.
\end{aligned}$$

Thus, $\mho^{\mathfrak{I}}_{\Upsilon}(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mho_{\Upsilon}(\flat,\varsigma)+\mho_{\Upsilon}(\hbar,\varsigma)}{2}$. Similarly, $\mho^{\mathfrak{I}}_{\Upsilon}(\flat\heartsuit_2\hbar,\varsigma) \succeq \frac{\mho_{\Upsilon}(\flat,\varsigma)+\mho_{\Upsilon}(\hbar,\varsigma)}{2}$ and $\mho^{\mathfrak{I}}_{\Upsilon}(\flat\heartsuit_3\hbar,\varsigma) \succeq \frac{\mho_{\Upsilon}(\flat,\varsigma)+\mho_{\Upsilon}(\hbar,\varsigma)}{2}$.
Similarly, $\mho^{\mathfrak{F}}_{\Upsilon}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_{\Upsilon}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Upsilon}(\hbar,\varsigma)\}$, $\mho^{\mathfrak{F}}_{\Upsilon}(\flat\heartsuit_2\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_{\Upsilon}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Upsilon}(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{F}}_{\Upsilon}(\flat\heartsuit_3\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_{\Upsilon}(\flat,\varsigma),\mho^{\mathfrak{F}}_{\Upsilon}(\hbar,\varsigma)\}$. Thus, Υ is a q-NSCVSBS of $\Lambda\times\Lambda$.
Conversely, assume that Υ is a q-NSCVSBS of $\Lambda\times\Lambda$, then for any $\flat=((\flat_1,\flat_2),\varsigma)$ and $\hbar=((\hbar_1,\hbar_2),\varsigma)$ are in $\Lambda\times\Lambda$. Now,

$$\begin{aligned}
\min\{\mathfrak{T}_{\Xi}(\flat_1\heartsuit_1\hbar_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2\heartsuit_1\hbar_2,\varsigma)\} &= \mathfrak{T}_{\Upsilon}(\flat_1\heartsuit_1\hbar_1,\flat_2\heartsuit_1\hbar_2,\varsigma)\\
&= \mathfrak{T}_{\Upsilon}[((\flat_1,\flat_2),\varsigma)\heartsuit_1((\hbar_1,\hbar_2),\varsigma)]\\
&= \mathfrak{T}_{\Upsilon}(\flat\heartsuit_1\hbar,\varsigma)\\
&\succeq \min\{\mathfrak{T}_{\Upsilon}(\flat,\varsigma),\mathfrak{T}_{\Upsilon}(\hbar,\varsigma)\}\\
&= \min\{\mathfrak{T}_{\Upsilon}((\flat_1,\flat_2),\varsigma)\},\mathfrak{T}_{\Upsilon}((\hbar_1,\hbar_2),\varsigma)\}\\
&= \min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2,\varsigma)\},\min\{\mathfrak{T}_{\Xi}(\hbar_1,\varsigma),\mathfrak{T}_{\Xi}(\hbar_2,\varsigma)\}\}.
\end{aligned}$$

If $\mathfrak{T}_{\Xi}(\flat_1\heartsuit_1\hbar_1,\varsigma) \preceq \mathfrak{T}_{\Xi}(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $\mathfrak{T}_{\Xi}(\flat_1,\varsigma) \preceq \mathfrak{T}_{\Xi}(\flat_2,\varsigma)$ and $\mathfrak{T}_{\Xi}(\hbar_1,\varsigma) \preceq \mathfrak{T}_{\Xi}(\hbar_2,\varsigma)$. We get $\mathfrak{T}_{\Xi}(\flat_1\heartsuit_1\hbar_1,\varsigma) \succeq \min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\hbar_1,\varsigma)\}$ for all $\flat_1,\hbar_1\in\Lambda$, and
$\min\{\mathfrak{T}_{\Xi}(\flat_1\heartsuit_2\hbar_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2\heartsuit_2\hbar_2,\varsigma)\} \succeq \min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2,\varsigma)\},\min\{\mathfrak{T}_{\Xi}(\hbar_1,\varsigma),\mathfrak{T}_{\Xi}(\hbar_2,\varsigma)\}\}$
If $\mathfrak{T}_{\Xi}(\flat_1\heartsuit_2\hbar_1,\varsigma) \preceq \mathfrak{T}_{\Xi}(\flat_2\heartsuit_2\hbar_2,\varsigma)$, then $\mathfrak{T}_{\Xi}(\flat_1\heartsuit_2\hbar_1,\varsigma) \succeq \min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\hbar_1,\varsigma)\}$.
$\min\{\mathfrak{T}_{\Xi}(\flat_1\heartsuit_3\hbar_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2\heartsuit_3\hbar_2,\varsigma)\} \succeq \min\{\min\{\mathfrak{T}_{\Xi}(\flat_1,\varsigma),\mathfrak{T}_{\Xi}(\flat_2,\varsigma)\},\min\{\mathfrak{T}_{\Xi}(\hbar_1,\varsigma),\mathfrak{T}_{\Xi}(\hbar_2,\varsigma)\}\}$.

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If $\mathfrak{T}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\preceq\mathfrak{T}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)$, then $\mathfrak{T}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq\min\{\mathfrak{T}_\Xi(\flat_1,\varsigma),\mathfrak{T}_\Xi(\hbar_1,\varsigma)\}$.

$$\begin{aligned}
&\min\{1-\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma),1-\mathfrak{F}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)\}\\
&=1-\mathfrak{F}_\Upsilon(\flat_1\heartsuit_1\hbar_1,\flat_2\heartsuit_1\hbar_2,\varsigma)\\
&=1-\mathfrak{F}_\Upsilon[((\flat_1,\flat_2),\varsigma)\heartsuit_1((\hbar_1,\hbar_2),\varsigma)]\\
&=1-\mathfrak{F}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\\
&\succeq\min\{1-\mathfrak{F}_\Upsilon(\flat,\varsigma),1-\mathfrak{F}_\Upsilon(\hbar,\varsigma)\}\\
&=\min\{1-\mathfrak{F}_\Upsilon((\flat_1,\flat_2),\varsigma)\},1-\mathfrak{F}_\Upsilon((\hbar_1,\hbar_2),\varsigma)\}\\
&=\min\{\min\{1-\mathfrak{F}_\Xi(\flat_1,\varsigma),1-\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\min\{1-\mathfrak{F}_\Xi(\hbar_1,\varsigma),1-\mathfrak{F}_\Xi(\hbar_2,\varsigma)\}\}.
\end{aligned}$$

If $1-\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\preceq1-\mathfrak{F}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $1-\mathfrak{F}_\Xi(\flat_1,\varsigma)\preceq1-\mathfrak{F}_\Xi(\flat_2,\varsigma)$ and $1-\mathfrak{F}_\Xi(\hbar_1,\varsigma)\preceq1-\mathfrak{F}_\Xi(\hbar_2,\varsigma)$. We get $1-\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\succeq\min\{1-\mathfrak{F}_\Xi(\flat_1,\varsigma),1-\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$ for all $\flat_1,\hbar_1\in\Lambda$, and $\min\{1-\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma),1-\mathfrak{F}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)\}\succeq\min\{\min\{1-\mathfrak{F}_\Xi(\flat_1,\varsigma),1-\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\min\{1-\mathfrak{F}_\Xi(\hbar_1,\varsigma),1-\mathfrak{F}_\Xi(\hbar_2,\varsigma)\}\}$.
If $1-\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\preceq1-\mathfrak{F}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)$, then $1-\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\succeq\min\{1-\mathfrak{F}_\Xi(\flat_1,\varsigma),1-\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$.
$\min\{1-\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma),1-\mathfrak{F}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)\}\succeq\min\{\min\{1-\mathfrak{F}_\Xi(\flat_1,\varsigma),1-\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\min\{1-\mathfrak{F}_\Xi(\hbar_1,\varsigma),1-\mathfrak{F}_\Xi(\hbar_2,\varsigma)\}\}$. If $1-\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\preceq1-\mathfrak{F}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)$, then $1-\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq\min\{1-\mathfrak{F}_\Xi(\flat_1,\varsigma),1-\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$.
Thus $\mho^{\mathfrak{T}}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{T}}_\Upsilon(\flat\heartsuit_2\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_\Upsilon(\flat\heartsuit_3\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$. Now,

$$\begin{aligned}
\frac{1}{2}\Big[\mathfrak{I}^l_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)+\mathfrak{I}^l_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)\Big]&=\mathfrak{I}^l_\Upsilon(\flat_1\heartsuit_1\hbar_1,\flat_2\heartsuit_1\hbar_2,\varsigma)\\
&=\mathfrak{I}^l_\Upsilon[((\flat_1,\flat_2),\varsigma)\heartsuit_1((\hbar_1,\hbar_2),\varsigma)]\\
&=\mathfrak{I}^l_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\\
&\succeq\frac{\mathfrak{I}^l_\Upsilon(\flat,\varsigma)+\mathfrak{I}^l_\Upsilon(\hbar,\varsigma)}{2}\\
&=\frac{\mathfrak{I}^l_\Upsilon((\flat_1,\flat_2),\varsigma)+\mathfrak{I}^l_\Upsilon((\hbar_1,\hbar_2),\varsigma)}{2}\\
&=\frac{1}{2}\left[\frac{\mathfrak{I}^l_\Xi(\flat_1,\varsigma)+\mathfrak{I}^l_\Xi(\flat_2,\varsigma)}{2}+\frac{\mathfrak{I}^l_\Xi(\hbar_1,\varsigma)+\mathfrak{I}^l_\Xi(\hbar_2,\varsigma)}{2}\right].
\end{aligned}$$

If $\mathfrak{I}^l_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\preceq\mathfrak{I}^l_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $\mathfrak{I}^l_\Xi(\flat_1,\varsigma)\preceq\mathfrak{I}^l_\Xi(\flat_2,\varsigma)$ and $\mathfrak{I}^l_\Xi(\hbar_1,\varsigma)\preceq\mathfrak{I}^l_\Xi(\hbar_2,\varsigma)$.
We get $\mathfrak{I}^l_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}^l_\Xi(\flat_1,\varsigma)+\mathfrak{I}^l_\Xi(\hbar_1,\varsigma)}{2}$. Similarly, $\mathfrak{I}^l_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}^l_\Xi(\flat_1,\varsigma)+\mathfrak{I}^l_\Xi(\hbar_1,\varsigma)}{2}$ and $\mathfrak{I}^l_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}^l_\Xi(\flat_1,\varsigma)+\mathfrak{I}^l_\Xi(\hbar_1,\varsigma)}{2}$.
Also, $\frac{1}{2}\Big[\mathfrak{I}^u_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)+\mathfrak{I}^u_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)\Big]\succeq\frac{1}{2}\left[\frac{\mathfrak{I}^u_\Xi(\flat_1,\varsigma)+\mathfrak{I}^u_\Xi(\flat_2,\varsigma)}{2}+\frac{\mathfrak{I}^u_\Xi(\hbar_1,\varsigma)+\mathfrak{I}^u_\Xi(\hbar_2,\varsigma)}{2}\right]$.
If $\mathfrak{I}^u_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\preceq\mathfrak{I}^u_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $\mathfrak{I}^u_\Xi(\flat_1,\varsigma)\preceq\mathfrak{I}^u_\Xi(\flat_2,\varsigma)$ and $\mathfrak{I}^u_\Xi(\hbar_1,\varsigma)\preceq\mathfrak{I}^u_\Xi(\hbar_2,\varsigma)$.
We get $\mathfrak{I}^u_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}^u_\Xi(\flat_1,\varsigma)+\mathfrak{I}^u_\Xi(\hbar_1,\varsigma)}{2}$ and $\mathfrak{I}^u_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}^u_\Xi(\flat_1,\varsigma)+\mathfrak{I}^u_\Xi(\hbar_1,\varsigma)}{2}$ and $\mathfrak{I}^u_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}^u_\Xi(\flat_1,\varsigma)+\mathfrak{I}^u_\Xi(\hbar_1,\varsigma)}{2}$.
Thus $\mho^{\mathfrak{I}}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\succeq\frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$. Similarly, $\mho^{\mathfrak{I}}_\Upsilon(\flat\heartsuit_2\hbar,\varsigma)\succeq\frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$ and $\mho^{\mathfrak{I}}_\Upsilon(\flat\heartsuit_3\hbar,\varsigma)\succeq\frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$. Similarly, $\max\{\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma),\mathfrak{F}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)\}\preceq\max\{\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\max\{\mathfrak{F}_\Xi(\hbar_1,\varsigma),\mathfrak{F}_\Xi(\hbar_2,\varsigma$
If $\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\succeq\mathfrak{F}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $\mathfrak{F}_\Xi(\flat_1,\varsigma)\succeq\mathfrak{F}_\Xi(\flat_2,\varsigma)$ and $\mathfrak{F}_\Xi(\hbar_1,\varsigma)\succeq\mathfrak{F}_\Xi(\hbar_2,\varsigma)$.
We get $\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\preceq\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$.
$\max\{\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma),\mathfrak{F}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)\}\preceq\max\{\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\max\{\mathfrak{F}_\Xi(\hbar_1,\varsigma),\mathfrak{F}_\Xi(\hbar_2,\varsigma)\}\}$.
If $\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\succeq\mathfrak{F}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)$, then $\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\preceq\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$.
$\max\{\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma),\mathfrak{F}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)\}\preceq\max\{\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\max\{\mathfrak{F}_\Xi(\hbar_1,\varsigma),\mathfrak{F}_\Xi(\hbar_2,\varsigma)\}\}$
If $\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq\mathfrak{F}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)$, then $\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\preceq\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$.
Also, Similarly to prove that $\max\{1-\mathfrak{T}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma),1-\mathfrak{T}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)\}\preceq\max\{\max\{1-\mathfrak{T}_\Xi(\flat_1,\varsigma),1-\mathfrak{T}_\Xi(\flat_2,\varsigma)\},\max\{1-\mathfrak{T}_\Xi(\hbar_1,\varsigma),1-\mathfrak{T}_\Xi(\hbar_2,\varsigma)\}\}$.
If $1-\mathfrak{T}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\succeq1-\mathfrak{T}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $1-\mathfrak{T}_\Xi(\flat_1,\varsigma)\succeq1-\mathfrak{T}_\Xi(\flat_2,\varsigma)$ and $1-\mathfrak{T}_\Xi(\hbar_1,\varsigma)\succeq1-\mathfrak{T}_\Xi(\hbar_2,\varsigma)$.
We get $1-\mathfrak{T}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\preceq\max\{1-\mathfrak{T}_\Xi(\flat_1,\varsigma),1-\mathfrak{T}_\Xi(\hbar_1,\varsigma)\}$.
$\max\{1-\mathfrak{T}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma),1-\mathfrak{T}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)\}\preceq\max\{\max\{1-\mathfrak{T}_\Xi(\flat_1,\varsigma),1-\mathfrak{T}_\Xi(\flat_2,\varsigma)\},\max\{1-\mathfrak{T}_\Xi(\hbar_1,\varsigma),1-$

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$\mathfrak{T}_\Xi(\hbar_2,\varsigma)\}\}$.
If $1-\mathfrak{T}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\succeq 1-\mathfrak{T}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)$, then $1-\mathfrak{T}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\preceq \max\{1-\mathfrak{T}_\Xi(\flat_1,\varsigma),1-\mathfrak{T}_\Xi(\hbar_1,\varsigma)\}$.
$\max\{1-\mathfrak{T}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma),1-\mathfrak{T}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)\}\preceq\max\{\max\{1-\mathfrak{T}_\Xi(\flat_1,\varsigma),1-\mathfrak{T}_\Xi(\flat_2,\varsigma)\},\max\{1-\mathfrak{T}_\Xi(\hbar_1,\varsigma),1-\mathfrak{T}_\Xi(\hbar_2,\varsigma)\}\}$.
If $1-\mathfrak{T}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq 1-\mathfrak{T}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)$, then $1-\mathfrak{T}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\preceq \max\{1-\mathfrak{T}_\Xi(\flat_1,\varsigma),1-\mathfrak{T}_\Xi(\hbar_1,\varsigma)\}$.
Hence, $\mho^{\mathfrak{F}}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$, $\mho^{\mathfrak{F}}_\Upsilon(\flat\heartsuit_2\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$ and
$\mho^{\mathfrak{F}}_\Upsilon(\flat\heartsuit_3\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$.
Assume that Υ is a q-NSCVSBS of $\Lambda\times\Lambda$, then for any $\flat=((\flat_1,\flat_2),\varsigma)$ and $\hbar=((\hbar_1,\hbar_2),\varsigma)$ are in $\Lambda\times\Lambda$. Now,

$$\begin{aligned}
\min\{\mathfrak{T}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma),\mathfrak{T}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)\} &= \mathfrak{T}_\Upsilon(\flat_1\heartsuit_1\hbar_1,\flat_2\heartsuit_1\hbar_2,\varsigma)\\
&= \mathfrak{T}_\Upsilon[((\flat_1,\flat_2),\varsigma)\heartsuit_1((\hbar_1,\hbar_2),\varsigma)]\\
&= \mathfrak{T}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\\
&\succeq \min\{\mathfrak{T}_\Upsilon(\flat,\varsigma),\mathfrak{T}_\Upsilon(\hbar,\varsigma)\}\\
&= \min\{\mathfrak{T}_\Upsilon((\flat_1,\flat_2),\varsigma)\},\mathfrak{T}_\Upsilon((\hbar_1,\hbar_2),\varsigma)\}\\
&= \min\{\min\{\mathfrak{T}_\Xi(\flat_1,\varsigma),\mathfrak{T}_\Xi(\flat_2,\varsigma)\},\min\{\mathfrak{T}_\Xi(\hbar_1,\varsigma),\mathfrak{T}_\Xi(\hbar_2,\varsigma)\}\}.
\end{aligned}$$

If $\mathfrak{T}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\preceq\mathfrak{T}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $\mathfrak{T}_\Xi(\flat_1,\varsigma)\preceq\mathfrak{T}_\Xi(\flat_2,\varsigma)$ and $\mathfrak{T}_\Xi(\hbar_1,\varsigma)\preceq\mathfrak{T}_\Xi(\hbar_2,\varsigma)$. We get $\mathfrak{T}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\succeq\min\{\mathfrak{T}_\Xi(\flat_1,\varsigma),\mathfrak{T}_\Xi(\hbar_1,\varsigma)\}$ for all $\flat_1,\hbar_1\in\Lambda$, and
$\min\{\mathfrak{T}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma),\mathfrak{T}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)\}\succeq\min\{\min\{\mathfrak{T}_\Xi(\flat_1,\varsigma),\mathfrak{T}_\Xi(\flat_2,\varsigma)\},\min\{\mathfrak{T}_\Xi(\hbar_1,\varsigma),\mathfrak{T}_\Xi(\hbar_2,\varsigma)\}\}$
If $\mathfrak{T}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\preceq\mathfrak{T}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)$, then $\mathfrak{T}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\succeq\min\{\mathfrak{T}_\Xi(\flat_1,\varsigma),\mathfrak{T}_\Xi(\hbar_1,\varsigma)\}$.
$\min\{\mathfrak{T}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma),\mathfrak{T}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)\}\succeq\min\{\min\{\mathfrak{T}_\Xi(\flat_1,\varsigma),\mathfrak{T}_\Xi(\flat_2,\varsigma)\},\min\{\mathfrak{T}_\Xi(\hbar_1,\varsigma),\mathfrak{T}_\Xi(\hbar_2,\varsigma)\}\}$.
If $\mathfrak{T}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\preceq\mathfrak{T}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)$, then $\mathfrak{T}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq\min\{\mathfrak{T}_\Xi(\flat_1,\varsigma),\mathfrak{T}_\Xi(\hbar_1,\varsigma)\}$.
Thus, $\mho^{\mathfrak{T}}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{T}}_\Upsilon(\flat\heartsuit_2\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_\Upsilon(\flat\heartsuit_3\hbar,\varsigma)\succeq\min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$.
Now,

$$\begin{aligned}
\frac{1}{2}\Big[\mathfrak{I}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)+\mathfrak{I}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)\Big] &= \mathfrak{I}_\Upsilon(\flat_1\heartsuit_1\hbar_1,\flat_2\heartsuit_1\hbar_2,\varsigma)\\
&= \mathfrak{I}_\Upsilon[((\flat_1,\flat_2),\varsigma)\heartsuit_1((\hbar_1,\hbar_2),\varsigma)]\\
&= \mathfrak{I}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\\
&\succeq \frac{\mathfrak{I}_\Upsilon(\flat,\varsigma)+\mathfrak{I}_\Upsilon(\hbar,\varsigma)}{2}\\
&= \frac{\mathfrak{I}_\Upsilon((\flat_1,\flat_2),\varsigma)+\mathfrak{I}_\Upsilon((\hbar_1,\hbar_2),\varsigma)}{2}\\
&= \frac{1}{2}\left[\frac{\mathfrak{I}_\Xi(\flat_1,\varsigma)+\mathfrak{I}_\Xi(\flat_2,\varsigma)}{2}+\frac{\mathfrak{I}_\Xi(\hbar_1,\varsigma)+\mathfrak{I}_\Xi(\hbar_2,\varsigma)}{2}\right].
\end{aligned}$$

If $\mathfrak{I}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\preceq\mathfrak{I}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $\mathfrak{I}_\Xi(\flat_1,\varsigma)\preceq\mathfrak{I}_\Xi(\flat_2,\varsigma)$ and $\mathfrak{I}_\Xi(\hbar_1,\varsigma)\preceq\mathfrak{I}_\Xi(\hbar_2,\varsigma)$.
We get $\mathfrak{I}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}_\Xi(\flat_1,\varsigma)+\mathfrak{I}_\Xi(\hbar_1,\varsigma)}{2}$. Similarly, $\mathfrak{I}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}_\Xi(\flat_1,\varsigma)+\mathfrak{I}_\Xi(\hbar_1,\varsigma)}{2}$ and $\mathfrak{I}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq\frac{\mathfrak{I}_\Xi(\flat_1,\varsigma)+\mathfrak{I}_\Xi(\hbar_1,\varsigma)}{2}$.
Thus $\mho^{\mathfrak{I}}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\succeq\frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$. Similarly, $\mho^{\mathfrak{I}}_\Upsilon(\flat\heartsuit_2\hbar,\varsigma)\succeq\frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$ and $\mho^{\mathfrak{I}}_\Upsilon(\flat\heartsuit_3\hbar,\varsigma)\succeq\frac{\mho_\Upsilon(\flat,\varsigma)+\mho_\Upsilon(\hbar,\varsigma)}{2}$.
Similarly, $\max\{\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma),\mathfrak{F}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)\}\preceq\max\{\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\max\{\mathfrak{F}_\Xi(\hbar_1,\varsigma),\mathfrak{F}_\Xi(\hbar_2,\varsigma)\}\}$.
If $\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\succeq\mathfrak{F}_\Xi(\flat_2\heartsuit_1\hbar_2,\varsigma)$, then $\mathfrak{F}_\Xi(\flat_1,\varsigma)\succeq\mathfrak{F}_\Xi(\flat_2,\varsigma)$ and $\mathfrak{F}_\Xi(\hbar_1,\varsigma)\succeq\mathfrak{F}_\Xi(\hbar_2,\varsigma)$.
We get $\mathfrak{F}_\Xi(\flat_1\heartsuit_1\hbar_1,\varsigma)\preceq\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$.
$\max\{\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma),\mathfrak{F}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)\}\preceq\max\{\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\max\{\mathfrak{F}_\Xi(\hbar_1,\varsigma),\mathfrak{F}_\Xi(\hbar_2,\varsigma)\}\}$.
If $\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\succeq\mathfrak{F}_\Xi(\flat_2\heartsuit_2\hbar_2,\varsigma)$, then $\mathfrak{F}_\Xi(\flat_1\heartsuit_2\hbar_1,\varsigma)\preceq\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$.
$\max\{\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma),\mathfrak{F}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)\}\preceq\max\{\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\flat_2,\varsigma)\},\max\{\mathfrak{F}_\Xi(\hbar_1,\varsigma),\mathfrak{F}_\Xi(\hbar_2,\varsigma)\}\}$
If $\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\succeq\mathfrak{F}_\Xi(\flat_2\heartsuit_3\hbar_2,\varsigma)$, then $\mathfrak{F}_\Xi(\flat_1\heartsuit_3\hbar_1,\varsigma)\preceq\max\{\mathfrak{F}_\Xi(\flat_1,\varsigma),\mathfrak{F}_\Xi(\hbar_1,\varsigma)\}$.
Hence, $\mho^{\mathfrak{F}}_\Upsilon(\flat\heartsuit_1\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$, $\mho^{\mathfrak{F}}_\Upsilon(\flat\heartsuit_2\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$ and
$\mho^{\mathfrak{F}}_\Upsilon(\flat\heartsuit_3\hbar,\varsigma)\preceq\max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$. Hence, Ξ is a q-NSCVSBS of Λ.

Theorem 3.8. *Let Ξ be the qNSV subset in Λ. Then $\mho=([\mathfrak{T}_\Xi,\mathfrak{T}_\Xi],[\mathfrak{I}_\Xi,\mathfrak{I}_\Xi],[\mathfrak{F}_\Xi,\mathfrak{F}_\Xi])$ is a q-NSCVSBS of Λ if and only if all non empty level set $\mho^{(\varrho_1,\varrho_2,s)}$ is a SBS of Λ for $\varrho_1,\varrho_2,s\in[0,1]$.*

Proof. Assume that $\mho$ is a q-NSCVSBS of Λ. For $\varrho_1,\varrho_2,s\in[0,1]$ and $\flat,\hbar\in\mho^{(\varrho_1,\varrho_2,s)}$. We have $\mathfrak{T}_\Xi(\flat,\varsigma)\succeq\varrho_1,\mathfrak{T}_\Xi(\hbar,\varsigma)\succeq\varrho_1$ and $1-\mathfrak{F}_\Xi(\flat,\varsigma)\succeq s,1-\mathfrak{F}_\Xi(\hbar,\varsigma)\succeq s$ and $\mathfrak{I}^l_\Xi(\flat,\varsigma)\succeq\varrho_2,\mathfrak{I}^l_\Xi(\hbar,\varsigma)\succeq\varrho_2$

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and $\mathfrak{I}^{u}_{\Xi}(\flat,\varsigma) \succeq \varrho_2, \mathfrak{I}^{u}_{\Xi}(\hbar,\varsigma) \succeq \varrho_2, 1-\mathfrak{T}_{\Xi}(\flat,\varsigma) \preceq \varrho_1, 1-\mathfrak{T}_{\Xi}(\hbar,\varsigma) \preceq \varrho_1$ and $\mathfrak{F}_{\Xi}(\flat,\varsigma) \preceq s, \mathfrak{F}_{\Xi}(\hbar,\varsigma) \preceq s$. Now, $\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\} \succeq \varrho_1$, $1-\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{1-\mathfrak{F}_{\Xi}(\flat,\varsigma),1-\mathfrak{F}_{\Xi}(\hbar,\varsigma)\} \succeq s$ and $\mathfrak{I}^{l}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^{l}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{l}_{\Xi}(\hbar,\varsigma)}{2} \succeq \varrho_2$, $\mathfrak{I}^{u}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^{u}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{u}_{\Xi}(\hbar,\varsigma)}{2} \succeq \varrho_2$ and $\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\} \preceq s$ and $1-\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{1-\mathfrak{T}_{\Xi}(\flat,\varsigma),1-\mathfrak{T}_{\Xi}(\hbar,\varsigma)\} \preceq \varrho_1$. This implies that $\flat\heartsuit_1\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$. Similarly, $\flat\heartsuit_2\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$ and $\flat\heartsuit_3\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$. Therefore $\mho^{(\varrho_1,\varrho_2,s)}$ is a SBS of Λ, where $\varrho_1,\varrho_2,s \in [0,1]$.

For $\varrho_1,\varrho_2,s \in [0,1]$ and $\flat,\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$. We have $\mathfrak{T}_{\Xi}(\flat,\varsigma) \succeq \varrho_1, \mathfrak{T}_{\Xi}(\hbar,\varsigma) \succeq \varrho_1$ and $\mathfrak{I}_{\Xi}(\flat,\varsigma) \succeq \varrho_2, \mathfrak{I}_{\Xi}(\hbar,\varsigma) \succeq \varrho_2$ and $\mathfrak{F}_{\Xi}(\flat,\varsigma) \preceq s, \mathfrak{F}_{\Xi}(\hbar,\varsigma) \preceq s$. Now, $\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\} \succeq \varrho_1$ and $\mathfrak{I}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}_{\Xi}(\flat,\varsigma)+\mathfrak{I}_{\Xi}(\hbar,\varsigma)}{2} \succeq \varrho_2$ and $\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\} \preceq s$. This implies that $\flat\heartsuit_1\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$. Similarly, $\flat\heartsuit_2\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$ and $\flat\heartsuit_3\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$. Therefore $\mho^{(\varrho_1,\varrho_2,s)}$ is a SBS of Λ, where $\varrho_1,\varrho_2,s \in [0,1]$.

Conversely, assume that $\mho^{(\varrho_1,\varrho_2,s)}$ is a SBS of Λ, where $\varrho_1,\varrho_2,s \in [0,1]$. Suppose if there exist $\flat,\hbar \in \Lambda$ such that $\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$, $1-\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \min\{1-\mathfrak{F}_{\Xi}(\flat,\varsigma),1-\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$, $\mathfrak{I}^{l}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \frac{\mathfrak{I}^{l}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{l}_{\Xi}(\hbar,\varsigma)}{2}$, $\mathfrak{I}^{u}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \frac{\mathfrak{I}^{u}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{u}_{\Xi}(\hbar,\varsigma)}{2}$ and $\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) > \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$. $1-\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) > \max\{1-\mathfrak{T}_{\Xi}(\flat,\varsigma),1-\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$. Select $\varrho_1,\varrho_2,s \in [0,1]$ such that $\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \varrho_1 \preceq \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$ and $1-\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \varrho_1 \preceq \min\{1-\mathfrak{F}_{\Xi}(\flat,\varsigma),1-\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$ and $\mathfrak{I}^{l}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \varrho_2 \preceq \frac{\mathfrak{I}^{l}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{l}_{\Xi}(\hbar,\varsigma)}{2}$ and $\mathfrak{I}^{u}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \varrho_2 \preceq \frac{\mathfrak{I}^{u}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{u}_{\Xi}(\hbar,\varsigma)}{2}$ and $\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) > s \succeq \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$, $1-\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) > s \succeq \max\{1-\mathfrak{T}_{\Xi}(\flat,\varsigma),1-\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$. Then $\flat,\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$, but $\flat\heartsuit_1\hbar \notin \mho^{(\varrho_1,\varrho_2,s)}$. This contradicts to that $\mho^{(\varrho_1,\varrho_2,s)}$ is a SBS of Λ. Hence, $\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$, $1-\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{1-\mathfrak{F}_{\Xi}(\flat,\varsigma),1-\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$, $\mathfrak{I}^{l}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^{l}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{l}_{\Xi}(\hbar,\varsigma)}{2}$, $\mathfrak{I}^{u}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^{u}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{u}_{\Xi}(\hbar,\varsigma)}{2}$ and $\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$ and $1-\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{1-\mathfrak{T}_{\Xi}(\flat,\varsigma),1-\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$. Similarly, $\heartsuit_2$ and $\heartsuit_3$ cases.

Let assume that $\mho^{(\varrho_1,\varrho_2,s)}$ is a SBS of Λ, where $\varrho_1,\varrho_2,s \in [0,1]$. Suppose if there exist $\flat,\hbar \in \Lambda$ such that $\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$, $\mathfrak{I}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \frac{\mathfrak{I}_{\Xi}(\flat,\varsigma)+\mathfrak{I}_{\Xi}(\hbar,\varsigma)}{2}$, $\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) > \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$. Select $\varrho_1,\varrho_2,s \in [0,1]$ such that $\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \varrho_1 \preceq \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$ and $\mathfrak{I}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) < \varrho_2 \preceq \frac{\mathfrak{I}_{\Xi}(\flat,\varsigma)+\mathfrak{I}_{\Xi}(\hbar,\varsigma)}{2}$ and $\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) > s \succeq \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$. Then $\flat,\hbar \in \mho^{(\varrho_1,\varrho_2,s)}$, but $\flat\heartsuit_1\hbar \notin \mho^{(\varrho_1,\varrho_2,s)}$. This contradicts to that $\mho^{(\varrho_1,\varrho_2,s)}$ is a SBS of Λ. Hence, $\mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\}$, $\mathfrak{I}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}_{\Xi}(\flat,\varsigma)+\mathfrak{I}_{\Xi}(\hbar,\varsigma)}{2}$ and $\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_{\Xi}(\flat,\varsigma),\mathfrak{F}_{\Xi}(\hbar,\varsigma)\}$. Similarly, $\heartsuit_2$ and $\heartsuit_3$ cases. Hence, $\mho = ([\mathfrak{T}_{\Xi},\mathfrak{T}_{\Xi}],[\mathfrak{I}_{\Xi},\mathfrak{I}_{\Xi}],[\mathfrak{F}_{\Xi},\mathfrak{F}_{\Xi}])$ is a q-NSCVSBS of Λ.

Definition 3.9. Let Ξ be the any q-NSCVSBS of Λ and $\varsigma \in \Lambda$. Then the pseudo qNSV coset $(\varsigma\Xi)^p$ is defined by

$$\begin{cases} (\varsigma\mathfrak{T}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)\mathfrak{T}_{\Xi}(\flat,\varsigma),\ 1-(\varsigma\mathfrak{F}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)(1-\mathfrak{F}_{\Xi})(\flat,\varsigma), \\ (\varsigma\mathfrak{I}^{l}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)\mathfrak{I}^{l}_{\Xi}(\flat,\varsigma),\ (\varsigma\mathfrak{I}^{u}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)\mathfrak{I}^{u}_{\Xi}(\flat,\varsigma), \\ (\varsigma\mathfrak{F}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)\mathfrak{F}_{\Xi}(\flat,\varsigma),\ 1-(\varsigma\mathfrak{T}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)(1-\mathfrak{T}_{\Xi})(\flat,\varsigma) \end{cases}$$

$$\begin{cases} (\varsigma\mathfrak{T}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)\mathfrak{T}_{\Xi}(\flat,\varsigma), \\ (\varsigma\mathfrak{I}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)\mathfrak{I}_{\Xi}(\flat,\varsigma), \\ (\varsigma\mathfrak{F}_{\Xi})^p(\flat,\varsigma) = p(\varsigma)\mathfrak{F}_{\Xi}(\flat,\varsigma) \end{cases}$$

each $\flat \in \Lambda, \varsigma \in Q \subseteq \lambda$ and for any non-empty set $p \in P$.

Theorem 3.10. *Let Ξ be the q-NSCVSBS of Λ, then the pseudo qNSV coset $(\varsigma\Xi)^p$ is a q-NSCVSBS of Λ.*

Proof. Let Ξ be the q-NSCVSBS of Λ and for each $\flat,\hbar \in \Lambda$. Now, $(\varsigma\mathfrak{T}_{\Xi})^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ \mathfrak{T}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq p(\varsigma)\ \min\{\mathfrak{T}_{\Xi}(\flat,\varsigma),\mathfrak{T}_{\Xi}(\hbar,\varsigma)\} = \min\{p(\varsigma)\ \mathfrak{T}_{\Xi}(\flat,\varsigma), p(\varsigma)\ \mathfrak{T}_{\Xi}(\hbar,\varsigma)\} = \min\{(\varsigma\mathfrak{T}_{\Xi})^p(\flat,\varsigma),(\varsigma\mathfrak{T}_{\Xi})^p(\hbar,\varsigma)\}$. Thus $(\varsigma\mathfrak{T}_{\Xi})^p(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{(\varsigma\mathfrak{T}_{\Xi})^p(\flat,\varsigma),(\varsigma\mathfrak{T}_{\Xi})^p(\hbar,\varsigma)\}$ and $1-(\varsigma\mathfrak{F}_{\Xi})^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ (1-\mathfrak{F}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma)) \succeq p(\varsigma)\ \min\{1-\mathfrak{F}_{\Xi}(\flat,\varsigma),1-\mathfrak{F}_{\Xi}(\hbar,\varsigma)\} = \min\{p(\varsigma)\ (1-\mathfrak{F}_{\Xi}(\flat,\varsigma)), p(\varsigma)\ (1-\mathfrak{F}_{\Xi}(\hbar,\varsigma))\} = \min\{1-(\varsigma\mathfrak{F}_{\Xi})^p(\flat,\varsigma), 1-(\varsigma\mathfrak{F}_{\Xi})^p(\hbar,\varsigma)\}$. Thus $1-(\varsigma\mathfrak{F}_{\Xi})^p(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{1-(\varsigma\mathfrak{F}_{\Xi})^p(\flat,\varsigma), 1-(\varsigma\mathfrak{F}_{\Xi})^p(\hbar,\varsigma)\}$.

Now, $(\varsigma\mathfrak{I}^{l}_{\Xi})^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ \mathfrak{I}^{l}_{\Xi}(\flat\heartsuit_1\hbar,\varsigma) \succeq p(\varsigma)\left[\frac{\mathfrak{I}^{l}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{l}_{\Xi}(\hbar,\varsigma)}{2}\right] = \frac{p(\varsigma)\ \mathfrak{I}^{l}_{\Xi}(\flat,\varsigma)+p(\varsigma)\ \mathfrak{I}^{l}_{\Xi}(\hbar,\varsigma)}{2} = \frac{(\varsigma\mathfrak{I}^{l}_{\Xi})^p(\flat,\varsigma)+(\varsigma\mathfrak{I}^{l}_{\Xi})^p(\hbar,\varsigma)}{2}$.

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Thus $(\varsigma\mathfrak{I}^l_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{(\varsigma\mathfrak{I}^l_\Xi)^p(\flat,\varsigma)+(\varsigma\mathfrak{I}^l_\Xi)^p(\hbar,\varsigma)}{2}$ and $(\varsigma\mathfrak{I}^u_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ \mathfrak{I}^u_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq p(\varsigma)\left[\frac{\mathfrak{I}^u_\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Xi(\hbar,\varsigma)}{2}\right] = \frac{p(\varsigma)\ \mathfrak{I}^u_\Xi(\flat,\varsigma)+p(\varsigma)\ \mathfrak{I}^u_\Xi(\hbar,\varsigma)}{2} = \frac{(\varsigma\mathfrak{I}^u_\Xi)^p(\flat,\varsigma)+(\varsigma\mathfrak{I}^u_\Xi)^p(\hbar,\varsigma)}{2}$. Thus $(\varsigma\mathfrak{I}^u_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{(\varsigma\mathfrak{I}^u_\Xi)^p(\flat,\varsigma)+(\varsigma\mathfrak{I}^u_\Xi)^p(\hbar,\varsigma)}{2}$. Now, $(\varsigma\mathfrak{F}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ \mathfrak{F}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \preceq p(\varsigma)\ \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\} = \max\{p(\varsigma)\ \mathfrak{F}_\Xi(\flat,\varsigma), p(\varsigma)\ \mathfrak{F}_\Xi(\hbar,\varsigma)\} = \max\{(\varsigma\mathfrak{F}_\Xi)^p(\flat,\varsigma),(\varsigma\mathfrak{F}_\Xi)^p(\hbar,\varsigma)\}$. Thus $(\varsigma\mathfrak{F}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{(\varsigma\mathfrak{F}_\Xi)^p(\flat,\varsigma),(\varsigma\mathfrak{F}_\Xi)^p(\hbar,\varsigma)\}$ and $1-(\varsigma\mathfrak{T}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ (1-\mathfrak{T}_\Xi(\flat\heartsuit_1\hbar,\varsigma)) \preceq p(\varsigma)\ \max\{1-\mathfrak{T}_\Xi(\flat,\varsigma), 1-\mathfrak{T}_\Xi(\hbar,\varsigma)\} = \max\{p(\varsigma)\ (1-\mathfrak{T}_\Xi(\flat,\varsigma)), p(\varsigma)\ (1-\mathfrak{T}_\Xi(\hbar,\varsigma))\} = \max\{1-(\varsigma\mathfrak{T}_\Xi)^p(\flat,\varsigma), 1-(\varsigma\mathfrak{T}_\Xi)^p(\hbar,\varsigma)\}$. Thus $1-(\varsigma\mathfrak{T}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{1-(\varsigma\mathfrak{T}_\Xi)^p(\flat,\varsigma), 1-(\varsigma\mathfrak{T}_\Xi)^p(\hbar,\varsigma)\}$.
Now, $(\varsigma\mathfrak{T}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ \mathfrak{T}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq p(\varsigma)\ \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\} = \min\{p(\varsigma)\ \mathfrak{T}_\Xi(\flat,\varsigma), p(\varsigma)\ \mathfrak{T}_\Xi(\hbar,\varsigma)\} = \min\{(\varsigma\mathfrak{T}_\Xi)^p(\flat,\varsigma),(\varsigma\mathfrak{T}_\Xi)^p(\hbar,\varsigma)\}$.
Thus $(\varsigma\mathfrak{T}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) \succeq \min\{(\varsigma\mathfrak{T}_\Xi)^p(\flat,\varsigma),(\varsigma\mathfrak{T}_\Xi)^p(\hbar,\varsigma)\}$. Now, $(\varsigma\mathfrak{I}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ \mathfrak{I}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \succeq p(\varsigma)\left[\frac{\mathfrak{I}_\Xi(\flat,\varsigma)+\mathfrak{I}_\Xi(\hbar,\varsigma)}{2}\right] = \frac{p(\varsigma)\ \mathfrak{I}_\Xi(\flat,\varsigma)+p(\varsigma)\ \mathfrak{I}_\Xi(\hbar,\varsigma)}{2} = \frac{(\varsigma\mathfrak{I}_\Xi)^p(\flat,\varsigma)+(\varsigma\mathfrak{I}_\Xi)^p(\hbar,\varsigma)}{2}$.
Thus $(\varsigma\mathfrak{I}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) \succeq \frac{(\varsigma\mathfrak{I}_\Xi)^p(\flat,\varsigma)+(\varsigma\mathfrak{I}_\Xi)^p(\hbar,\varsigma)}{2}$.
Now, $(\varsigma\mathfrak{F}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) = p(\varsigma)\ \mathfrak{F}_\Xi(\flat\heartsuit_1\hbar,\varsigma) \preceq p(\varsigma)\ \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\} = \max\{p(\varsigma)\ \mathfrak{F}_\Xi(\flat,\varsigma), p(\varsigma)\ \mathfrak{F}_\Xi(\hbar,\varsigma)\} = \max\{(\varsigma\mathfrak{F}_\Xi)^p(\flat,\varsigma),(\varsigma\mathfrak{F}_\Xi)^p(\hbar,\varsigma)\}$. Thus $(\varsigma\mathfrak{F}_\Xi)^p(\flat\heartsuit_1\hbar,\varsigma) \preceq \max\{(\varsigma\mathfrak{F}_\Xi)^p(\flat,\varsigma),(\varsigma\mathfrak{F}_\Xi)^p(\hbar,\varsigma)\}$. Similarly, $\heartsuit_2$ and $\heartsuit_3$ cases. Hence, $(\varsigma\Xi)^p$ is a q-NSCVSBS of Λ.

Definition 3.11. Let $(\Lambda_1,\sharp_1,\sharp_2,\sharp_3)$ and $(\Lambda_2,\triangle_1,\triangle_2,\triangle_3)$ be the bisemirings. Let $\Upsilon:\Lambda_1\to\Lambda_2$ and Ξ be an q-NSCVSBS in Λ_1, Υ be an q-NSCVSBS in $\Upsilon(\Lambda_1)=\Lambda_2$, the image of VS is defined as $\mho_{\Xi(V)}(\hbar,\varsigma) = [\mathfrak{T}_{\Xi(V)}(\hbar,\varsigma), 1-\mathfrak{F}_{\Xi(V)}(\hbar,\varsigma)], [\mathfrak{I}^l_{\Xi(V)}(\hbar,\varsigma),\mathfrak{I}^u_{\Xi(V)}(\hbar,\varsigma)], [\mathfrak{F}_{\Xi(V)}(\hbar,\varsigma), 1-\mathfrak{T}_{\Xi(V)}(\hbar,\varsigma)]$ where $\mathfrak{T}_{\Xi(V)}(\hbar,\varsigma) = \mathfrak{T}_\Upsilon\Xi(\hbar,\varsigma)$, $\mathfrak{I}^l_{\Xi(V)}(\hbar,\varsigma) = \mathfrak{I}^l_\Upsilon\Xi(\hbar,\varsigma)$, $\mathfrak{I}^u_{\Xi(V)}(\hbar,\varsigma) = \mathfrak{I}^u_\Upsilon\Xi(\hbar,\varsigma)$ and $\mathfrak{F}_{\Xi(V)}(\hbar,\varsigma) = \mathfrak{F}_\Upsilon\Xi(\hbar,\varsigma)$ and $\mathfrak{T}_{\Xi(V)}(\hbar,\varsigma) = \mathfrak{T}_\Upsilon\Xi(\hbar,\varsigma)$, $\mathfrak{I}_{\Xi(V)}(\hbar,\varsigma) = \mathfrak{I}_\Upsilon\Xi(\hbar,\varsigma)$ and $\mathfrak{F}_{\Xi(V)}(\hbar,\varsigma) = \mathfrak{F}_\Upsilon\Xi(\hbar,\varsigma)$.

Definition 3.12. Let $(\Lambda_1,\sharp_1,\sharp_2,\sharp_3)$ and $(\Lambda_2,\triangle_1,\triangle_2,\triangle_3)$ be the bisemirings. Let $\Xi:\Lambda_1\to\Lambda_2$ be any function. Let Υ be a VS in $\Xi(\Lambda_1)=\Lambda_2$. Then the inverse image of Υ, Ξ^{-1} is the VS in Λ_1 by $\mho_{\Xi^{-1}(V)}(\flat,\varsigma) = [\mathfrak{T}_{\Xi^{-1}(V)}(\flat,\varsigma), 1-\mathfrak{F}_{\Xi^{-1}(V)}(\flat,\varsigma)], [\mathfrak{I}^l_{\Xi^{-1}(V)}(\flat,\varsigma),\mathfrak{I}^u_{\Xi^{-1}(V)}(\flat,\varsigma)], [\mathfrak{F}_{\Xi^{-1}(V)}(\flat,\varsigma), 1-\mathfrak{T}_{\Xi^{-1}(V)}(\flat,\varsigma)]$,
where $\mathfrak{T}_{\Xi^{-1}(V)}(\flat,\varsigma) = \mathfrak{T}_\Upsilon(\Xi^{-1}(\flat,\varsigma))$, $\mathfrak{I}^l_{\Xi^{-1}(V)}(\flat,\varsigma) = \mathfrak{I}^l_\Upsilon(\Xi^{-1}(\flat,\varsigma))$, $\mathfrak{I}^u_{\Xi^{-1}(V)}(\flat,\varsigma) = \mathfrak{I}^u_\Upsilon(\Xi^{-1}(\flat,\varsigma))$, $\mathfrak{F}_{\Xi^{-1}(V)}(\flat,\varsigma) = \mathfrak{F}_\Upsilon(\Xi^{-1}(\flat,\varsigma))$ and $\mathfrak{T}_{\Xi^{-1}(V)}(\flat,\varsigma) = \mathfrak{T}_\Upsilon(\Xi^{-1}(\flat,\varsigma))$, $\mathfrak{I}_{\Xi^{-1}(V)}(\flat,\varsigma) = \mathfrak{I}_\Upsilon(\Xi^{-1}(\flat,\varsigma))$ and $\mathfrak{F}_{\Xi^{-1}(V)}(\flat,\varsigma) = \mathfrak{F}_\Upsilon(\Xi^{-1}(\flat,\varsigma))$.

Theorem 3.13. *Every homomorphic image of q-NSCVSBS of Λ_1 is a q-NSCVSBS of Λ_2.*

Proof. Let $\Xi:\Lambda_1\to\Lambda_2$ be a homomorphism. Now, $\Xi(\flat\sharp_1\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)$, $\Xi(\flat\sharp_2\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)$ and $\Xi(\flat\sharp_3\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)$ for all $\flat,\hbar\in\Lambda_1$. Let $V=\Xi(\Xi)$, Ξ is a q-NSCVSBS of Λ_1. Let $\Xi(\flat,\varsigma),\Xi(\hbar,\varsigma)\in\Lambda_2$, $\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\} = \min\{\mathfrak{T}_\Upsilon\Xi(\flat,\varsigma),\mathfrak{T}_\Upsilon\Xi(\hbar,\varsigma)\}$ and $1-\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq 1-\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \min\{1-\mathfrak{F}_\Xi(\flat,\varsigma), 1-\mathfrak{F}_\Xi(\hbar,\varsigma)\} = \min\{1-\mathfrak{F}_\Upsilon\Xi(\flat,\varsigma), 1-\mathfrak{F}_\Upsilon\Xi(\hbar,\varsigma)\}$. Thus $\mho^{\mathfrak{T}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon\Xi(\hbar,\varsigma)\}$.
Similarly, $\mho^{\mathfrak{T}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon\Xi(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon\Xi(\hbar,\varsigma)\}$.
Now, $\mathfrak{I}^l_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{I}^l_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^l_\Xi(\flat,\varsigma)+\mathfrak{I}^l_\Xi(\hbar,\varsigma)}{2} = \frac{\mathfrak{I}^l_\Upsilon\Xi(\flat,\varsigma)+\mathfrak{I}^l_\Upsilon\Xi(\hbar,\varsigma)}{2}$ and $\mathfrak{I}^u_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{I}^u_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^u_\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Xi(\hbar,\varsigma)}{2} = \frac{\mathfrak{I}^u_\Upsilon\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Upsilon\Xi(\hbar,\varsigma)}{2}$. Thus $\mho^{\mathfrak{I}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \frac{\mho^{\mathfrak{I}}_\Upsilon\Xi(\flat,\varsigma)+\mho^{\mathfrak{I}}_\Upsilon\Xi(\hbar,\varsigma)}{2}$. Similarly, $\mho^{\mathfrak{I}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{I}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{I}}_\Upsilon\Xi(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{I}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{I}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{I}}_\Upsilon\Xi(\hbar,\varsigma)\}$.
Now, $\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\} = \max\{\mathfrak{F}_\Upsilon\Xi(\flat,\varsigma),\mathfrak{F}_\Upsilon\Xi(\hbar,\varsigma)\}$ and $1-\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq 1-\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq \max\{1-\mathfrak{T}_\Xi(\flat,\varsigma), 1-\mathfrak{T}_\Xi(\hbar,\varsigma)\} = \max\{1-\mathfrak{T}_\Upsilon\Xi(\flat,\varsigma), 1-\mathfrak{T}_\Upsilon\Xi(\hbar,\varsigma)\}$. Thus $\mho^{\mathfrak{F}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon\Xi(\hbar,\varsigma)\}$. Similarly,
$\mho^{\mathfrak{F}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon\Xi(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{F}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon\Xi(\hbar,\varsigma)\}$.
Let $V=\Xi(\Xi)$, Ξ is a q-NSCVSBS of Λ_1. Let $\Xi(\flat,\varsigma),\Xi(\hbar,\varsigma)\in\Lambda_2$, $\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma),\mathfrak{T}_\Xi(\hbar,\varsigma)\} = \min\{\mathfrak{T}_\Upsilon\Xi(\flat,\varsigma),\mathfrak{T}_\Upsilon\Xi(\hbar,\varsigma)\}$. Thus $\mho^{\mathfrak{T}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon\Xi(\hbar,\varsigma)\}$.
Similarly, $\mho^{\mathfrak{T}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon\Xi(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{T}}_\Upsilon\Xi(\hbar,\varsigma)\}$.
Now, $\mathfrak{I}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{I}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}_\Xi(\flat,\varsigma)+\mathfrak{I}_\Xi(\hbar,\varsigma)}{2} = \frac{\mathfrak{I}_\Upsilon\Xi(\flat,\varsigma)+\mathfrak{I}_\Upsilon\Xi(\hbar,\varsigma)}{2}$. Thus, $\mho^{\mathfrak{I}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \frac{\mho^{\mathfrak{I}}_\Upsilon\Xi(\flat,\varsigma)+\mho^{\mathfrak{I}}_\Upsilon\Xi(\hbar,\varsigma)}{2}$. Similarly, $\mho^{\mathfrak{I}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{I}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{I}}_\Upsilon\Xi(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{I}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)) \succeq \min\{\mho^{\mathfrak{I}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{I}}_\Upsilon\Xi(\hbar,\varsigma)\}$. Now, $\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma),\mathfrak{F}_\Xi(\hbar,\varsigma)\} = \max\{\mathfrak{F}_\Upsilon\Xi(\flat,\varsigma),\mathfrak{F}_\Upsilon\Xi(\hbar,\varsigma)\}$. Thus, $\mho^{\mathfrak{F}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon\Xi(\hbar,\varsigma)\}$.
Similarly, $\mho^{\mathfrak{F}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon\Xi(\hbar,\varsigma)\}$ and
$\mho^{\mathfrak{F}}_\Upsilon(\Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon\Xi(\flat,\varsigma),\mho^{\mathfrak{F}}_\Upsilon\Xi(\hbar,\varsigma)\}$. Hence, Υ is a q-NSCVSBS of Λ_2.

Theorem 3.14. *Every homomorphic pre-image of q-NSCVSBS of Λ_2 is a q-NSCVSBS of Λ_1.*

Proof. Let $\Xi : \Lambda_1 \to \Lambda_2$ and $\Xi(\flat\sharp_1\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma), \Xi(\flat\sharp_2\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)$ and $\Xi(\flat\sharp_3\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)$ for all $\flat,\hbar \in \Lambda_1$. Let $V = \Xi(\Xi)$, where Υ is any q-NSCVSBS of Λ_2. Let $\flat,\hbar \in \Lambda_1$. Now, $\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) = \mathfrak{T}_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = \mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \min\{\mathfrak{T}_\Upsilon\Xi(\flat,\varsigma), \mathfrak{T}_\Upsilon\Xi(\hbar,\varsigma)\} = \min\{\mathfrak{T}_\Xi(\flat,\varsigma), \mathfrak{T}_\Xi(\hbar,\varsigma)\}$. Thus $\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma), \mathfrak{T}_\Xi(\hbar,\varsigma)\}$ and $1-\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) = 1-\mathfrak{F}_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = 1-\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \min\{1-\mathfrak{F}_\Upsilon\Xi(\flat,\varsigma), 1-\mathfrak{F}_\Upsilon\Xi(\hbar,\varsigma)\} = \min\{1-\mathfrak{F}_\Xi(\flat,\varsigma), 1-\mathfrak{F}_\Xi(\hbar,\varsigma)\}$. Thus $1-\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \min\{1-\mathfrak{F}_\Xi(\flat,\varsigma), 1-\mathfrak{F}_\Xi(\hbar,\varsigma)\}$. Hence, $\mho^{\mathfrak{T}}_\Upsilon(\flat\sharp_1\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$.
Similarly, $\mho^{\mathfrak{T}}_\Upsilon(\flat\sharp_2\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_\Upsilon(\flat\sharp_3\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$.
Now, $\mathfrak{I}^l_\Xi(\flat\sharp_1\hbar,\varsigma) = \mathfrak{I}^l_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = \mathfrak{I}^l_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \frac{\mathfrak{I}^l_\Upsilon\Xi(\flat,\varsigma)+\mathfrak{I}^l_\Upsilon\Xi(\hbar,\varsigma)}{2} = \frac{\mathfrak{I}^l_\Xi(\flat,\varsigma)+\mathfrak{I}^l_\Xi(\hbar,\varsigma)}{2}$. Thus $\mathfrak{I}^l_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^l_\Xi(\flat,\varsigma)+\mathfrak{I}^l_\Xi(\hbar,\varsigma)}{2}$ and $\mathfrak{I}^u_\Xi(\flat\sharp_1\hbar,\varsigma) = \mathfrak{I}^u_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = \mathfrak{I}^u_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \frac{\mathfrak{I}^u_\Upsilon\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Upsilon\Xi(\hbar,\varsigma)}{2} = \frac{\mathfrak{I}^u_\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Xi(\hbar,\varsigma)}{2}$. Thus $\mathfrak{I}^u_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}^u_\Xi(\flat,\varsigma)+\mathfrak{I}^u_\Xi(\hbar,\varsigma)}{2}$. Hence, $\mho^{\mathfrak{I}}_\Upsilon(\flat\sharp_1\hbar,\varsigma) \succeq \frac{\mho^{\mathfrak{I}}_\Upsilon(\flat,\varsigma)+\mho^{\mathfrak{I}}_\Upsilon(\hbar,\varsigma)}{2}$.
Similarly, $\mho^{\mathfrak{I}}_\Upsilon(\flat\sharp_2\hbar,\varsigma) \succeq \frac{\mho^{\mathfrak{I}}_\Upsilon(\flat,\varsigma)+\mho^{\mathfrak{I}}_\Upsilon(\hbar,\varsigma)}{2}$ and $\mho^{\mathfrak{I}}_\Upsilon(\flat\sharp_3\hbar,\varsigma) \succeq \frac{\mho^{\mathfrak{I}}_\Upsilon(\flat,\varsigma)+\mho^{\mathfrak{I}}_\Upsilon(\hbar,\varsigma)}{2}$.
Now, $\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) = \mathfrak{F}_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = \mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \max\{\mathfrak{F}_\Upsilon\Xi(\flat,\varsigma), \mathfrak{F}_\Upsilon\Xi(\hbar,\varsigma)\} = \max\{\mathfrak{F}_\Xi(\flat,\varsigma), \mathfrak{F}_\Xi(\hbar,\varsigma)\}$. Thus $\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma), \mathfrak{F}_\Xi(\hbar,\varsigma)\}$ and $1-\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) = 1-\mathfrak{T}_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = 1-\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \max\{1-\mathfrak{T}_\Upsilon\Xi(\flat,\varsigma), 1-\mathfrak{T}_\Upsilon\Xi(\hbar,\varsigma)\} = \max\{1-\mathfrak{T}_\Xi(\flat,\varsigma), 1-\mathfrak{T}_\Xi(\hbar,\varsigma)\}$.
Thus $1-\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq \max\{1-\mathfrak{T}_\Xi(\flat,\varsigma), 1-\mathfrak{T}_\Xi(\hbar,\varsigma)\}$.
Hence, $\mho^{\mathfrak{F}}_\Upsilon(\flat\sharp_1\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$.
Similarly, $\mho^{\mathfrak{F}}_\Upsilon(\flat\sharp_2\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{F}}_\Upsilon(\flat\sharp_3\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$.
Now, $\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) = \mathfrak{T}_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = \mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \min\{\mathfrak{T}_\Upsilon\Xi(\flat,\varsigma), \mathfrak{T}_\Upsilon\Xi(\hbar,\varsigma)\} = \min\{\mathfrak{T}_\Xi(\flat,\varsigma), \mathfrak{T}_\Xi(\hbar,\varsigma)\}$. Thus $\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma), \mathfrak{T}_\Xi(\hbar,\varsigma)\}$. Hence, $\mho^{\mathfrak{T}}_\Upsilon(\flat\sharp_1\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{T}}_\Upsilon(\flat\sharp_2\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{T}}_\Upsilon(\flat\sharp_3\hbar,\varsigma) \succeq \min\{\mho^{\mathfrak{T}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{T}}_\Upsilon(\hbar,\varsigma)\}$. Now, $\mathfrak{I}_\Xi(\flat\sharp_1\hbar,\varsigma) = \mathfrak{I}_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = \mathfrak{I}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \frac{\mathfrak{I}_\Upsilon\Xi(\flat,\varsigma)+\mathfrak{I}_\Upsilon\Xi(\hbar,\varsigma)}{2} = \frac{\mathfrak{I}_\Xi(\flat,\varsigma)+\mathfrak{I}_\Xi(\hbar,\varsigma)}{2}$.
Thus $\mathfrak{I}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \frac{\mathfrak{I}_\Xi(\flat,\varsigma)+\mathfrak{I}_\Xi(\hbar,\varsigma)}{2}$. Hence, $\mho^{\mathfrak{I}}_\Upsilon(\flat\sharp_1\hbar,\varsigma) \succeq \frac{\mho^{\mathfrak{I}}_\Upsilon(\flat,\varsigma)+\mho^{\mathfrak{I}}_\Upsilon(\hbar,\varsigma)}{2}$. Similarly, $\mho^{\mathfrak{I}}_\Upsilon(\flat\sharp_2\hbar,\varsigma) \succeq \frac{\mho^{\mathfrak{I}}_\Upsilon(\flat,\varsigma)+\mho^{\mathfrak{I}}_\Upsilon(\hbar,\varsigma)}{2}$ and $\mho^{\mathfrak{I}}_\Upsilon(\flat\sharp_3\hbar,\varsigma) \succeq \frac{\mho^{\mathfrak{I}}_\Upsilon(\flat,\varsigma)+\mho^{\mathfrak{I}}_\Upsilon(\hbar,\varsigma)}{2}$.
Now, $\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) = \mathfrak{F}_\Upsilon(\Xi(\flat\sharp_1\hbar,\varsigma)) = \mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \max\{\mathfrak{F}_\Upsilon\Xi(\flat,\varsigma), \mathfrak{F}_\Upsilon\Xi(\hbar,\varsigma)\} = \max\{\mathfrak{F}_\Xi(\flat,\varsigma), \mathfrak{F}_\Xi(\hbar,\varsigma)\}$. Thus $\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq \max\{\mathfrak{F}_\Xi(\flat,\varsigma), \mathfrak{F}_\Xi(\hbar,\varsigma)\}$.
Hence, $\mho^{\mathfrak{F}}_\Upsilon(\flat\sharp_1\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$. Similarly, $\mho^{\mathfrak{F}}_\Upsilon(\flat\sharp_2\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$ and $\mho^{\mathfrak{F}}_\Upsilon(\flat\sharp_3\hbar,\varsigma) \preceq \max\{\mho^{\mathfrak{F}}_\Upsilon(\flat,\varsigma), \mho^{\mathfrak{F}}_\Upsilon(\hbar,\varsigma)\}$. Hence, Ξ is a q-NSCVSBS of Λ_1.

Theorem 3.15. *If $\Xi : \Lambda_1 \to \Lambda_2$ is a homomorphism, then $\Xi(\Xi_{(\varrho_1,\varrho_2,s)})$ is a level SBS of q-NSCVSBS Υ of Λ_2.*

Proof. Let $\Xi : \Lambda_1 \to \Lambda_2$ be a homomorphism and $\Xi(\flat\sharp_1\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma), \Xi(\flat\sharp_2\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)$ and $\Xi(\flat\sharp_3\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)$ for all $\flat,\hbar \in \Lambda_1$. Let $V = \Xi(\Xi)$, Ξ is a q-NSCVSBS of Λ_1. By Theorem 3.13, Υ is a q-NSCVSBS of Λ_2. Let $\Xi_{(\varrho_1,\varrho_2,s)}$ be any level SBS of Ξ. Suppose that $\flat,\hbar \in \Xi_{(\varrho_1,\varrho_2,s)}$. Then $\Xi(\flat\sharp_1\hbar,\varsigma), \Xi(\flat\sharp_2\hbar,\varsigma)$ and $\Xi(\flat\sharp_3\hbar,\varsigma) \in \Xi_{(\varrho_1,\varrho_2,s)}$. Now, $\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)) = \mathfrak{T}_\Xi(\flat,\varsigma) \succeq \varrho_1, \mathfrak{T}_\Upsilon(\Xi(\hbar,\varsigma)) = \mathfrak{T}_\Xi(\hbar,\varsigma) \succeq \varrho_1$. Thus $\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \varrho_1$ and $1-\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)) = 1-\mathfrak{F}_\Xi(\flat,\varsigma) \succeq s, 1-\mathfrak{F}_\Upsilon(\Xi(\hbar,\varsigma)) = 1-\mathfrak{F}_\Xi(\hbar,\varsigma) \succeq s$. Thus $1-\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle\Xi(\hbar,\varsigma)) \succeq 1-\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq s$. Now, $\mathfrak{I}^l_\Upsilon(\Xi(\flat,\varsigma)) = \mathfrak{I}^l_\Xi(\flat,\varsigma) \succeq \varrho_2, \mathfrak{I}^l_\Upsilon(\Xi(\hbar,\varsigma)) = \mathfrak{I}^l_\Xi(\hbar,\varsigma) \succeq \varrho_2$. Thus $\mathfrak{I}^l_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{I}^l_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \varrho_2$ and $\mathfrak{I}^u_\Upsilon(\Xi(\flat,\varsigma)) = \mathfrak{I}^u_\Xi(\flat,\varsigma) \succeq \varrho_2, \mathfrak{I}^u_\Upsilon(\Xi(\hbar,\varsigma)) = \mathfrak{I}^u_\Xi(\hbar,\varsigma) \succeq \varrho_2$. Thus $\mathfrak{I}^u_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{I}^u_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \varrho_2$. Now, $\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)) = \mathfrak{F}_\Xi(\flat,\varsigma) \preceq s, \mathfrak{F}_\Upsilon(\Xi(\hbar,\varsigma)) = \mathfrak{F}_\Xi(\hbar,\varsigma) \preceq s$. Thus $\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq s$ and $1-\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)) = 1-\mathfrak{T}_\Xi(\flat,\varsigma) \preceq \varrho_1, 1-\mathfrak{T}_\Upsilon(\Xi(\hbar,\varsigma)) = 1-\mathfrak{T}_\Xi(\hbar,\varsigma) \preceq \varrho_1$. Thus $1-\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle\Xi(\hbar,\varsigma)) \preceq 1-\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq \varrho_1$, for all $\Xi(\flat,\varsigma), \Xi(\hbar,\varsigma) \in \Lambda_2$.
Now, $\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)) = \mathfrak{T}_\Xi(\flat,\varsigma) \succeq \varrho_1, \mathfrak{T}_\Upsilon(\Xi(\hbar,\varsigma)) = \mathfrak{T}_\Xi(\hbar,\varsigma) \succeq \varrho_1$.
Thus $\mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \varrho_1$.
Now, $\mathfrak{I}_\Upsilon(\Xi(\flat,\varsigma)) = \mathfrak{I}_\Xi(\flat,\varsigma) \succeq \varrho_2, \mathfrak{I}_\Upsilon(\Xi(\hbar,\varsigma)) = \mathfrak{I}_\Xi(\hbar,\varsigma) \succeq \varrho_2$.
Thus $\mathfrak{I}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \succeq \mathfrak{I}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \varrho_2$. Now, $\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)) = \mathfrak{F}_\Xi(\flat,\varsigma) \preceq s, \mathfrak{F}_\Upsilon(\Xi(\hbar,\varsigma)) = \mathfrak{F}_\Xi(\hbar,\varsigma) \preceq s$. Thus $\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma)) \preceq \mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \preceq s$, for all $\Xi(\flat,\varsigma), \Xi(\hbar,\varsigma) \in \Lambda_2$. Similarly to prove other operations. Hence proved.

Theorem 3.16. *If $\Xi : \Lambda_1 \to \Lambda_2$ is any homomorphism, then $\Xi_{(\varrho_1,\varrho_2,s)}$ is a level SBS of q-NSCVSBS Ξ of Λ_1.*

Proof. Let $\Xi : \Lambda_1 \to \Lambda_2$ be a homomorphism and $\Xi(\flat\sharp_1\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_1\Xi(\hbar,\varsigma), \Xi(\flat\sharp_2\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_2\Xi(\hbar,\varsigma)$ and $\Xi(\flat\sharp_3\hbar,\varsigma) = \Xi(\flat,\varsigma)\triangle_3\Xi(\hbar,\varsigma)$ for all $\flat,\hbar \in \Lambda_1$. Let $V = \Xi(\Xi)$, Υ is a q-NSCVSBS of Λ_2. By Theorem 3.14, Ξ is a q-NSCVSBS of Λ_1. Let $\Xi(\Xi_{(\varrho_1,\varrho_2,s)})$ be a level SBS of Υ. Suppose that $\Xi(\flat,\varsigma), \Xi(\hbar,\varsigma) \in \Xi(\Xi_{(\varrho_1,\varrho_2,s)})$. Then $\Xi(\flat\sharp_1\hbar,\varsigma), \Xi(\flat\sharp_2\hbar,\varsigma)$ and $\Xi(\flat\sharp_3\hbar,\varsigma) \in \Xi(\Xi_{(\varrho_1,\varrho_2,s)})$. Now, $\mathfrak{T}_\Xi(\flat,\varsigma) = \mathfrak{T}_\Upsilon(\Xi(\flat,\varsigma)) \succeq \varrho_1, \mathfrak{T}_\Xi(\hbar,\varsigma) = \mathfrak{T}_\Upsilon(\Xi(\hbar,\varsigma)) \succeq \varrho_1$. Thus $\mathfrak{T}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \min\{\mathfrak{T}_\Xi(\flat,\varsigma), \mathfrak{T}_\Xi(\hbar,\varsigma)\} \succeq \varrho_1$ and $1-\mathfrak{F}_\Xi(\flat,\varsigma) = 1-\mathfrak{F}_\Upsilon(\Xi(\flat,\varsigma)) \succeq s, 1-\mathfrak{F}_\Xi(\hbar,\varsigma) = 1-\mathfrak{F}_\Upsilon(\Xi(\hbar,\varsigma)) \succeq s$. Thus $1-\mathfrak{F}_\Xi(\flat\sharp_1\hbar,\varsigma) \succeq \min\{1-\mathfrak{F}_\Xi(\flat,\varsigma), 1-$

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$\mathfrak{F}_{\Xi}(\hbar, \varsigma)\} \succeq s$. Now, $\mathfrak{I}^{l}_{\Xi}(\flat, \varsigma) = \mathfrak{I}^{l}_{\Upsilon}(\Xi(\flat, \varsigma)) \succeq \varrho_2, \mathfrak{I}^{l}_{\Xi}(\hbar, \varsigma) = \mathfrak{I}^{l}_{\Upsilon}(\Xi(\hbar, \varsigma)) \succeq \varrho_2$. Thus $\mathfrak{I}^{l}_{\Xi}(\flat\sharp_1\hbar, \varsigma) \succeq \frac{\mathfrak{I}^{l}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{l}_{\Xi}(\hbar,\varsigma)}{2} \succeq \varrho_2$ and $\mathfrak{I}^{u}_{\Xi}(\flat, \varsigma) = \mathfrak{I}^{u}_{\Upsilon}(\Xi(\flat, \varsigma)) \succeq \varrho_2, \mathfrak{I}^{u}_{\Xi}(\hbar, \varsigma) = \mathfrak{I}^{u}_{\Upsilon}(\Xi(\hbar, \varsigma)) \succeq \varrho_2$. Thus $\mathfrak{I}^{u}_{\Xi}(\flat\sharp_1\hbar, \varsigma) \succeq \frac{\mathfrak{I}^{u}_{\Xi}(\flat,\varsigma)+\mathfrak{I}^{u}_{\Xi}(\hbar,\varsigma)}{2} \succeq \varrho_2$. Now, $\mathfrak{F}_{\Xi}(\flat, \varsigma) = \mathfrak{F}_{\Upsilon}(\Xi(\flat, \varsigma)) \preceq s, \mathfrak{F}_{\Xi}(\hbar, \varsigma) = \mathfrak{F}_{\Upsilon}(\Xi(\hbar, \varsigma)) \preceq s$. Thus $\mathfrak{F}_{\Xi}(\flat\sharp_1\hbar, \varsigma) = \mathfrak{F}_{\Upsilon}(\Xi(\flat, \varsigma)\triangle_1\Xi(\hbar, \varsigma)) \preceq \max\{\mathfrak{F}_{\Xi}(\flat, \varsigma), \mathfrak{F}_{\Xi}(\hbar, \varsigma)\} \preceq s$ and $1 - \mathfrak{T}_{\Xi}(\flat, \varsigma) = 1 - \mathfrak{T}_{\Upsilon}(\Xi(\flat, \varsigma)) \preceq \varrho_1, 1 - \mathfrak{T}_{\Xi}(\hbar, \varsigma) = 1 - \mathfrak{T}_{\Upsilon}(\Xi(\hbar, \varsigma)) \preceq \varrho_1$. Thus $1 - \mathfrak{T}_{\Xi}(\flat\sharp_1\hbar, \varsigma) = 1 - \mathfrak{T}_{\Upsilon}(\Xi(\flat, \varsigma)\triangle_1\Xi(\hbar, \varsigma)) \preceq \max\{1 - \mathfrak{T}_{\Xi}(\flat, \varsigma), 1 - \mathfrak{T}_{\Xi}(\hbar, \varsigma)\} \preceq \varrho_1$, for all $\flat, \hbar \in \Lambda_1$.
Now, $\mathfrak{T}_{\Xi}(\flat, \varsigma) = \mathfrak{T}_{\Upsilon}(\Xi(\flat, \varsigma)) \succeq \varrho_1, \mathfrak{T}_{\Xi}(\hbar, \varsigma) = \mathfrak{T}_{\Upsilon}(\Xi(\hbar, \varsigma)) \succeq \varrho_1$.
Thus $\mathfrak{T}_{\Xi}(\flat\sharp_1\hbar, \varsigma) \succeq \min\{\mathfrak{T}_{\Xi}(\flat, \varsigma), \mathfrak{T}_{\Xi}(\hbar, \varsigma)\} \succeq \varrho_1$. Now, $\mathfrak{I}_{\Xi}(\flat, \varsigma) = \mathfrak{I}_{\Upsilon}(\Xi(\flat, \varsigma)) \succeq \varrho_2, \mathfrak{I}_{\Xi}(\hbar, \varsigma) = \mathfrak{I}_{\Upsilon}(\Xi(\hbar, \varsigma)) \succeq \varrho_2$. Thus $\mathfrak{I}_{\Xi}(\flat\sharp_1\hbar, \varsigma) \succeq \frac{\mathfrak{I}_{\Xi}(\flat,\varsigma)+\mathfrak{I}_{\Xi}(\hbar,\varsigma)}{2} \succeq \varrho_2$. Now, $\mathfrak{F}_{\Xi}(\flat, \varsigma) = \mathfrak{F}_{\Upsilon}(\Xi(\flat, \varsigma)) \preceq s, \mathfrak{F}_{\Xi}(\hbar, \varsigma) = \mathfrak{F}_{\Upsilon}(\Xi(\hbar, \varsigma)) \preceq s$.
Thus $\mathfrak{F}_{\Xi}(\flat\sharp_1\hbar, \varsigma) = \mathfrak{F}_{\Upsilon}(\Xi(\flat, \varsigma)\triangle_1\Xi(\hbar, \varsigma)) \preceq \max\{\mathfrak{F}_{\Xi}(\flat, \varsigma), \mathfrak{F}_{\Xi}(\hbar, \varsigma)\} \preceq s$, for all $\flat, \hbar \in \Lambda_1$. Similarly to prove other two operations. Hence proved.

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References

[1] Zadeh, L. A.; Fuzzy sets, Information and Control 1965, **8**, no. 3, 338-353.

[2] Gau, W. L.; Buehrer, D. J.; Vague sets, IEEE Transactions on Systems, Man and Cybernetics 1993, **23**, no. 2, 610-613.

[3] Atanassov, K; Intuitionistic fuzzy sets, Fuzzy Sets and Systems 1986, **20**, no. 1, 87-96.

[4] Smarandache, F.; A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability (fourth edition), Rehoboth American Research Press, 2008.

[5] Smarandache, F.; Neutrosophic set-a generalization of the intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics 2005, **24**, no. 3, 287-297.

[6] Ahsan, J.; Saifullah, K.; Khan, F. Fuzzy semirings, Fuzzy Sets and systems 1993, **60**, 309-320.

[7] SG Quek, H Garg, G Selvachandran, K Arulmozhi,VIKOR and TOPSIS framework with a truthful-distance measure for the (t, s)-regulated interval-valued neutrosophic soft set, Soft Computing, 1-27, 2023.

[8] K Arulmozhi, A Iampan, Multi criteria group decision making based on VIKOR and TOPSIS methods for Fermatean fuzzy soft with aggregation operators, ICIC Express Letters 16 (10), 1129–1138, 2022.

[9] M Palanikumar, K Arulmozhi, MCGDM based on TOPSIS and VIKOR using Pythagorean neutrosophic soft with aggregation operators, Neutrosophic Sets and Systems,, 538-555, 2022.

[10] Palanikumar, M., Arulmozhi, K, On intuitionistic fuzzy normal subbisemiring of bisemiring, Nonlinear Studies 2021, 28(3), 717–721.

[11] Sen, M.K.; Ghosh, S.; Ghosh, S. An introduction to bisemirings. Southeast Asian Bulletin of Mathematics, 2004, 28(3), 547-559.

[12] Biswas, R.; Vague groups, International Journal of Computational Cognition 2006, **4**, no. 2, 20-23.

[13] Golan, S. J; Semirings and their applications, Kluwer Academic Publishers, London, 1999.

[14] Hussian, F.; Hashism, R. M.; Khan, A.; Naeem, M.; Generalization of bisemirings, International Journal of Computer Science and Information Security 2016, **14**, no. 9, 275-289.

Received: October 24, 2023 Revised: February 22, 2024 Accepted: May 07, 2024

[15] Vandiver, H.S. Note on a simple type of algebra in which the cancellation law of addition does not hold. Bulletin of the American Mathematical Society, 1934, 40(12), 914-920.

[16] Glazek, K. A guide to the literature on semirings and their applications in mathematics and information sciences. Springer, Dordrecht, Netherlands, 2002.

[17] Jun, Y.B.; Song, S.Z. Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets. Scientiae Mathematicae Japonicae Online, 2008, 427-437.

[18] Lee, K.J. Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras. Bulletin of the Malaysian Mathematical Sciences Society, 2009, 32(3), 361-373.

[19] M Palanikumar, S Broumi, Square root Diophantine neutrosophic normal interval-valued sets and their aggregated operators in application to multiple attribute decision making, International Journal of Neutrosophic Science, 4, 2022.

[20] M Palanikumar, K Arulmozhi, Novel possibility Pythagorean interval valued fuzzy soft set method for a decision making, TWMS J. App. and Eng. Math. V.13, N.1, 2023, pp. 327-340.

[21] Muhiuddin, G.; Harizavi, H.; Jun, Y.B. Bipolar-valued fuzzy soft hyper BCK ideals in hyper BCK algebras. Discrete Mathematics, Algorithms and Applications, 2020, 12(2), 2050018.

[22] Zararsz, Z.; Riaz, M. Bipolar fuzzy metric spaces with application. Comp. Appl. Math. 2022, 41(49).

[23] Selvachandran, G.; Salleh, A.R. Algebraic Hyperstructures of vague soft sets associated with hyperrings and hyperideals. The Scientific World Journal, 2015, 2015, 780121.

[24] Jun, Y.B.; Kim, H.S.; Lee, K.J. Bipolar fuzzy translations in BCK/BCI-algebras. Journal of the Chungcheong Mathematical Society, 2009, 22(3), 399-408.

[25] Jun, Y.B.; Park, C.H. Filters of BCH-algebras based on bipolar-valued fuzzy sets. International Mathematical Forum, 2009, 4(13), 631-643.

[26] Hussain, F. Factor bisemirings. Italian Journal of Pure and Applied Mathematics, 2015, 34, 81-88.

[27] F. Al-Sharqi, A. Al-Quran and Z. M. Rodzi, Multi-Attribute Group Decision-Making Based on Aggregation Operator and Score Function of Bipolar Neutrosophic Hypersoft Environment, Neutrosophic Sets and Systems, 61(1), 465-492, 2023.

[28] A. Al-Quran, F. Al-Sharqi, A. U. Rahman and Z. M. Rodzi, The q-rung orthopair fuzzy-valued neutrosophic sets: Axiomatic properties, aggregation operators and applications. AIMS Mathematics, 9(2), 5038-5070, 2024.

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