On Some Novel Results About Weak Fuzzy Complex Integers

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Abstract

This paper is dedicated to studying for the first time the concept of weak fuzzy complex integers division and units, where we present a full classification of weak fuzzy complex integer units with necessary and sufficient conditions for division in the set of weak fuzzy complex integers. On the other hand, we provide an algorithm for solving weak fuzzy complex linear congruencies with many related examples that explain how algorithms work.

Keywords: weak fuzzy complex integer; weak fuzzy complex unit; weak fuzzy complex congruence.

1. Introduction and basics

The concept of weak fuzzy complex numbers has been defined as a new two-dimensional generalization of real numbers [1] in a similar algebraic building as split-complex numbers [12].

This new algebraic class of numbers has been used in the theory of vector spaces [11], and inner products [2].

The importance and interest of studying weak fuzzy complex integers come from the fact that the set of weak fuzzy complex integers is not an algebraic ring (it is not closed concerning multiplication) even though it contains the integer ring Z.

The study of the theory of weak fuzzy complex Diophantine equations began with solving Pythagoras Diophantine equations $x^2+y^2=z^2, and\ x^2+y^2+z^2=t^2$ which called weak fuzzy complex Pythagoras triples/quadruples [5-6]. Also, the solutions of weak fuzzy complex linear Diophantine equations in two variables were studied in [9].

Recently, many authors have used weak fuzzy complex numbers in classifying A-curves [7-8], matrix theory [3], and computer applications [4].

These previous studies have prompted us to close an important research gap about the foundations of number theory and number theoretical concepts in the set of weak fuzzy complex integers, such as weak fuzzy complex division, weak fuzzy complex ordering and units, and the solutions of weak fuzzy complex linear congruencies. We recall some basic definitions:

Definition: The set of *Weak Fuzzy Complex numbers* was defined as follows, where *'J'* is the Weak Fuzzy Complex operator ($J \notin \boldsymbol{R}$):

$\boldsymbol{F}_J = \{x_0 + x_1 J\ ;\ x_0\ , x_1 \in \boldsymbol{R}\ , J^2 = t \in\ ]0,1[\ \}$.

Definition:

Let $X = x_0 + x_1 J, Y = y_0 + y_1 J \in \boldsymbol{F}_J$, where $x_0\ , x_1, y_0, y_1 \in \boldsymbol{R}$

- Addition: $X + Y = (x_0 + y_0) + (x_1 + y_1)J$.
- Multiplication $X \times Y = (x_0 y_0 + x_1 y_1 t) + (x_0 y_1 + x_1 y_0)J$.
- The conjugate of X is: $\bar{X} = x_0 - x_1 J$

Doi: https://doi.org/10.54216/JNFS.080202
Received: November 27, 2023 Revised: January 09, 2024 Accepted: April 22, 2024

The set of weak fuzzy complex integers is defined as follows:
$\boldsymbol{Z}_w = \{x_0 + x_1 J\ ;\ x_0, x_1 \in \boldsymbol{Z}\ , J^2 = t \in]0,1[\ \}$. This set is not a ring that is because it is not closed for multiplication.

Main Concepts and Discussion

Definition:

Let $A = a_0 + a_1 J, B = b_0 + b_1 J \in \mathbb{Z}_w$, where $a_i, b_i \in \mathbb{Z}$, then $A|B$ if there exists $C = c_0 + c_1 J \in \mathbb{Z}_w$ such that $A.C = B$, where $J^2 = t \in]0,1[$.

Theorem:

Let $A, B \in \mathbb{Z}_w$, then $A|B$ if and only if:
There exists $c_0, c_1 \in \mathbb{Z}$ such that:

1) $(a_0 - a_1\sqrt{t})(c_0 - c_1\sqrt{t}) = b_0 - b_1\sqrt{t}$.
2) $(a_0 + a_1\sqrt{t})(c_0 + c_1\sqrt{t}) = b_0 + b_1\sqrt{t}$; $J^2 = t \in]0,1[$.

Proof:

Assume that $A|B$, then there exists $C = c_0 + c_1 J \in \mathbb{Z}_w$ such that A.C=B.
This is equivalent to: $(a_0 + a_1 J)(c_0 + c_1 J) = b_0 + b_1 J$, hence:

$$\begin{cases} a_0 c_0 + a_1 c_1 t = b_0 & (1) \\ a_0 c_1 + a_1 c_0 = b_1 & (2) \end{cases}$$

We multiply (2) by $\sqrt{t}$ and compute $(1) - \sqrt{t}(2)$, $(1) + \sqrt{t}(2)$, hence:

$$\begin{cases} (a_0 + a_1\sqrt{t})(c_0 + c_1\sqrt{t}) = b_0 + b_1\sqrt{t} \\ (a_0 - a_1\sqrt{t})(c_0 - c_1\sqrt{t}) = b_0 - b_1\sqrt{t} \end{cases}$$

For the converse, suppose that there exists $c_0, c_1 \in \mathbb{Z}$ such that

$$\begin{cases} (a_0 + a_1\sqrt{t})(c_0 + c_1\sqrt{t}) = b_0 + b_1\sqrt{t} & (1) \\ (a_0 - a_1\sqrt{t})(c_0 - c_1\sqrt{t}) = b_0 - b_1\sqrt{t} & (2) \end{cases}$$

Thus

$$\begin{cases} a_0 c_0 + a_1 c_1 t + \sqrt{t}(a_0 c_1 + a_1 c_0) = b_0 + b_1\sqrt{t} & (1) \\ a_0 c_0 + a_1 c_1 t - \sqrt{t}(a_0 c_1 - a_1 c_0) = b_0 - b_1\sqrt{t} & (2) \end{cases}$$

By adding (1)+(2), and subtracting $\frac{1}{2\sqrt{t}}[(1) - (2)]$, we get

$$\begin{cases} a_0 c_0 + a_1 c_1 t = b_0 & (1)' \\ a_0 c_1 + a_1 c_0 t = b_1 & (2)' \end{cases}$$

Hence: $(a_0 + a_1 J)(c_0 + c_1 J) = b_0 + b_1 J$, thus $A|B$.

Definition:

Let $A = a_0 + a_1 J, B = b_0 + b_1 J \in \mathbb{Z}_w$, we say that:

$A \geq B$ if and only if: $\begin{cases} a_0 - a_1\sqrt{t} \geq b_0 - b_1\sqrt{t} \\ a_0 + a_1\sqrt{t} \geq b_0 + b_1\sqrt{t} \end{cases}$

Theorem:

$(\mathbb{Z}_w, \geq)$ is partially ordered set.

Proof:

Consider $A = a_0 + a_1 J, B = b_0 + b_1 J, C = c_0 + c_1 J \in \mathbb{Z}_w$, then:
$A \geq A$, that is because:

$$\begin{cases} a_0 - a_1\sqrt{t} \geq a_0 - a_1\sqrt{t} \\ a_0 + a_1\sqrt{t} \geq a_0 + a_1\sqrt{t} \end{cases}$$

If $A \geq B$ and $B \geq A$, then: $\begin{cases} a_0 - a_1\sqrt{t} \geq b_0 - b_1\sqrt{t} \\ a_0 + a_1\sqrt{t} \geq b_0 + b_1\sqrt{t} \end{cases}$ and

$$\begin{cases} a_0 + a_1\sqrt{t} \leq b_0 + b_1\sqrt{t} \\ a_0 - a_1\sqrt{t} \leq b_0 - b_1\sqrt{t} \end{cases}$$

Hence:

$$\begin{cases} a_0 + a_1\sqrt{t} = b_0 + b_1\sqrt{t} \\ a_0 - a_1\sqrt{t} = b_0 - b_1\sqrt{t} \end{cases}$$

Thus $a_0 = b_0$, $a_1 = b_1$, so that A=B.

If $A \geq B$ and $B \geq C$, then: $\begin{cases} a_0 + a_1\sqrt{t} \geq b_0 + b_1\sqrt{t} \geq c_0 + c_1\sqrt{t} \\ a_0 - a_1\sqrt{t} \geq b_0 - b_1\sqrt{t} \geq c_0 - c_1\sqrt{t} \end{cases}$

Thus $A \geq C$

Doi: https://doi.org/10.54216/JNFS.080202
Received: November 27, 2023 Revised: January 09, 2024 Accepted: April 22, 2024

Remark:

Let $A = a_0 + a_1 J \in \mathbb{Z}_w$, then $A \geq 0$ if $\begin{cases} a_0 + a_1\sqrt{t} \geq 0 \\ a_0 - a_1\sqrt{t} \geq 0 \end{cases}$.

Also, A > 0 if $\begin{cases} a_0 + a_1\sqrt{t} > 0 \\ a_0 - a_1\sqrt{t} > 0 \end{cases}$

Example:

For $J^2 = t = \frac{1}{3}$, then A=3-J > 0, that is because:

$$\begin{cases} 3 + (-1)\sqrt{t} = 3 - \frac{1}{\sqrt{3}} > 0 \\ 3 - (-1)\sqrt{t} = 3 + \frac{1}{\sqrt{3}} > 0 \end{cases}$$

Definition:

Let $A = a + bJ \in \mathbb{Z}_w$, then A is a unit if and only if:
$A^{-1} = \frac{a}{a^2-b^2t} - \frac{b}{a^2-b^2t}J \in \mathbb{Z}_w$. The set of units in $\mathbb{Z}_w$ is denoted by $\cup\,(Z_w)$.

Theorem:

If $J^2 = t \notin Q$, then $\cup\,(Z_w) = \{1, -1\}$.

Proof:

Assume that $J^2 = t \notin Q$, then: $A = a + bJ \in Z_w$ is a unit if and only if:

$$\begin{cases} \dfrac{a}{a^2 - b^2 t} \in \mathbb{Z} \\ \dfrac{b}{a^2 - b^2 t} \in \mathbb{Z} \end{cases}$$

Since $a, b \in \mathbb{Z}$, then $a^2 - b^2 t \in \mathbb{Z}$, which is possible if and only if b=0, $a \in \{1, -1\}$, hence $\cup\,(Z_w) = \{1, -1\}$.

Theorem:

Let $A = a + bJ \in Z_w$, with $J^2 = t = \frac{m}{n} \in Q\, \cap]0,1[$, $gcd(m,n) = 1$, if A is a unit, then $n|b^2$.

Proof:

A is a unit implies that $a^2 - tb^2 = a^2 - \frac{m}{n}b^2 \in \mathbb{Z}$, so that $n|b^2$.

Remark:

If $J^2 = t = \frac{m}{n}$, where $n = {p_i}^{\alpha_i}$; p_i is a prime, then $n|b$ if $A = a + bJ$ is a unit.

Theorem:

Let $J^2 = t = \frac{m^2}{n^2} \in Q\, \cap]0,1[$; $\begin{cases} gcd(m,n) = 1, \\ m > 1 \\ n = {p_i}^{\alpha_i};\ p_i \ \ is\ a\ prime \end{cases}$

Then, $\cup\,(Z_w) = \{1, -1\}$.

Proof:

Assume that $X = a + bJ \in \cup\,(Z_w)$, hence:

$$\begin{cases} a^2 - tb^2 | a \\ a^2 - tb^2 | b \\ n | b \end{cases} \Rightarrow \begin{cases} (a - \frac{m}{n}b)(a + \frac{m}{n}b) \Big| a \\ (a - \frac{m}{n}b)(a + \frac{m}{n}b) \Big| b \end{cases} \Rightarrow \begin{cases} \left|a - \frac{m}{n}b\right| \leq |a| \\ \left|a + \frac{m}{n}b\right| \leq |b| \end{cases};\ a - \frac{m}{n}b \in \mathbb{Z}.$$

If a > b, b >0, then $\left|a + \frac{m}{n}b\right| = a + \frac{m}{n}b \leq a$

Thus b=0, $a \in \{-1,1\}$

If a < 0, b < 0, then $\left|a + \frac{m}{n}b\right| = -a - \frac{m}{n}b \leq -a$

Thus $-\frac{m}{n}b \leq 0$, so that b=0, $a \in \{-1,1\}$

If a > 0 , b < 0, then $\left|a - \frac{m}{n}b\right| = a - \frac{m}{n}b \leq a$,

Thus $-\frac{m}{n}b \leq 0$, so that b=0, $a \in \{-1,1\}$.

If a < 0 , b > 0, then $\left|a - \frac{m}{n}b\right| = -a + \frac{m}{n}b \leq -a$,

So that $\frac{m}{n}b \leq 0$, hence b=0, $a \in \{-1,1\}$.

If a = 0 , then $-\frac{m^2}{n^2}b^2|b$, hence: $\frac{m^2}{n^2}b^2|b$

So there exists $q \in \mathbb{Z}$ such that $q\frac{m^2}{n^2}b^2 = b;\ b \neq 0$.

Thus: $qm^2b = n^2$, so that $m^2|n^2$ which is a contradiction.

This means that $\cup\,(Z_w) = \{1, -1\}$.

Doi: https://doi.org/10.54216/JNFS.080202
Received: November 27, 2023 Revised: January 09, 2024 Accepted: April 22, 2024

Theorem:

For $J^2 = t = \frac{1}{n^2}$; $n = {p_i}^{\alpha_i}$; p_i is a prime, hence:
$\cup (Z_w) = \{1, -1, p_iJ, {p_i}^2J, \dots, {p_i}^{2\alpha_i}J, -p_iJ, -{p_i}^2J, \dots, -{p_i}^{2\alpha_i}J\}$.
Proof:
The only possible case is a=0, hence:
$X = bJ \in \cup (Z_w)$, and $qb = n^2, q \in \mathbb{Z}$, thus:
$b|n^2$, hence: $\cup (Z_w) = \{1, -1, p_iJ, \dots, {p_i}^{2\alpha_i}J, -p_iJ, \dots, -{p_i}^{2\alpha_i}J\}$

Theorem:

For $J^2 = t = \frac{m}{n} \in Q \cap]0,1[$; $gcd(m, n) = 1, m > 1$.
Then $\cup (Z_w) = \{1, -1\}$.
Proof:

We have $\begin{cases} a^2 - tb^2 | a \\ a^2 - tb^2 | b \end{cases} \Rightarrow \begin{cases} |(a - \sqrt{t}\, b)(a + \sqrt{t}\, b)| \leq |a| \\ |(a - \sqrt{t}\, b)(a + \sqrt{t}\, b)| \leq |b| \end{cases}$

If a > 0, b >0, then $|a + \sqrt{t}b| \leq |a| \Rightarrow a + \sqrt{t}b \leq a \Rightarrow \sqrt{t}b \leq 0$,
Thus $a \in \{-1,1\}, b = 0$
If a < 0, b < 0, then $|a + \sqrt{t}b| \leq |a| \Rightarrow -a - \sqrt{t}b \leq -a \Rightarrow -\sqrt{t}b \leq 0$,
Then b=0, $a \in \{-1,1\}$.
If a > 0, b < 0, then $|a - \sqrt{t}b| = a - \sqrt{t}b \leq a \Rightarrow -\sqrt{t}b \leq 0$
Then b=0, $a \in \{-1,1\}$.
If a < 0, b > 0, then $|a - \sqrt{t}b| \leq |a| \Rightarrow -a + \sqrt{t}b \leq -a \Rightarrow \sqrt{t}b \leq 0$,
Then b=0, $a \in \{-1,1\}$.
If a=0, then: $-tb^2|b$, so there exists $q \in \mathbb{Z}$; $-qtb^2 = b$, hence -q t b=1, thus:
$-q\frac{m}{n}b = 1 \Rightarrow -qmb = n$, thus $m|n$ which is a contradiction,
so that $\cup (Z_w) = \{1, -1\}$.

Theorem:

Let $J^2 = t = \frac{1}{n} \in Q \cap]0,1[$, then:
$\cup (Z_w) = \{1, -1, \mp\frac{n}{q}J;\ q|n\}$.

Proof:
If $a \neq 0$, we get $a = 1, a = -1, b = 0$ by a similar argument of the previous theorem.
For $a = 0$, then: $-tb^2|b$, so there exists $q \in \mathbb{Z}$ such that:
$-q\, t\, b^2 = b \Rightarrow -q\, t\, b = 1 \Rightarrow \frac{-q}{n}b = 1 \Rightarrow b = \frac{-n}{q}$; $q|n$.

Example:

For
$J^2 = t = \frac{1}{30}$,$\cup (Z_w) =$
$\{1, -1, 30J, -30J, 15J, -15J, 10J, -10J, 5J, -5J, 6J, -6J, 3J, -3J, 2J, -2J, J, -J\}$.

Example:

For $J^2 = t = \frac{2}{35}$, then $\cup (Z_w) = \{1, -1\}$.

Definition:

Let $A = a + bJ, B = c + dJ, C = m + nJ \in Z_w$, we say that:
$A \equiv B\ (mod\ C)$ if and only if: $C|B - A$, i.e. there exists $x, y \in Z$ such that:

$$\begin{cases} (m - n\sqrt{t})(x - y\sqrt{y}) = (a - c) - \sqrt{t}(b - d) \\ (m + n\sqrt{t})(x + y\sqrt{y}) = (a + c) + \sqrt{t}(b + d) \end{cases}$$

Where $J^2 = t \in]0,1[$.

Remark:

If $A, B, C, D \in Z_w$, and $A \equiv B\ (mod\ C)$, then:
1] $A + D \equiv B + D(mod\ C)$
2] $A - D \equiv B - D(mod\ C)$
3] $AD \equiv BD(mod\ C)$

Definition:

Let $A, B \in Z_w$, we say that A is invertible modulo B if there exists $C \in Z_w$ such that: $A.C \equiv 1(mod\ B)$.
We denote it By $A^{-1}(mod\ B) = C$.

Definition:

The linear congruence in one variable is defined as follows:

Doi: https://doi.org/10.54216/JNFS.080202
Received: November 27, 2023 Revised: January 09, 2024 Accepted: April 22, 2024

$AX \equiv B(mod\ C);\ A, B,\ C \in \mathbb{Z}_w$

Remark:

Finding the inverse of A modulo B is equivalent to solving the linear congruence $AX \equiv 1(mod\ B)$

Theorem:

Consider $\mathbb{Z}_w = \{a + bJ;\ a, b \in \mathbb{Z}, J^2 = t \in]0,1[$, and the linear congruence in one variable $AX \equiv B(mod\ C)$; where

$A = a_0 + a_1J, B = b_0 + b_1J, C = c_0 + c_1J, X = x_0 + x_1J \in \mathbb{Z}_w$, hence

For $t \notin Q$, each solution of $AX \equiv B(mod\ B)$ must be a solution of the following classical congruencies:

$$\begin{cases} a_0x_0 \equiv b_0(mod\ c_0) \\ a_1x_1 \equiv 0(mod\ c_1) \end{cases}$$

For $t = \frac{m}{n} \in Q;\ gcd(m, n) = 1$, each solution of $AX \equiv B(mod\ B)$ must be a solution of:

$$\begin{cases} na_0x_0 + ma_1x_1 \equiv nb_0(mod\ d) \\ \quad a_1x_0 + a_0x_1 \equiv b_1(mod\ d) \\ \qquad d = gcd\ (c_0, c_1) \end{cases}$$

Proof:

Assume that $X = x_0 + x_1J$ is a solution of $AX \equiv B(mod\ C)$, then there exists $M = m_0 + m_1J \in \mathbb{Z}_w$ such that:

$MC = AX - B$, thus: $AX = B + MC$, which is equivalent to

$$\begin{cases} (a_0 - a_1\sqrt{t})\ (x_0 - x_1\sqrt{t}) = b_0 - b_1\sqrt{t} + (m_0 - m_1\sqrt{t})(c_0 - c_1\sqrt{t}) \quad (1) \\ (a_0 + a_1\sqrt{t})\ (x_0 + x_1\sqrt{t}) = b_0 + b_1\sqrt{t} + (m_0 + m_1\sqrt{t})(c_0 + c_1\sqrt{t}) \quad (2) \end{cases}$$

Which is equivalent to:

$$\begin{cases} a_0x_0 + a_1x_1t + \sqrt{t}(-a_1x_0 - a_0x_1) = (b_0 + m_0c_0) + m_1c_1t + \sqrt{t}\ (-m_1c_0 - m_0c_1 - b_1) \quad (1) \\ \quad a_0x_0 + a_1x_1t + \sqrt{t}(a_1x_0 + a_0x_1) = (b_0 + m_0c_0) + m_1c_1t + \sqrt{t}\ (m_1c_0 + m_0c_1 + b_1) \quad (2) \end{cases}$$

By adding (1)' to (2)', and subtracting $\frac{1}{2\sqrt{t}}[(2)' - (1)']$, we get

$$\begin{cases} a_0x_0 + a_1x_1t = b_0 + m_0c_0 + m_1c_1t \quad (I) \\ a_1x_0 + a_0x_1 = m_1c_0 + m_0c_1 + b_1 \quad (II) \end{cases}$$

If $t \in \mathbb{R}$ and $t \notin Q$, then:

(I) Implies: $\begin{cases} a_0x_0 = b_0 + m_0c_0 \\ \quad a_1x_1 = m_1c_1 \end{cases}$

Hence x_0 is a solution of $a_0x_0 \equiv b_0(mod\ c_0)$, and x_1 is a solution of $a_1x_1 \equiv 0(mod\ c_1)$.

If $t = \frac{m}{n} \in Q;\ gcd\ (m, n) = 1$, then (I), (II) become:

$$\begin{cases} na_0x_0 + a_1mx_1 = nb_0 + nm_0c_0 + m_1mc_1 \quad (I)' \\ \quad a_1x_0 + a_0x_1 = m_1c_0 + m_0c_1 + b_1 \quad (II)'' \end{cases}$$

Consider $d = gcd(c_0, c_1)$, hence:

$$\begin{cases} d|nm_0c_0 + m_1mc_1 \\ \quad d|m_1c_0 + m_0c_1 \end{cases}$$

Thus:

$$\begin{cases} na_0x_0 + ma_1x_1 \equiv nb_0(mod\ d) \\ \quad a_1x_0 + a_0x_1 \equiv b_1(mod\ d) \end{cases}$$

So that, the proof holds.

Algorithms for solving weak fuzzy linear congruence in one variable:

1] $AX \equiv B(mod\ C)$ in the case of $J^2 = t \notin Q$:

Assume that (m_0, m_1) is a solution of the following linear Diophantine equation in two variables (m_0, m_1).

$({a_1}^2c_0 - a_0a_1c_1)m_0 + ({a_0}^2c_1 - a_0a_1c_0)m_1 = a_1(a_0b_1 - a_1b_0)(\varphi)$,

Hence: ${a_1}^2c_0m_0 - a_0a_1c_1m_0 + {a_0}^2c_1m_1 - a_0a_1c_0m_1 = a_1a_0b_1 - {a_1}^2b_0$.

Then: ${a_1}^2b_0 + {a_1}^2c_0m_0 + {a_0}^2c_1m_1 = a_0a_1c_0m_1 + a_0a_1m_0c_1 + a_0a_1b_1$,

So that, by dividing both sides with $a_0a_1 \neq 0$, we get:

$a_1\frac{b_0 + a_1c_0m_0}{a_0} + a_0\frac{c_1m_1}{a_1} = m_1c_0 + m_0c_1 + b_1$.

If $X = x_0 + x_1J$ is a solution of $AX \equiv B(mod\ C)$, then we get directly from the previous theorem:

$$\begin{cases} a_0x_0 = b_0 + c_0m_0 \\ \quad a_1x_1 = c_1m_1 \end{cases} \Leftrightarrow \begin{cases} x_0 = \dfrac{b_0 + c_0m_0}{a_0} \\ x_1 = \dfrac{c_1m_1}{a_1} \end{cases}$$

Thus: $a_1x_0 + a_0x_1 = m_1c_0 + m_0c_1 + b_1$, which is the second Diophantine equation generated by $AX \equiv B(mod\ C)$.

From this discussion, in the case of $t \notin Q$, we can generat the solutions of the weak fuzzy linear congruence $AX \equiv B(mod\ C)$.

By following these steps:

Step (1):

Solve the classical linear Diophantine equation (φ).

Step (2):

The Pair (m_0, m_1) generates a solution of the congruence.

If

$$\begin{cases} a_0 | b_0 + m_0c_0 \\ a_1 | m_1c_1 \end{cases}$$

And $x_0 = \frac{b_0+m_0c_0}{a_0}$, $x_1 = \frac{m_1c_1}{a_1}$

Example:

Consider the following weak fuzzy complex linear congruence:

$(5 + 3J)X \equiv 3 + 6J(mod\ 2 + 3J)$, we have: for $J^2 = t = \frac{1}{e}$

$\begin{cases} a_0 = 5 \\ a_1 = 3 \end{cases}$, $\begin{cases} b_0 = 3 \\ b_1 = 6 \end{cases}$, $\begin{cases} c_0 = 2 \\ c_1 = 3 \end{cases}$

The Diophantine equation (Q) is: $-27m_0 + 45m_1 = 63 \Leftrightarrow -3m_0 + 5m_1 = 7$.

For the solution $(m_0, m_1) = (1,2)$, we get: $x_0 = \frac{3+2}{5} = 1$, $x_1 = \frac{6}{3} = 2$, hence X=1+2J.

For the solution $(m_0, m_1) = (6,5)$, we get: $x_0 = \frac{3+12}{5} = 3$, $x_1 = \frac{15}{3} = 5$, hence X=3+5J.

For $(m_0, m_1) = (11,8)$, we get: $x_0 = \frac{3+22}{5} = 5$, $x_1 = \frac{24}{3} = 8$, hence X=5+8J.

2] $AX \equiv B(mod\ C)$ in the case of $t = \frac{m}{n} \in Q \cap]0,1[$; $gcd\ (m, n) = 1$.

It is clear that we must solve the linear system:

$$\begin{cases} na_0x_0 + ma_1x_1 \equiv nb_0(mod\ d) \\ a_1x_0 + a_0x_1 \equiv b_1(mod\ d) \end{cases} \qquad d = gcd(c_0, c_1)$$

If $gcd(n\,a_0{}^2 - m\,a_1{}^2, d) = 1$, it has a unique solution modulo (d) the solution is:

$\begin{pmatrix} x_0{}' \\ x_1{}' \end{pmatrix} = (n\,a_0{}^2 - m\,a_1{}^2)^{-1}(mod\ d).\begin{pmatrix} a_0 & -ma_1 \\ -a_1 & na_0 \end{pmatrix}\begin{pmatrix} nb_0 \\ b_1 \end{pmatrix}$.

So that $x_0 = x_0{}' + q_1d$, $x_1 = x_1{}' + q_2d$.

We substitute $\begin{cases} x_0 = x_0{}' + q_1d \\ x_1 = x_1{}' + q_2d \end{cases}$ in the linear system (I)', (II)'

Then we accept the values of (q_1, q_2) that generate only integer values of (m_0, m_1).

Example:

For $J^2 = t = \frac{1}{2}$, m=1, n=2. Consider:

$(3 + 4J)X \equiv (2 + J)(mod\ 9 + 6J)$, we have:

$\begin{cases} a_0 = 3 \\ a_1 = 4 \end{cases}$, $\begin{cases} b_0 = 2 \\ b_1 = 1 \end{cases}$, $\begin{cases} c_0 = 9 \\ c_1 = 6 \end{cases}$, $d = gcd(9,6) = 3$.

$$\begin{cases} na_0x_0 + ma_1x_1 \equiv nb_0(mod\ d) \\ a_1x_0 + a_0x_1 \equiv b_1(mod\ d) \end{cases} \Leftrightarrow \begin{cases} 6x_0 + 4x_1 \equiv 4(mod\ 3) \\ 4x_0 + 3x_1 \equiv 1(mod\ 3) \end{cases}$$

$gcd(n\,a_0{}^2 - m\,a_1{}^2, d) = gcd\ (2,3) = 1$.

$(n\,a_0{}^2 - m\,a_1{}^2)^{-1}(mod\ 3) = (2)^{-1}(mod\ 3) = 2$.

$\begin{pmatrix} x_0{}' \\ x_1{}' \end{pmatrix} = 2.\begin{pmatrix} 3 & -4 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 8 \\ -10 \end{pmatrix} = \begin{pmatrix} 16 \\ -20 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(mod\ 3)$

Thus: $\begin{cases} x_0 = 1 + 3q_1 \\ x_1 = 1 + 3q_2 \end{cases}$; $q_1, q_2 \in \mathbb{Z}$.

Now, we use the linear system (I)', (II)':

$$\begin{cases} na_0x_0 + a_1mx_1 = nb_0 + nm_0c_0 + m_1mc_1 & (I)' \\ a_1x_0 + a_0x_1 = m_1c_0 + m_0c_1 + b_1 & (II)' \end{cases}$$

Hence:

$$\begin{cases} 18m_0 + 6m_1 = 6(1 + 3q_1) + 4(1 + 3q_2) - 4 & (I)' \\ 6m_0 + 9m_1 = 4(1 + 3q_1) + 3(1 + 3q_2) - 1 & (II)' \end{cases}$$

Which is equivalent to:

Doi: https://doi.org/10.54216/JNFS.080202
Received: November 27, 2023 Revised: January 09, 2024 Accepted: April 22, 2024

$$\begin{cases} 18m_0 + 6m_1 = 6 + 18q_1 + 12q_2 & (I)' \\ 6m_0 + 9m_1 = 6 + 12q_1 + 9q_2 & (II)' \end{cases}$$

Hence.

$$\begin{cases} 3m_0 + m_1 = 1 + 3q_1 + 2q_2 & (I)' \\ 2m_0 + 3m_1 = 2 + 4q_1 + 3q_2 & (II)' \end{cases}$$

The solution of (I)', (II)' Is:

$$\begin{cases} m_0 = \frac{1}{7}(1 + 5q_1 + 3q_2) \\ m_1 = \frac{1}{7}(4 + 6q_1 + 5q_2) \end{cases}$$

The accepted values of (q_1, q_2) are the values that have the following property:

$$\begin{cases} 1 + 5q_1 + 3q_2 \equiv 0(mod\ 7) \\ 4 + 6q_1 + 5q_2 \equiv 0(mod\ 7) \end{cases}$$

This implies: $\begin{cases} 5q_1 + 3q_2 \equiv 6(mod\ 7) & (1) \\ 6q_1 + 5q_2 \equiv 3(mod\ 7) & (2) \end{cases}$

$\Delta = \begin{vmatrix} 5 & 3 \\ 6 & 5 \end{vmatrix} = 25 - 18 = 7 \equiv 0(mod\ 7)$.

This means that the congruence (1) equivalents (2).

It is easy to see that (2) holds from (1) by multiplying with 4, thus: $5q_1 + 3q_2 \equiv 6(mod\ 7)$

For example, $q_1 = 0, q_2 = 2$ is a solution, thus:

$x_0 = 1, x_1 = 7, X = x_0 + x_1 J = 1 + 7J$ is a solution of $(3 + 4J)X \equiv (2 + J)(mod\ 9 + 6J)$

We can see that $m_0 + m_1 J = 1 + 2J$, and:

(3+4J) (1+7J) – (2+J) = (9+6J) (1+2J).

Also, $q_1 = 1, q_2 = 5$ is another solution, thus

$x_0 = 1 + 3 = 4, x_1 = 1 + 15 = 16, m_0 = \frac{1}{7}(1 + 5 + 15) = 3, m_1 = \frac{1}{7}(4 + 6 + 25) = 5$, hence $X = x_0 + x_1 J = 4 + 16J$ is a solution of $(3 + 4J)X \equiv (2 + J)(mod\ 9 + 6J)$.

We can see easily that: (3+4J) (4+16J) – (2+J)=(9+6J)(3+5J)

Remark:

if $gcd(n\, {a_0}^2 - m\, {a_1}^2, d) \neq 1$, then we must examine the solvability of

$$\begin{cases} na_0x_0 + ma_1x_1 \equiv nb_0(mod\ d) \\ a_1x_0 + a_0x_1 \equiv nb_1(mod\ d) \end{cases}$$

And if it has solutions, we can continue the other steps.

Example:

For $J^2 = t = \frac{1}{2}, (4 + 2J)X \equiv (3 + J)(mod\ 2 + 6J)$

$$\begin{cases} a_0 = 4 \\ a_1 = 2 \end{cases}, \quad \begin{cases} b_0 = 3 \\ b_1 = 1 \end{cases}, \begin{cases} c_0 = 2 \\ c_1 = 6 \end{cases}, \quad d = gcd(c_0, c_1) = 2, \begin{cases} m = 1 \\ n = 2 \end{cases}$$

$$n\, {a_0}^2 - m\, {a_1}^2 = 32 - 4 = 28, gcd\ (28{,}2) = 2 \neq 1$$

$$\begin{cases} na_0x_0 + ma_1x_1 \equiv nb_0(mod\ d) \\ a_1x_0 + a_0x_1 \equiv b_1(mod\ d) \end{cases} \Leftrightarrow \begin{cases} 8x_0 + 2x_1 \equiv 6(mod\ 2) & (1) \\ 2x_0 + 4x_1 \equiv 1(mod\ 2) & (2) \end{cases}$$

It is clear that congruence (2) has no solutions, thus

$(4 + 2J)X \equiv (3 + J)\ (mod 2 + 6J)$ is not solvable.

Conclusion and future directions

In this paper, we have studied for the first time the concept of weak fuzzy complex integers division and units, where we presented a full classification of weak fuzzy complex integer units with necessary and sufficient conditions for division in the set of weak fuzzy complex integers. On the other hand, we provided an algorithm for solving weak fuzzy complex linear congruencies with many related examples that explain how algorithms work.

As a future research direction, we aim to study how weak fuzzy complex integers can be used in generalizing some public-key crypto-algorithms in a similar way to other generalized systems of integers used in [10].

References

[1] Hatip, A., "An Introduction To Weak Fuzzy Complex Numbers ", Galoitica Journal Of Mathematical Structures and Applications, Doi: https://doi.org/10.54216/GJMSA.030101, Vol. 03, No. 01, PP. 08-13, 2023.

[2] Ali, R., "On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

[3] Yaser Ahmad Alhasan, Abuobida Mohammed A. Alfahal, Raja Abdullah Abdulfatah Giorgio Nordo, Musaddak M. Abdul Zahra, "On Some Novel Results About Weak Fuzzy Complex Matrices" ,DOI: https://doi.org/10.54216/IJNS.210112 , 2023.

[4] Razouk,L., Mahmoud, S., Ali, M., "A Computer Program For The System Of Weak Fuzzy Complex Numbers And Their Arithmetic Operations Using Python", Galoitica Journal Of Mathematical Structures and Applications, Doi: https://doi.org/10.54216/GJMSA.080104 , Vol. 08, No. 01, PP. 45-51, 2023.

[5] Alfahal, A., Abobala, M., Alhasan, Y., & Abdulfatah, R. (2023). Generating Weak Fuzzy Complex and Anti Weak Fuzzy Complex Integer Solutions for Pythagoras Diophantine Equation $X^2 + Y^2 = Z^2$. *International Journal of Neutrosophic Science (IJNS*), Vol. 22, No. 01, PP. 08-14. https://doi.org/10.54216/IJNS.220201

[6] Galarza, F., Flores, M., Rivero, D., & Abobala, M. (2023). On Weak Fuzzy Complex Pythagoras Quadruples. *International Journal of Neutrosophic Science (IJNS*), Vol. 22, No. 02, PP. 108-113. https://doi.org/10.54216/IJNS.220209

[7] Alhasan, Y., Xu, L., Abdulfatah, R., & Alfahal, A. (2023). The Geometrical Characterization for The Solutions of a Vectorial Equation By Using Weak Fuzzy Complex Numbers and Other Generalizations Of Real Numbers. *International Journal of Neutrosophic Science (IJNS*), Vol. 21, No. 04, PP. 155-159. https://doi.org/10.54216/IJNS.210415.

[8] Abdallah Shihadeh, Wael Mahmoud M. Salameh, Malik Bataineh, Hassan Al-Tarawneh, Ayman Alahmade, Abdallah Al-Husban. (2024). On the Geometry of Weak Fuzzy Complex Numbers and Applications to the Classification of Some A-Curves. International Journal of Neutrosophic Science, 23 (4), 369-375 (**Doi** : https://doi.org/10.54216/IJNS.230428).

[9] Mayada Abualhomos, Wael Mahmoud M. Salameh, Malik Bataineh, Mowafaq Omar Al-Qadri, Ayman Alahmade, Abdallah Al-Husban. (2024). An Effective Algorithm for Solving Weak Fuzzy Complex Diophantine Equations in Two Variables. International Journal of Neutrosophic Science, 23 (4), 386-394 (**Doi** : https://doi.org/10.54216/IJNS.230431).

[10] Abobala, M., Sankari, H., & Zeina, M. (2024). On novel security systems based on the 2-cyclic refined integers and the foundations of 2-cyclic refined number theory. *Journal of Fuzzy Extension and Applications*, *5*(1), 69-85. doi: 10.22105/jfea.2024.423818.1321.

[11] Ahmed Hatip. (2023). On The Fuzzy Weak Complex Vector Spaces. Galoitica: Journal of Mathematical Structures and Applications, 3 (1), 29-32 (**Doi** : https://doi.org/10.54216/GJMSA.030105).

[12] Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.

Doi: https://doi.org/10.54216/JNFS.080202
Received: November 27, 2023 Revised: January 09, 2024 Accepted: April 22, 2024