On the Development of Fuzzy Estimators for Life Time Distributions based on Censored Fuzzy Life Times

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Abstract

Lifetime analyses comprise the techniques dealing with observations obtained from the occurrence of a specified event(s). In most of the situations dealing with lifetime observations, some units are recorded as censored observations. Dealing with censored observations makes these techniques unique. Countless standard statistical tools are available for inference based on censored lifetime observations. These classical techniques consider lifetime observations as precise numbers and ignore the uncertainty of single observations. Whereas in practical applications it is not possible to measure life times as precise numbers, they are always more or less nonprecise. The imprecision in measurements can be covered by neutrosophic set. Fuzzy estimators for life time distributions potentially use neutrosophic system to model and analyze the inherent uncertainties and neutalities present in the data and the parameter estimates. This study aimed to obtain estimators for the Weibull parameters and two exponential parameters based on the up-to-date fuzzy number approach, a special case for neutrosophic set. The suggested estimators incorporate fuzziness in addition to random variation, which makes these estimators more realistic. The same techniques need to be extended to fuzzy and neutrosophic sets.

Keywords: Characterizing function; Fuzzy numbers; Life time; Non-precise data; Neutrosophic sets

1 introduction

Statistics is a science of decision-making, mainly based on numerical observations. The validity of statistical inference are based on the quality of the observed data. In daily life situations, almost all of the variables of interest are of continuous, and the obtained measurements are recorded in the form of precise numbers, vectors or functions.

The classical statistical techniques are designed in such a way that deal with precise numbers of the random variables, and only consider variation among the precise observations. However, the real-world data entails two types of uncertainties: one is modelled through stochastic models, and the other is fuzziness, which is usually ignored but should be considered and modeled through *fuzzy numbers* [1].

For all the situations, it is not possible for the sampling unit to be precisely classified or presented in the form of precise numbers but rather involve some uncertainty or imprecision. The most common examples are starfish, bacteria, light intensity, water level of water because of fluctuation, life time, etc. Such situations are commonly used in daily life language, such as low or high blood pressure, cold and very cold, tall or short, etc. The same was furthermore extended to neutrosophic set by introducing further indeterminacy.

https://doi.org/10.54216/IJNS.240217
Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

In the same way, measurements of continuous variables such as the height of an object, life time of a unit, speed of a vehicle, etc., cannot be measured in the form of precise numbers, but they are always more or less fuzzy [1].

Statistics are now widely used in almost every field of life. In medical and engineering fields, the general response of interest is the waiting time, from entry to the study until a specified event occurs, called *life time*. The most common specified events are death in medicine and failure in the engineering field. In [2], it has been presented that life times are no more precise numbers but fuzzy.

In the said branches of statistics, it is not very convenient to wait until the failure of all units. It is obvious to terminate the experiment due to time or other constraints, and the units that still stay at the termination time are called censored observations.

For type I censoring, the termination time of an experiment is fixed in advance, and the surviving items at the time of termination are right censored.

For type II censoring, the experiment will be terminated on the r^{th} failure/death, and the rest of the items that stay are censored [3].

The Weibull distribution is one of the most popular distributions to model time to event data. Its density is defined by:

$$f(t|\,\tau,\beta) = \frac{\beta}{\tau}\left(\frac{t}{\tau}\right)^{\beta-1} exp\left\{-\left(\frac{t}{\tau}\right)^{\beta}\right\} \quad \forall\, t>0,\;\; \tau>0,\;\; \beta>0$$

$\tau:\ Scale\ Parameter;\ also\ called\ characteristic\ life\ time$

$\beta:\ Shape\ Parameter.$

In the same manner two parameters exponential distribution is one of the most popular distributions, specially the memory-less property makes it very unique and applicable in practical life time analysis. Its density is defined by:

$$f(t|\,,\lambda) = \lambda e^{-\lambda(t-\mu)} \quad \forall\, t \geq \mu \geq 0, \quad \lambda > 0$$

Several estimators for the parameters of the Weibull and two parameter exponeyail distribution are suggested by different researchers like parameter estimation of Weibull distribution [4], MLE of Weibull parameters for complete and censored observations [5], estimation for the two parameter Weibull distribution for type I censoring[6], estimation of Weibull parameters with type II highly censored data [7], life testing [8], estimation of the parameters of two parameter exponential distribution from censored samples [9], applications of exponential models to problems in cancer research [10]

All these estimators are for data in the form of precise numbers and ignore the fuzziness of the individual life time observations. Life time is a continuous variable, and in [2] it has been shown that life times are no more precise numbers, and should be modeled through fuzzy number approach.

In statistical inference maximum likelihood estimation for the unknown parameters has significant importance. It is based on the objective to maximize the likelihood function in the parametric space based on observed data. It has a lot of applications in almost every field especially estimation technique for censored data. Furthermore, the properties of reparameterization and its approximate normal distributions makes it more applicable and unique.

Therefore, this study is aimed to make a generalized maximum likelihood estimation of Weibull and exponential parameters using censored fuzzy life time data.

Fuzzy Set Theory

According to [1] some concepts of fuzzy set theory are explained below.
Fuzzy Numbers

Let t^* is denoting fuzzy number, characterized by function of one real variable called *characterizing function* $\xi(\cdot)$ with the following conditions:

https://doi.org/10.54216/IJNS.240217

Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

1. $\xi : \mathbb{R} \to [0\,,\,1]$.
2. The so-called δ-cut obtained by $C_\delta(t^*) := \{t \in \mathbb{R} : \xi\,(t) \geq \delta\} \quad \forall \delta \in (0\,,\,1]$, is a finite union of compact intervals i.e.
$$C_\delta(t^*) = \bigcup_{j=1}^{k_\delta} [c_{\delta,j}\,,\, d_{\delta,j}] \neq \emptyset$$
.
3. $supp[\xi(\cdot)] := [t \in \mathbb{R} : \xi\,(t) > 0\,] \subseteq [a\,,\,b]$ is bounded support of $\xi(\cdot)$.

The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$.
If all δ-cuts of a fuzzy number are non-empty closed bounded intervals, the corresponding fuzzy number is called *fuzzy interval*.

Remark: The family $(C_\delta(t^*);\ \delta \in (0\,,\,1])$ is nested, i.e. for $\delta_1 < \delta_2$ we have $C_{\delta_1}(t^*) \supseteq C_{\delta_2}(t^*)$.

Lemma: Denoting by $\mathbb{1}_A\,(\cdot)$ the indicator function of a set $A \subseteq \mathbb{R}$, for any characterizing function $\xi(\cdot)$ of a fuzzy number the following is valid:

$$\xi\,(t) = \max\left\{\delta{\cdot}\mathbb{1}_{C_\delta(t^*)}(t) :\ \delta \in [0\,,\,1]\right\} \qquad \forall t \in \mathbb{R}.$$

Remark: It should be noted that not all nested families $(A_\delta\,;\ \delta \in (0\,,\,1])$ of finite unions of compact intervals are the δ-cuts of a fuzzy number. But the following construction lemma holds:

Construction Lemma: Let $(A_\delta\,;\ \delta \in (0\,,\,1])$ with $A_\delta = \bigcup_{j=1}^{k_\delta} [c_{\delta,j}\,,\, d_{\delta,j}]$ be a nested family of non-empty subsets of $\mathbb{R}$. Then the characterizing function of the generated fuzzy number is given by

$$\xi\,(t) = \sup\left\{\delta{\cdot}\mathbb{1}_{A_\delta}(t)\ :\ \delta \in [0\,,\,1]\right\} \qquad \forall t \in \mathbb{R}.$$

[11].

Realizing the importance of fuzziness some work has been done like, [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24] in addition to these neutrosophic sets were applied in many fields like, [25], [26], [27], but yet in most of the situations it is ignored. the same can be extended to other field like [28], [29].

2 Fuzzy Maximum Likelihood Estimators of Weibull distribution

Let n is denoting total number of observations, for type II censoring, r is denoting the total number of failures. The ordered life times are denoted by $(t_{1:n},\, t_{2:n},\, ...,\, t_{r:n})$, then the log-likelihood function for the Weibull distribution is denoted by $l\,(\tau, \beta;\ t_{1:n},\, t_{2:n},\, ...,\, t_{r:n})$, and can be written as:

$$l = const + r\ln(\beta) - r\beta\ln(\tau) + (\beta - 1)\sum_{i=1}^{r}\ln(t_{i:n}) - \frac{1}{\tau^\beta}\left\{\sum_{i=1}^{r}\ln(t_{i:n}) + (n-r)t_{r:n}^{\beta}\right\}$$

For precise life time data, the maximum likelihood estimators $\hat{\tau}$ and $\hat{\beta}$ of the parameters are the solution of the following equations:

$$\hat{\tau} = \left[\frac{1}{r}\left\{\sum_{i=1}^{r} t_{i:n}^{\hat{\beta}} + (n-r)t_{r:n}^{\hat{\beta}}\right\}\right]^{\frac{1}{\hat{\beta}}} \tag{1}$$

and

https://doi.org/10.54216/IJNS.240217

Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

$$\frac{1}{\hat{\beta}} = \left[\frac{\sum_{i=1}^{r} t_{i:n}^{\hat{\beta}} \ln(t_{i:n}) + (n-r) t_{r:n}^{\hat{\beta}} \ln(t_{r:n})}{\sum_{i=1}^{r} t_{i:n}^{\hat{\beta}} + (n-r) t_{r:n}^{\hat{\beta}}} - \frac{1}{r} \sum_{i=1}^{r} \ln(t_{i:n}) \right] \tag{2}$$

The right hand side of equation (2) is a monotone increasing function in β, and the left hand side is a decreasing function. It follows that where they intersect will be the value of $\hat{\beta}$. Then putting that value in equation (1) to obtain the estimate $\hat{\tau}$. For type I censoring $t_{r:n}$ should be replaced by the pre-specified censoring time [5].

The given log-likelihood function, and maximum likelihood estimators need to be generalized for fuzzy observations t_1^*, t_2^*, ..., t_n^*, i.e. the generalized fuzzy estimators are defined through the extension principle. They are denoted by $\hat{\tau}^*$, $\hat{\beta}^*$ and are constructed in the following way:

The generating family of intervals $(A_\delta; \delta \in (0, 1])$ for the fuzzy number $\hat{\tau}^*$ are obtained by

$$A_\delta\left(\hat{\tau^*}\right) = \left[\min_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \hat{\tau} \, , \max_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \hat{\tau} \right],$$

where $\hat{\tau}$ is the solution of the MLE equations above.

In a similar way the generating family of intervals $(B_\delta; \delta \in (0, 1])$ for the fuzzy estimate $\hat{\beta}^*$ can be obtained:

$$B_\delta\left(\hat{\beta}^*\right) = \left[\min_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \hat{\beta} \, , \max_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \hat{\beta} \right]$$

with $(\underline{t} = t_{1:n}, t_{2:n}, ..., t_{r:n})$.

The characterizing functions $\varphi(\cdot)$ of $\hat{\tau}^*$ and $\psi(\cdot)$ of $\hat{\beta}^*$ are given by the construction lemma in the following way:

$$\varphi(\tau) = \sup\left\{\delta \cdot \mathbb{1}_{A_\delta}(\tau) \, : \, \delta \in [0, 1]\right\} \quad \forall\, \tau \in [0, \infty)$$

$$\psi(\beta) = \sup\left\{\delta \cdot \mathbb{1}_{B_\delta}(\beta) \, : \, \delta \in [0, 1]\right\} \quad \forall\, \beta \in [0, \infty)$$

The above mathematical calculations can be done approximately by the following algorithm:

1. The values for δ are taken from 0 to 1 with an increment $\triangle \in (0, 1)$.
2. For a given value of δ all δ-cuts of the fuzzy observations are determined.
3. Taking values from the δ-cuts to get hypothetical classical samples.
4. From these hypothetical classical samples at a given level δ, calculate the classical estimates.
5. In order to construct the generalized (fuzzy) estimators take minimum and maximum values from these estimates and consider it as the end points of the family $(A_\delta$ and B_δ ; $\delta \in (0, 1])$ of generating intervals A_δ and B_δ of the characterizing functions of the fuzzy estimators at a given level of δ.
6. Steps 2-5 are performed for each estimator for $\delta = 0\,(\triangle)\,1$.
7. From all these generating intervals A_δ and B_δ obtained for each δ, $\left(\text{i.e. } \delta = 0\,(\triangle)\,1\right)$ through the above mentioned Construction Lemma the characterizing functions of the fuzzy estimates of the parameters are obtained approximately.

Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

Using the above mentioned algorithm the characterizing functions of the fuzzy parameter estimators of the Weibull distribution under various conditions are constructed.

In figure 1, figure 4, and figure 7 the characterizing functions of failure times are given. Since the censored observations are not observed completely therefore their characterizing functions are ignored. As an example the characterizing functions of the fuzzy samples and fuzzy estimators are explained in figure 1 to figure 9.

In figure 1 a type I censoring fuzzy sample of 8 observations consisting of 7 failures and 1 censored observation is considered. The characterizing functions of the fuzzy parameter estimates of the Weibull distribution are given in figure 2 and figure 3.

Figure 1: Sample of Type I censoring with fuzzy censoring time

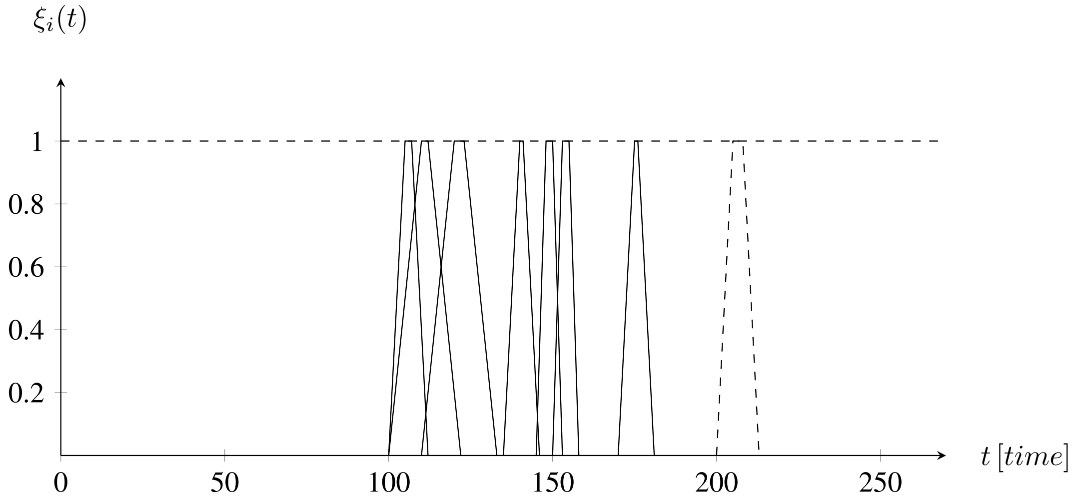
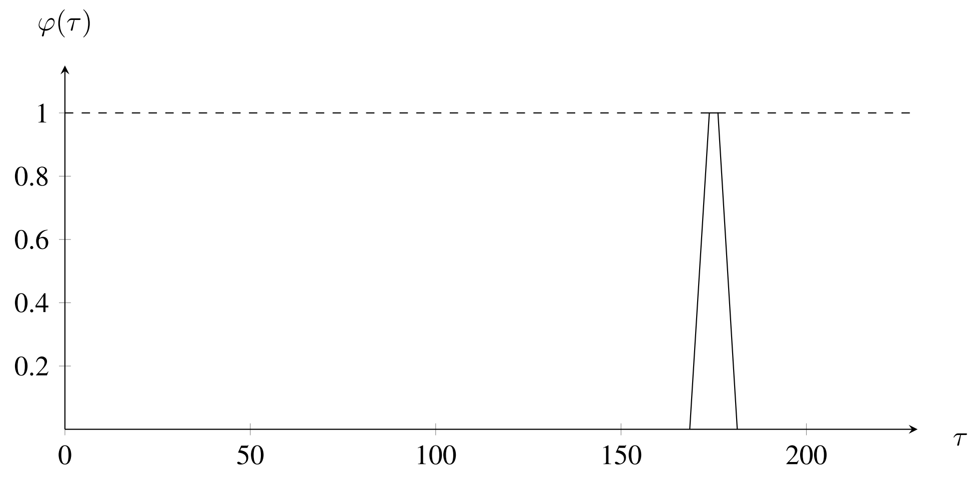
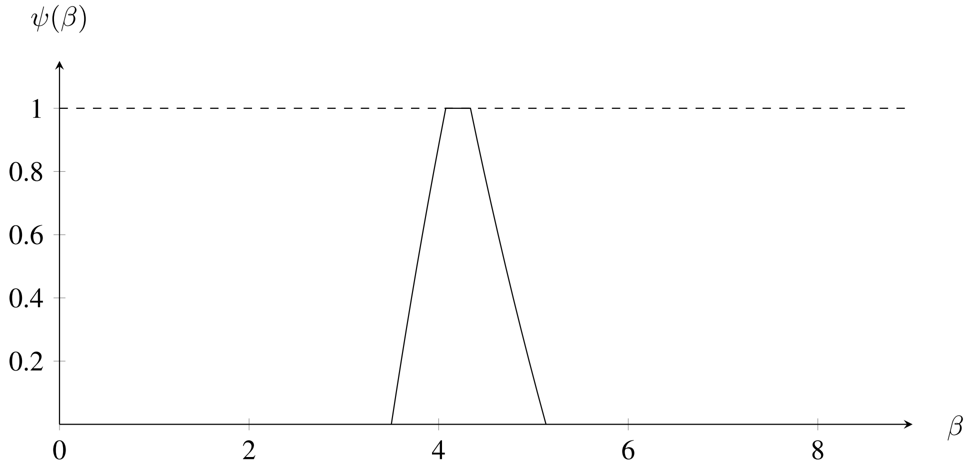


Figure 2: Characterizing function of $\hat{\tau}^*$ for type I censoring data with fuzzy censoring time from figure 1.



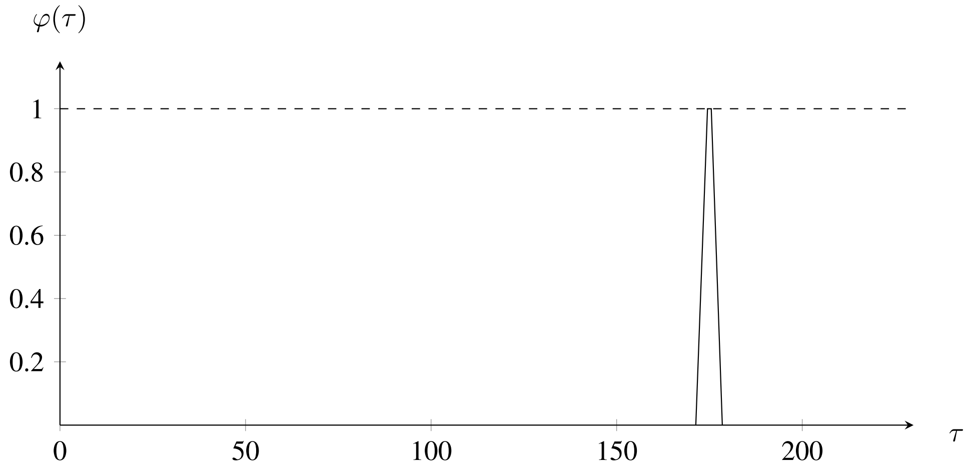
Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

Figure 3: Characterizing function of $\hat{\beta}^*$ for type I censoring with fuzzy censoring time



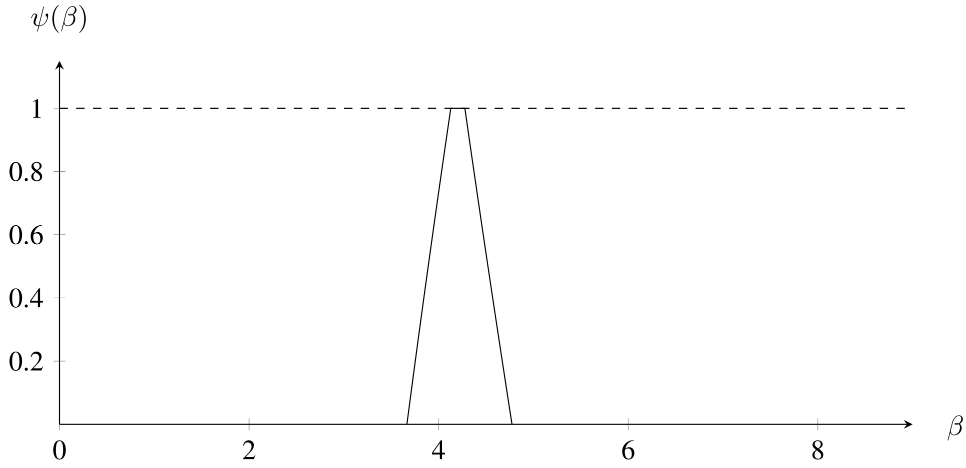
Next the termination time of the experiment is assumed to be a precise number. Given below in figure 4 is a sample of 8 units with 7 failures and 1 censored observation, with precise censoring time. Figure 5 and figure 6 show the characterizing functions of the fuzzy estimates of the Weibull parameters with fuzzy failure times and precise censoring time from figure 4.

Figure 4: Characterizing function of $\hat{\tau}^*$ for type I censoring data from figure 4 with precise censoring time



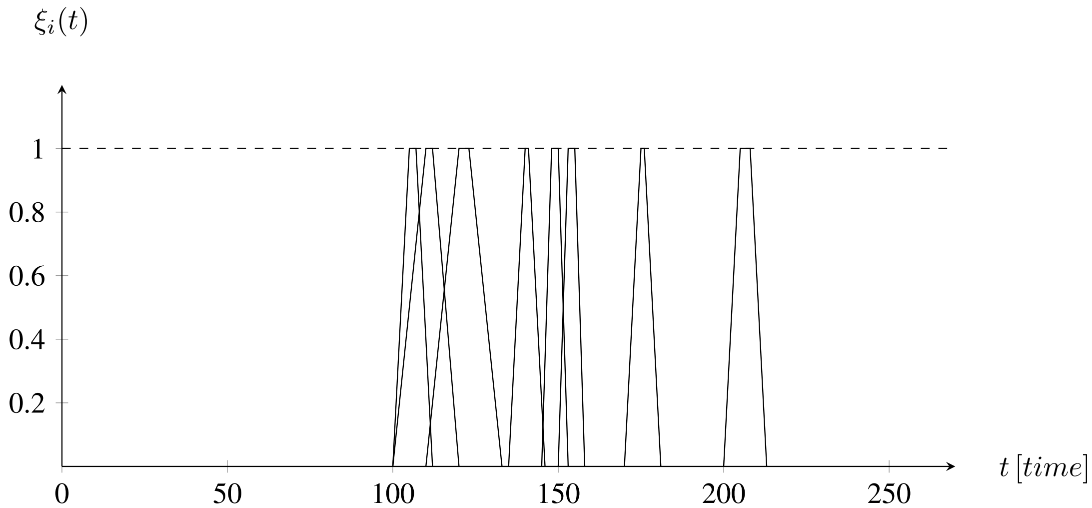
Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

Figure 5: Characterizing function of $\hat{\beta}^*$ for type I censoring with precise censoring time



For type II censoring considering 8 fuzzy failure times and 2 censored observations are given in figure 7. The characterizing functions of the fuzzy estimates are given in figure 8 and figure 9.

Figure 6: Sample of Type II censoring with fuzzy failure times



https://doi.org/10.54216/IJNS.240217

Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

Figure 7: Characterizing function of $\hat{\tau}^*$ for type II censoring with fuzzy failure times from figure 7.

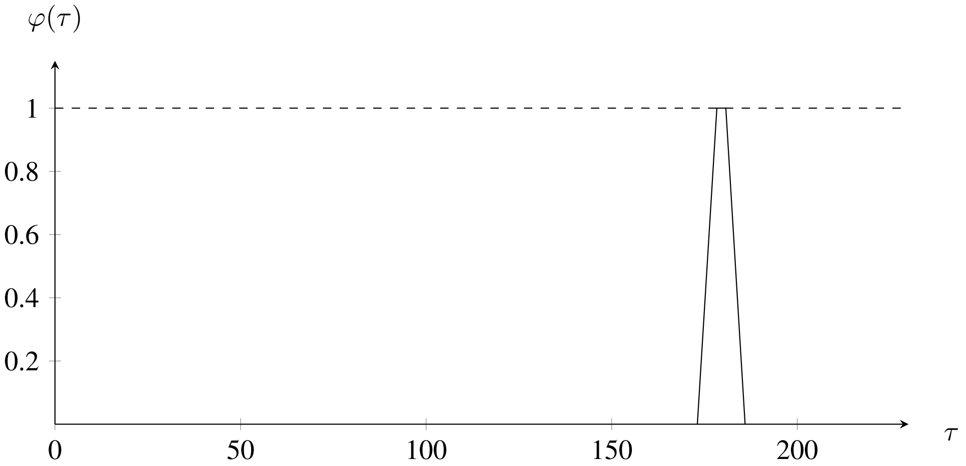
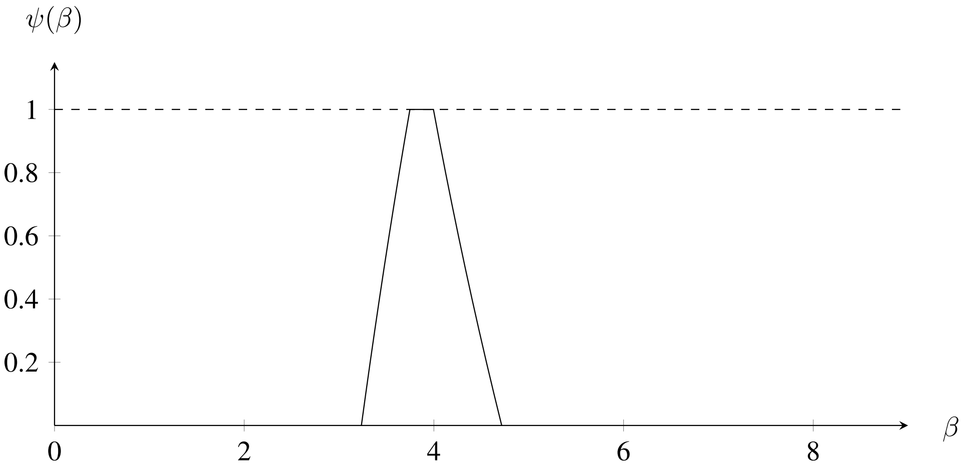


Figure 8: Characterizing function of $\hat{\beta}^*$ for type II censoring with fuzzy failure times



3 Fuzzy Maximum Likelihood Estimators of Exponential distribution

For the parameter estimation of two parameter exponential distribution, in case of censoring, r precise failures times are denoted by $\Big(t_1, \, t_2, \, ..., \, t_r\Big)$, and $\Big(t_{r+1}, \, t_{r+2}, \, ..., \, t_n\Big)$ are censored observations respectively, makes a total of n observations. The maximum likelihood estimators for the parameters are denoted by $\hat{\lambda}$ and $\hat{\mu}$.

These estimators are defined as

$$\hat{\lambda} = \left[\frac{r-1}{\sum_{i=1}^{r} t_i + \sum_{i=r+1}^{n} t_i^{+} - n(t_{(1)})} \right] \tag{3}$$

and

$$\hat{\mu} = \left[max\Big((t_{(1)} - \frac{1}{n\hat{\lambda}}), 0 \Big) \right] \tag{4}$$

for detail see [30].

https://doi.org/10.54216/IJNS.240217
Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

For fuzzy observations t_1^*, t_2^*, ..., t_r^*, i.e. the generalized fuzzy estimators are defined through the generalized theorem, and are denoted by $\hat{\lambda}^*$ and $\hat{\mu}^*$.

Let t_1^*, t_2^*, ..., t_r^* are denoting fuzzy life time observations with corresponding lower and upper ends of the δ-cuts, i.e. $C_\delta(t_i^*) = [\underline{t}_{i,\delta}\,,\, \bar{t}_{i,\delta}] \quad \forall\delta \in (0\,,\,1]$.

Based on fuzzy life times lower and upper ends of the δ-cuts of generalized estimator $\hat{\lambda}^*$ are defined as

$$\underline{\lambda}_\delta = \left[\frac{r-1}{\sum_{i=1}^{r} \bar{t}_{i,\delta} + \sum_{i=r+1}^{n} t_i^+ - n(\underline{t}_{(1,\delta)})}\right] \quad \forall\delta \in (0\,,\,1]. \tag{5}$$

and

$$\overline{\lambda}_\delta = \left[\frac{r-1}{\sum_{i=1}^{r} \underline{t}_{i,\delta} + \sum_{i=r+1}^{n} t_i^+ - n(\bar{t}_{(1,\delta)})}\right] \quad \forall\delta \in (0\,,\,1]. \tag{6}$$

In the same way lower and upper ends of the δ-cuts of generalized estimator $\hat{\mu}^*$ are defined as

$$\underline{\mu} = \left[max\Big((\underline{t}_{(1,\delta)} - \frac{1}{n\underline{\lambda}}), 0\Big)\right] \quad \forall\delta \in (0\,,\,1]. \tag{7}$$

and

$$\overline{\mu} = \left[max\Big((\bar{t}_{(1,\delta)} - \frac{1}{n\overline{\lambda}_\delta}), 0\Big)\right] \quad \forall\delta \in (0\,,\,1]. \tag{8}$$

The generating families of intervals of the fuzzy estimators $\hat{\lambda}^*$ and $\hat{\mu}^*$ are denoted as $A_\delta(\hat{\lambda}^*)$ and $A_\delta(\hat{\mu}^*)$, where

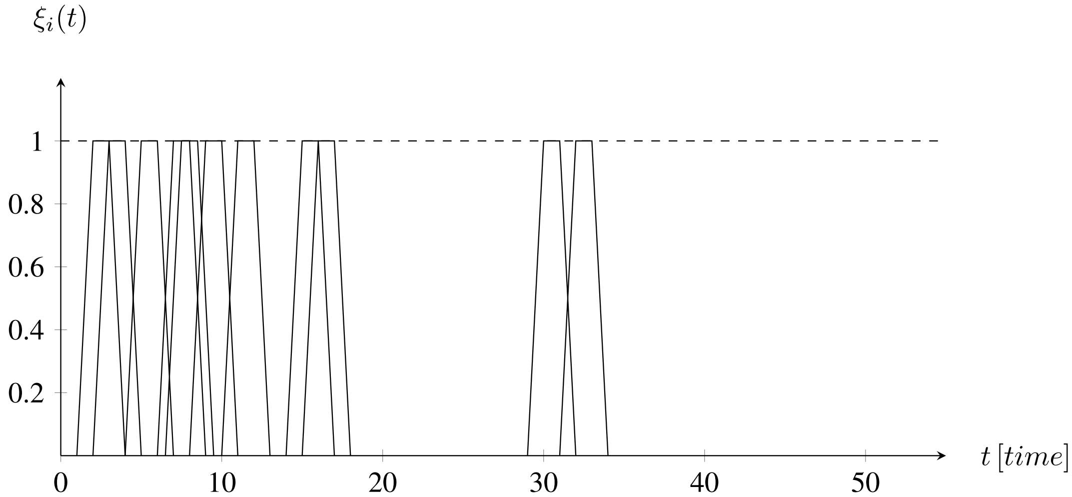
$$\Big(A_\delta(\hat{\lambda}^*) = [\underline{\lambda}_\delta\,,\,\overline{\lambda}_\delta] \qquad \forall\delta \in (0\,,\,1]\Big) \tag{9}$$

and

$$\Big(A_\delta(\hat{\mu}^*) = [\underline{\mu}_\delta\,,\,\overline{\mu}_\delta] \qquad \forall\delta \in (0\,,\,1]\Big) \tag{10}$$

Total 19 observations are taken out of which 8 are censored, i.e. $t_i^+ = (3,\, 8,\, 13,\, 21,\, 26,\, 35,\, 44,\, 45)$, and characterizing function of the 11 fuzzy life times are given in below figure 10.

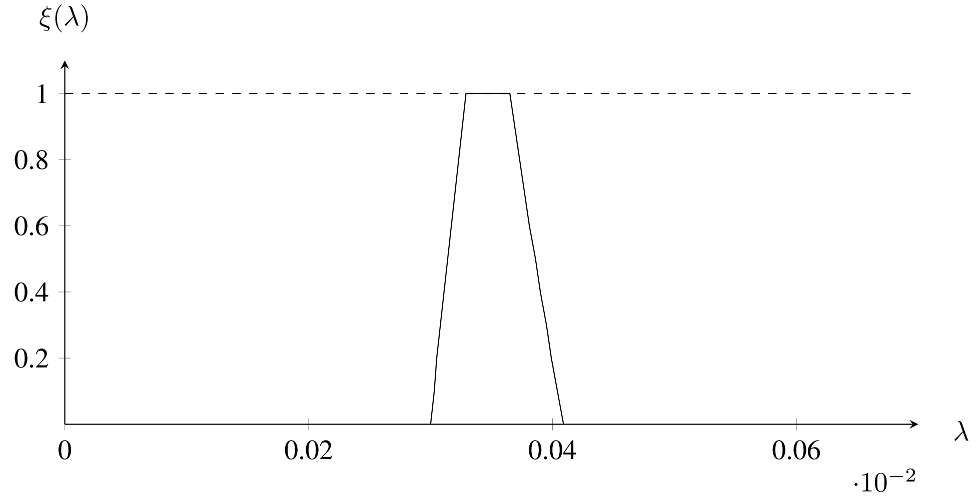
Figure 9: Sample of fuzzy life time



https://doi.org/10.54216/IJNS.240217
Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

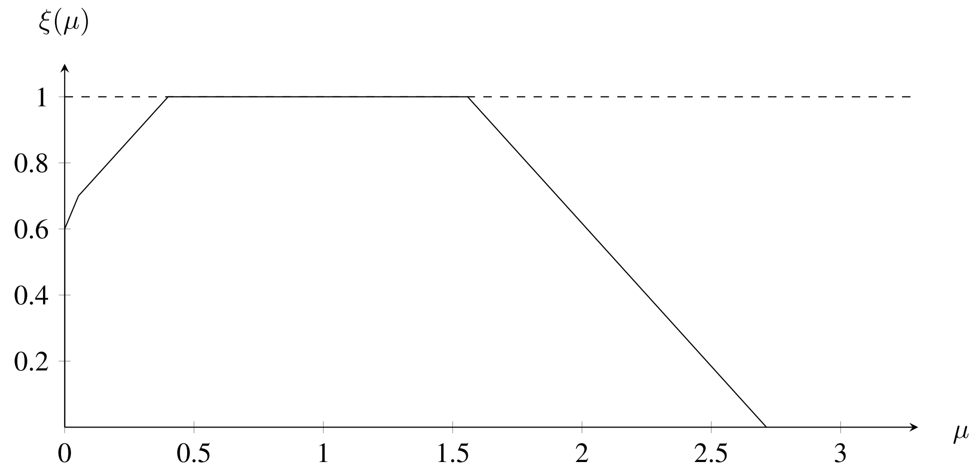
From the generating families of intervals given in equation (9) and equation (10), using the construction lemma, characterizing functions of the fuzzy estimates $\hat{\lambda}^*$ and $\hat{\mu}^*$ are obtained.

Figure 10: Characterizing function of the fuzzy estimator $\hat{\lambda}^*$



The above figures shows the characterizing function of fuzzy estimates $\hat{\lambda}^*$, which are based on the all the available information in the form of fuzziness and stochastic variation.

Figure 11: Characterizing function of the fuzzy estimator $\hat{\mu}^*$



The above figures shows the characterizing function of fuzzy estimates $\hat{\mu}^*$, which are based on the all the available information in the form of fuzziness and stochastic variation.

4 Conclusion

In practical situations, we cannot measure a real continuous variable as a precise number, and the measurements are more or less fuzzy. Life time is a continuous phenomenon, but the available standard statistical tools consider life time observations as precise numbers and make inferences based on them. By doing so, we lose information and obtain misleading inferences. Therefore, fuzzy number techniques are more suitable for inference of lifetime observations. In this paper, we considered hypothetical fuzzy life time observations and calculated the characterizing functions of the fuzzy parameter estimates for

https://doi.org/10.54216/IJNS.240217

Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024

the Weibull and exponential distributions based on censored fuzzy life time data. The proposed estimation procedures have a significant number of applications in life time analysis, reliability analysis, repair systems, with a constant hazard rate, flexible hazard rates, and especially for small sample sizes of field and laboratory test life time data. Furthermore, where the waiting time have indeterminacy the same techniques need to be extended to neutrosophic sets.

Acknowledgment: Major part of the paper is available at "'Report No. Forschungsbericht SM-2014-2 November, 2014; 17 pages"
https://publik.tuwien.ac.at/showentry.php?ID=232586lang=2'
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https://doi.org/10.54216/IJNS.240217
Received: March 11, 2024 Revised: April 07, 2024 Accepted: April 29, 2024