



## Arithmetic Operations on Generalized Pentagonal Fuzzy Numbers

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### Abstract

Fuzzy concepts have been widely used to treat imprecision in many fields of natural and social sciences. In most of the natural science fields such as applied mathematics, physics, chemistry, and engineering, triangular and trapezoidal fuzzy numbers are commonly used and arithmetic operations on those numbers are studied in detail. On the other hand, in engineering and social science fields such as sociology and psychology, while treating the uncertainties, these numbers are not applicable and fuzzy numbers with more parameters and clear definitions of their arithmetic operations are needed. In order to fill this gap in the literature, in this study we propose the generalized pentagonal fuzzy numbers, and we define fuzzy arithmetic operations based on both extension and the function principle.

**Keywords:** fuzzy number; fuzzy arithmetic; extension principle; Chen's function principle; pentagonal.

### 1 Introduction

When uncertainty becomes a matter of debate, conventional approaches to treat uncertainty focus on probability theory. However, the probability-based approaches may not be sufficient to reflect all the uncertainties that may arise in real world applications. Modelers may face some difficulties while trying to build a valid model of a system, in which the measurements can not be determined precisely. Since some uncertainty within systems can not be considered appropriately using the concepts of probability theory, fuzzy set theory has been used in the modeling of systems since 1980's. Fuzzy set theory, originally introduced by Zadeh in 1965,<sup>8</sup> provides a framework for considering parameters that are vaguely or unclearly defined or whose values are imprecise or determined based on the subjective beliefs of individuals.

In 1978, Dubois and Prade<sup>4</sup> proposed the concept of fuzzy number as a quantity whose value is imprecise and can be defined as a fuzzy subset of the real line.

In most of the natural science fields, triangular and trapezoidal fuzzy numbers are the most commonly used measuring tools to evaluate the uncertainty in a physical system. However, in most of the cases in social sciences, triangular and trapezoidal fuzzy numbers,<sup>7,10</sup> may not be enough to measure the attributes usually associated with opinions leading to an ordinal information which can be represented by more than four different points on the real line. Therefore, even trapezoidal fuzzy numbers can not be enough to represent such cases arising from social science measurements.

In this study, we propose the notion of pentagonal fuzzy numbers in order to fill this gap in literature. Moreover, we present arithmetic operations on the proposed normal generalized pentagonal fuzzy numbers (NGPtFN's). The main arithmetic operations such as; addition, subtraction, multiplication, and division are defined based on two essential principles from the literature. First, we introduce these main arithmetic operations based on a very well-known extension principle proposed by Zadeh in 1975,<sup>9,6</sup> It was noted by Maturro in 2009,<sup>5</sup> arithmetic operations based on extension principle present some drawbacks in the applications of social sciences examples. Therefore, we also define all arithmetic operations on NGPtFN's based on another principle called function principle, which is proposed by Chen in 1985<sup>1</sup> for triangular and trapezoidal fuzzy numbers.

## 2 Preliminaries

In this section, we will recall some basic notions of fuzzy set theory and fuzzy numbers.

**Definition 2.1.** Let  $X$  be a universal set. A fuzzy set  $\tilde{A}$  of  $X$  is defined as a set of pairs.  $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ . Here,  $\mu_{\tilde{A}} : X \rightarrow [0, 1]$  is a mapping and called the membership function, and for every  $x \in X$ , the real number  $\mu_{\tilde{A}}(x)$  is called the membership degree of  $x \in X$ .

**Definition 2.2.** A fuzzy set  $\tilde{A}$  is normalized if there exists an element  $x \in X$  so that  $\mu_{\tilde{A}}(x) = 1$ .

**Definition 2.3.** A normalized convex fuzzy set  $\tilde{A}$  of real line  $\mathbf{R}$  is called a fuzzy number if:

- i) there exists exactly one  $x_0 \in \mathbf{R}$  with  $\mu_{\tilde{A}}(x_0) = 1$
- ii) the function  $\mu_{\tilde{A}}(x)$  is piecewise continuous.

**Definition 2.4.** (*Extension Principle*)<sup>9</sup> Let  $f : X \rightarrow Y$  be a function,  $\tilde{A}$  be a fuzzy set defined on  $X$ . Then the extension principle gives result to define a fuzzy set  $\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x), x \in X\}$  over  $Y$  where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & f^{-1}(y) \neq \emptyset \\ 0, & f^{-1}(y) = \emptyset \end{cases}$$

Note also that one can use a more generalized form of the extension principle given in the following definition:

**Definition 2.5.** Let  $X$  be a Cartesian product of universes,  $X = \prod_{i=1}^n X_i$ ,  $\tilde{A}_i$  be a fuzzy set over  $X_i$  for every  $1 \leq i \leq n$ , and  $f$  be a function from  $X$  to a universe  $Y$ . Then the extension principle gives result to define a fuzzy set  $\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, \dots, x_n), (x_1, \dots, x_n) \in X\}$  over  $Y$  where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x_1, \dots, x_n) \in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), \dots, \mu_{\tilde{A}_n}(x_n)\}, & f^{-1}(y) \neq \emptyset \\ 0, & f^{-1}(y) = \emptyset \end{cases}$$

**Definition 2.6.**<sup>1</sup> A fuzzy number  $\tilde{A}$  with membership function  $\mu_{\tilde{A}}$  as:

$$\mu_{\tilde{A}}(x) = \begin{cases} w \frac{x-c}{a-c}, & c \leq x \leq a \\ w, & a \leq x \leq b \\ w \frac{x-d}{b-d}, & b \leq x \leq d \\ 0, & \text{otherwise} \end{cases}$$

is called a trapezoidal fuzzy number. Usually, this type of fuzzy numbers are denoted as  $\tilde{A} = (c, a, b, d; w)$ , where  $c, a, b, d, w$  are real numbers and  $0 < w \leq 1$ .

**Definition 2.7.** (*Chen's Function Principle*)<sup>1</sup> Let  $g$  be a function from  $n$ -dimensional Euclidean space  $\mathbf{R}^n$  into real line  $\mathbf{R}$ ,  $f_g$  is a corresponding function from  $n$ -dimensional fuzzy numbers into a fuzzy number, and  $\tilde{A}_i = (c_i, a_i, b_i, d_i; w_i)$ , be trapezoidal fuzzy numbers for  $i = 1, 2, \dots, n$ . The fuzzy number  $\tilde{B}$  in  $\mathbf{R}$  induced trapezoidal fuzzy numbers  $\tilde{A}_i$ 's through function  $g$  is also another trapezoidal fuzzy number and defined as:  $f_g(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \tilde{B} = (c, s, t, d; w)$ , where  $w = \min\{w_1, \dots, w_n\}$ ,  $s_i = \min\{x \mid f_{A_i}(x) \geq w\}$ ,  $t_i = \max\{x \mid f_{A_i}(x) \geq w\}$ ,  $T = \{g(x_1, \dots, x_n) \mid x_i = c_i \text{ or } d_i, i = 1, 2, \dots, n\}$ ,  $T_1 = \{g(x_1, \dots, x_n) \mid x_i = s_i \text{ or } t_i, i = 1, 2, \dots, n\}$ ,  $c = \min T$ ,  $s = \min T_1$ ,  $t = \max T_1$ ,  $d = \max T$ , when  $\min T \leq \min T_1$ , and  $\max T_1 \leq \max T$ .

### 3 Generalized Pentagonal Fuzzy Numbers

**Definition 3.1.** A fuzzy number  $\tilde{A}_w = (a_1, a_2, a_3, a_4, a_5; w_1, w_2, w)$  is said to be a generalized pentagonal fuzzy number (GPtFN) if its membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} w_1 \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ w_1 + (w - w_1) \frac{x - a_2}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ w + (w_2 - w) \frac{x - a_3}{a_4 - a_3}, & a_3 \leq x \leq a_4 \\ w_2 - w_2 \frac{x - a_4}{a_5 - a_4}, & a_4 \leq x \leq a_5 \\ 0, & \text{otherwise} \end{cases}$$

[xtick=data, axis x line=center, xticklabels= $a_1, a_2, a_3, a_4, a_5$ , ytick=0,0.3,1,0.7,0, yticklabels=0, $w_1, w, w_2$ , axis y line=left, enlargelimits=0.05] coordinates (1,0) (2.5,0.3) (3.7,1) (4.5,0.7) (5.2,0);

Figure 1: A GPtFN

**Definition 3.2.** A generalized pentagonal fuzzy number denoted by  $\tilde{A}_w$  can also be defined as  $\tilde{A}_w = (P_1(j), Q_1(k), Q_2(l), P_2(m))$  for  $j \in [0, w_1], k \in [w_1, w], l \in [w_2, w]$ , and  $m \in [0, w_2]$  where

- (i)  $P_1(j)$  is a bounded left continuous non decreasing function over  $[0, w_1]$
- (ii)  $Q_1(k)$  is a bounded left continuous non decreasing function over  $[w_1, w]$
- (iii)  $Q_2(l)$  is a bounded left continuous non increasing function over  $[w_2, w]$
- (iv)  $P_2(m)$  is a bounded left continuous non increasing function over  $[0, w_2]$ .

**Definition 3.3.** If  $w = 1$  then the generalized pentagonal fuzzy number is called a normal generalized pentagonal fuzzy number *NGPtFN*. Here  $w$  is the maximum membership value, and for convenience the normal generalized pentagonal fuzzy numbers are denoted by  $\tilde{A}$ , and  $\tilde{A} = (a_1, a_2, a_3, a_4, a_5; w_1, w_2)$ .

**Definition 3.4.** A positive GPtFN is denoted as  $\tilde{A}_w = (a_1, a_2, a_3, a_4, a_5; w_1, w_2, w)$  where all  $a_i > 0$  for all  $i = 1, 2, 3, 4, 5$ .

**Definition 3.5.** A negative GPtFN is denoted as  $\tilde{A}_w = (a_1, a_2, a_3, a_4, a_5; w_1, w_2, w)$  where all  $a_i < 0$  for all  $i = 1, 2, 3, 4, 5$ .

**Definition 3.6.** Let  $\tilde{A}_w = (a_1, a_2, a_3, a_4, a_5; w_1, w_2, w)$  and  $\tilde{B}_{w'} = (b_1, b_2, b_3, b_4, b_5; w'_1, w'_2, w')$  be two GPtF numbers. Then  $\tilde{A}$  is identically equal to  $\tilde{B}$  if and only if  $w = w', a_i = b_i$ , and  $w_j = w'_j$  for all  $i = 1, 2, 3, 4, 5$ , and  $j = 1, 2$ .

**Definition 3.7.** For a given fuzzy set  $\tilde{A}$ , the crisp set  $A_\alpha$  defined as:  $A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$  is called the  $\alpha$ -cut set. Note that one can define the  $\alpha$ -cut set  $(A_w)_\alpha$  for a GPtFN  $\tilde{A}_w$  with  $w_1 \leq w_2$  as:

$$(A_w)_\alpha = \begin{cases} [P_1^{-1}(\alpha), P_2^{-1}(\alpha)], & 0 \leq \alpha \leq w_1 \\ [Q_1^{-1}(\alpha), P_2^{-1}(\alpha)], & w_1 < \alpha < w_2 \\ [Q_1^{-1}(\alpha), Q_2^{-1}(\alpha)], & w_2 \leq \alpha \leq w. \end{cases}$$

Since the  $\alpha$ -cut sets are crisp sets we can also obtain those as follows:

Consider  $P_1(x) = \alpha$ . Then:

$$x = a_1 + \frac{\alpha}{w_1}(a_2 - a_1).$$

Similarly if  $P_2(x) = \alpha$ , then:

$$x = a_4 + \frac{w_2 - \alpha}{w_2}(a_5 - a_4).$$

Consider  $Q_1(x) = \alpha$ . This implies:

$$x = a_2 + \frac{\alpha - w_1}{w - w_1}(a_3 - a_2).$$

And finally, if  $Q_2(x) = \alpha$ , then:

$$x = a_3 + \frac{\alpha - w}{w_2 - w}(a_4 - a_3).$$

Hence:

$$(A_w)_\alpha = \begin{cases} [a_1 + \frac{\alpha}{w_1}(a_2 - a_1), a_4 + \frac{w_2 - \alpha}{w_2}(a_5 - a_4)], & 0 \leq \alpha \leq w_1 \\ [a_2 + \frac{\alpha - w_1}{w - w_1}(a_3 - a_2), a_4 + \frac{a_5 - a_4}{w_2}(w_2 - \alpha)], & w_1 < \alpha < w_2 \\ [a_2 + \frac{\alpha - w_1}{w - w_1}(a_3 - a_2), a_3 + \frac{\alpha - w}{w_2 - w}(a_4 - a_3)], & w_2 \leq \alpha \leq 1. \end{cases}$$

Furthermore the  $\alpha$ -cut set  $(A_w)_\alpha$  for a GPtFN  $\tilde{A}_w$  with  $w_2 \leq w_1$  can be defined similarly.

To provide simplicity, for the rest of the study, we only consider the normal generalized pentagonal fuzzy numbers *NGPtFN*.

#### 4 Arithmetic Operations on NGPtFN's

In this section we discuss the main arithmetic operations (addition, subtraction, multiplication, and division) for NGPtFN's. Definitions of these operations are based upon both the extension and the function principles.

By definition, the extension principle can be adopted to the operations on NGPtFN's without any modifications. However the function principle can not be used for the operations on NGPtFN's. The following definition is the new version of the function principle that will be used in this study.

**Definition 4.1.** (*Function Principle for NGPtFN's*) Let  $g$  be a function from  $n$ -dimensional Euclidean space  $\mathbf{R}^n$  into real line  $\mathbf{R}$ , and  $f_g$  be the corresponding function from  $n$ -dimensional NGPtFN's into a NGPtFN. The normal generalized pentagonal fuzzy number  $\tilde{B}$  induced from  $n$  normal generalized pentagonal fuzzy numbers  $\tilde{A}_i = (a_1^i, a_2^i, a_3^i, a_4^i, a_5^i; w_1^i, w_2^i)$  through  $g$  is given as follows:

$$f_g(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \tilde{B} = (b_1, b_2, b_3, b_4, b_5; w_1, w_2) \text{ where:}$$

$$\begin{aligned} w_1 &= \min\{w_1^i\}, \\ w_2 &= \min\{w_2^i\}, \\ l_{A_i} &= \min\{x \mid \mu_{\tilde{A}_i}(x) \geq w_1\}, \\ r_{A_i} &= \max\{x \mid \mu_{\tilde{A}_i}(x) \geq w_2\}, \\ T_1 &= \{g(x_1, x_2, \dots, x_n) \mid x_i = a_1^i \text{ or } a_5^i, i = 1, 2, \dots, n\} \\ T_2 &= \{g(x_1, x_2, \dots, x_n) \mid x_i = l_{A_i} \text{ or } r_{A_i}, i = 1, 2, \dots, n\} \\ T_3 &= \{g(x_1, x_2, \dots, x_n) \mid x_i = a_3^i, i = 1, 2, \dots, n\} \\ b_1 &= \min T_1 \\ b_2 &= \min T_2 \\ b_3 &= T_3 \\ b_4 &= \max T_2 \\ b_5 &= \max T_1 \end{aligned}$$

Let  $\tilde{A}_w = (a_1, a_2, a_3, a_4, a_5; w_1, w_2)$  and  $\tilde{B}_{w'} = (b_1, b_2, b_3, b_4, b_5; w'_1, w'_2)$  be two NGPtFN's with the following membership functions, respectively:

$$\mu_{\tilde{A}}(x) = \begin{cases} w_1 \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ w_1 + (1 - w_1) \frac{x - a_2}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 1 + (w_2 - 1) \frac{x - a_3}{a_4 - a_3}, & a_3 \leq x \leq a_4 \\ w_2 - w_2 \frac{x - a_4}{a_5 - a_4}, & a_4 \leq x \leq a_5 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu_{\tilde{B}}(x) = \begin{cases} w'_1 \frac{x - b_1}{b_2 - b_1}, & b_1 \leq x \leq b_2 \\ w'_1 + (1 - w'_1) \frac{x - b_2}{b_3 - b_2}, & b_2 \leq x \leq b_3 \\ 1 + (w'_2 - 1) \frac{x - b_3}{b_4 - b_3}, & b_3 \leq x \leq b_4 \\ w'_2 - w'_2 \frac{x - b_4}{b_5 - b_4}, & b_4 \leq x \leq b_5 \\ 0, & \text{otherwise} \end{cases}$$

In the following sections, the arithmetic operations on these two NGPtFN's will be explored.

#### 4.1 Addition of two NGPtFN's

##### 4.1.1 Addition of two NGPtFN's based on extension principle

Let  $\tilde{A}(+) \tilde{B} = \tilde{C}$  where  $\mu_{\tilde{C}}(z) = \sup(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \mid x + y = z)$ .

Let  $w''_1 = \min(w_1, w'_1)$ , and  $w''_2 = \min(w_2, w'_2)$ . Then the membership function  $\mu_{\tilde{C}}$  for  $\tilde{C}$  is defined as following:

$$\mu_{\tilde{C}}(z) = \begin{cases} \sup(\min(w_1 \frac{x - a_1}{a_2 - a_1}, w'_1 \frac{y - b_1}{b_2 - b_1}) \mid x + y = z), & a_1 \leq x \leq a_2, b_1 \leq y \leq b_2 \\ \sup(\min(w_1 + (1 - w_1) \frac{x - a_2}{a_3 - a_2}, w'_1 + (1 - w'_1) \frac{y - b_2}{b_3 - b_2}) \mid x + y = z), & a_2 \leq x \leq a_3, b_2 \leq y \leq b_3 \\ \sup(\min(1 + (w_2 - 1) \frac{x - a_3}{a_4 - a_3}, 1 + (w'_2 - 1) \frac{y - b_3}{b_4 - b_3}) \mid x + y = z), & a_3 \leq x \leq a_4, b_3 \leq y \leq b_4 \\ \sup(\min(w_2 - w_2 \frac{x - a_4}{a_5 - a_4}, w'_2 - w'_2 \frac{y - b_4}{b_5 - b_4}) \mid x + y = z), & a_4 \leq x \leq a_5, b_4 \leq y \leq b_5 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Note that the membership function of  $\tilde{C}$  given in 1 is indeed equal to:

$$\mu_{\tilde{C}}(z) = \begin{cases} w''_1 \frac{z - a_1 - b_1}{a_2 + b_2 - a_1 - b_1}, & a_1 + b_1 \leq z \leq a_2 + b_2 \\ w''_1 + (1 - w''_1) \frac{z - a_2 - b_2}{a_3 + b_3 - a_2 - b_2}, & a_2 + b_2 \leq z \leq a_3 + b_3 \\ 1 + (w''_2 - 1) \frac{z - a_3 - b_3}{a_4 + b_4 - a_3 - b_3}, & a_3 + b_3 \leq z \leq a_4 + b_4 \\ w''_2 - w''_2 \frac{z - a_4 - b_4}{a_5 + b_5 - a_4 - b_4}, & a_4 + b_4 \leq z \leq a_5 + b_5 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

It should be noted that the addition of two NGPtFN's  $\tilde{A}$  and  $\tilde{B}$  is another NGPtFN  $\tilde{C} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5; w''_1, w''_2)$  with the membership function given in 2.

[ xtick=data, axis x line=center, xticklabels= $a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5$ , ytick=0,0.8,1,0.7,0, yticklabels=0, $w_1'', 1, w_2''$ , axis y line=left, enlargelimits=0.05] coordinates (1.5,0) (2.5,0.8) (3.5,1) (4.8,0.7) (6,0);

Figure 2: Rough sketch of membership function for  $\tilde{A}(+) \tilde{B}$

**4.1.2 Addition of two NGPtFN’s based on function principle**

Under the function principle the sum of NGPtF numbers  $\tilde{A}$  and  $\tilde{B}$  is another NGPtFN  $\tilde{A} \oplus \tilde{B} = \tilde{C} = (c_1, c_2, c_3, c_4, c_5; w_1'', w_2'')$ .

Here we first construct:

$$\begin{aligned}
 l_A &= \min\{x \mid \mu_{\tilde{A}}(x) \geq w_1''\} \\
 l_B &= \min\{x \mid \mu_{\tilde{B}}(x) \geq w_1''\} \\
 r_A &= \max\{x \mid \mu_{\tilde{A}}(x) \geq w_2''\} \\
 r_B &= \max\{x \mid \mu_{\tilde{B}}(x) \geq w_2''\}, \text{ and } w_1'' = \min\{w_1, w_1'\}, w_2'' = \min\{w_2, w_2'\}. \text{ Then:} \\
 T_1 &= \{a_1 + b_1, a_1 + b_5, a_5 + b_1, a_5 + b_5\} \\
 T_2 &= \{l_A + l_B, l_A + r_B, r_A + l_B, r_A + r_B\} \\
 T_3 &= a_3 + b_3. \text{ Finally:} \\
 c_1 &= \min T_1, c_2 = \min T_2, c_3 = T_3, c_4 = \max T_2, \text{ and } c_5 = \max T_1.
 \end{aligned}$$

**4.2 Multiplication of a NGPtFN With a Scalar**

**4.2.1 Multiplication of a NGPtFN with a scalar based on extension principle**

Under the extension principle the multiplication of a NGPtFN  $\tilde{A}$  with a scalar  $k$  is another NGPtFN  $k(\cdot)\tilde{A} = \tilde{C} = (c_1, c_2, c_3, c_4, c_5; w_1, w_2)$ .

Here the definition of  $\tilde{C}$  depends on the constant  $k$ .

Case1 :  $k \geq 0$

$$\mu_{\tilde{C}}(z) = \begin{cases} \sup(\min(w_1 \frac{x - a_1}{a_2 - a_1} \mid k.x = z)), & a_1 \leq x \leq a_2 \\ \sup(\min(w_1 + (1 - w_1) \frac{x - a_2}{a_3 - a_2} \mid k.x = z)), & a_2 \leq x \leq a_3 \\ \sup(\min(1 + (w_2 - 1) \frac{x - a_3}{a_4 - a_3} \mid k.x = z)), & a_3 \leq x \leq a_4 \\ \sup(\min(w_2 - w_2 \frac{x - a_4}{a_5 - a_4} \mid k.x = z)), & a_4 \leq x \leq a_5 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Note that the membership function of  $\tilde{C}$  given in 3 is indeed equal to:

$$\mu_{\tilde{C}}(z) = \begin{cases} w_1 \frac{z - k.a_1}{k.a_2 - k.a_1}, & k.a_1 \leq z \leq k.a_2 \\ w_1 + (1 - w_1) \frac{z - k.a_2}{k.a_3 - k.a_2}, & k.a_2 \leq z \leq k.a_3 \\ 1 + (w_2 - 1) \frac{z - k.a_3}{k.a_4 - k.a_3}, & k.a_3 \leq z \leq k.a_4 \\ w_2 - w_2 \frac{z - k.a_4}{k.a_5 - k.a_4}, & k.a_4 \leq z \leq k.a_5 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

It should be noted that the multiplication of a NGPtFN  $\tilde{A}$  with a constant  $k \geq 0$  is another NGPtFN given by  $\tilde{C} = (k.a_1, k.a_2, k.a_3, k.a_4, k.a_5; w_1, w_2)$  with the membership function given in 4.

[ xtick=data, axis x line=center, xticklabels= $k.a_1, k.a_2, k.a_3, k.a_4, k.a_5$ , ytick=0,0.3,1,0.6,0, yticklabels=0, $w_1, 1, w_2$ , axis y line=left, enlargelimits=0.05] coordinates (1.5,0) (2.5,0.3) (3.5,1) (4.8,0.6) (6,0);

Figure 3: Rough sketch for  $k(\cdot)\tilde{A}$  where  $k > 0$

Case2 :  $k < 0$

$$\mu_{\tilde{C}}(z) = \begin{cases} \sup(\min(w_2 \frac{a_4 - x}{a_5 - a_4} + w_2 \mid k.x = z)), & a_5 \leq x \leq a_4 \\ \sup(\min(w_2 + (1 - w_2) \frac{x - a_4}{a_3 - a_4} \mid k.x = z)), & a_4 \leq x \leq a_3 \\ \sup(\min(1 + (w_1 - 1) \frac{x - a_3}{a_2 - a_3} + 1 \mid k.x = z)), & a_3 \leq x \leq a_2 \\ \sup(\min(w_1 + w_1 \frac{a_2 - x}{a_1 - a_2} \mid k.x = z)), & a_2 \leq x \leq a_1 \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Note that the membership function of  $\tilde{C}$  given in 5 is indeed equal to:

$$\mu_{\tilde{C}}(z) = \begin{cases} w_2 \frac{k.a_4 - z}{k.a_5 - k.a_4}, & k.a_5 \leq z \leq k.a_4 \\ w_2 + (1 - w_2) \frac{z - k.a_4}{k.a_3 - k.a_4}, & k.a_4 \leq z \leq k.a_3 \\ 1 + (w_1 - 1) \frac{z - k.a_3}{k.a_2 - k.a_3} + 1, & k.a_3 \leq z \leq k.a_2 \\ w_1 - w_1 \frac{k.a_2 - z}{k.a_1 - k.a_2}, & k.a_2 \leq z \leq k.a_1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

It should be noted that the multiplication of a NGPtFN  $\tilde{A}$  with a constant  $k < 0$  is another NGPtFN given by  $\tilde{C} = (k.a_5, k.a_4, k.a_3, k.a_2, k.a_1; w_1, w_2)$  with the membership function given in 6.

[ xtick=data, axis x line=center, xticklabels= $k.a_1, k.a_2, k.a_3, k.a_4, k.a_5$ , ytick=0,0.3,1,0.6,0, yticklabels=0, $w_1, 1, w_2$ , axis y line=center, enlargelimits=0.05] coordinates (-1.5,0) (-2.5,0.3) (-3.5,1) (-4.8,0.6) (-6,0) ;

Figure 4: Rough sketch for  $k(\cdot)\tilde{A}$  where  $k < 0$

### 4.2.2 Multiplication of a NGPtFN with a scalar based on function principle

Under the function principle the multiplication of a NGPtFN  $\tilde{A}$  with a scalar  $k$  is another NGPtFN

$$k \odot \tilde{A} = \tilde{C} = (c_1, c_2, c_3, c_4, c_5; w''_1, w''_2).$$

Here the definition of  $\tilde{C}$  depends on the constant  $k$ .

Case1 :  $k \geq 0$  First construct:

$$l_A = \min\{x \mid \mu_{\tilde{A}}(x) \geq w''_1\} = a_2$$

$$r_A = \max\{x \mid \mu_{\tilde{A}}(x) \geq w''_2\} = a_4, \text{ and } w''_1 = w_1, w''_2 = w_2. \text{ Then:}$$

$$T_1 = \{k.a_1, k.a_5\}$$

$$T_2 = \{k.a_2, k.a_4\}$$

$$T_3 = a_3. \text{ Finally:}$$

$$c_1 = \min T_1 = k.a_1, c_2 = \min T_2 = k.a_2, c_3 = k.a_3, c_4 = \max T_2 = k.a_4, \text{ and } c_5 = \max T_1 = k.a_5.$$

Case2 :  $k < 0$  First construct:

$$l_A = \min\{x \mid \mu_{\tilde{A}}(x) \geq w''_1\} = a_2$$

$$r_A = \max\{x \mid \mu_{\tilde{A}}(x) \geq w''_2\} = a_4, \text{ and } w''_1 = w_1, w''_2 = w_2. \text{ Then:}$$

$$T_1 = \{k.a_1, k.a_5\}$$

$$T_2 = \{k.a_2, k.a_4\}$$

$$T_3 = a_3. \text{ Finally:}$$

$$c_1 = \min T_1 = k.a_5, c_2 = \min T_2 = k.a_4, c_3 = k.a_3, c_4 = \max T_2 = k.a_2, \text{ and } c_5 = \max T_1 = k.a_1.$$

### 4.3 Substraction of two NGPtFN's

#### 4.3.1 Substraction of two NGPtFN's based on extension principle

Let  $\tilde{A}(-)\tilde{B} = \tilde{C}$  where  $\tilde{C} = \tilde{A}(+)(-1)(\cdot)\tilde{B}$ . Let  $w''_1 = \min(w_1, w'_1)$ , and  $w''_2 = \min(w_2, w'_2)$ . Then the membership function  $\mu_{\tilde{C}}$  for  $\tilde{C}$  is defined as following:

$$\mu_{\tilde{C}}(z) = \begin{cases} \sup(\min(w_1 \frac{x-a_1}{a_2-a_1}, w'_1 \frac{y-b_1}{b_2-b_1}) | x-y=z), & a_1 \leq x \leq a_2, -b_5 \leq y \leq -b_4 \\ \sup(\min(w_1 + (1-w_1) \frac{x-a_2}{a_3-a_2}, w'_1 + (1-w'_1) \frac{y-b_2}{b_3-b_2}) | x-y=z), & a_2 \leq x \leq a_3, -b_4 \leq y \leq -b_3 \\ \sup(\min(1 + (w_2-1) \frac{x-a_3}{a_4-a_3}, 1 + (w'_2-1) \frac{y-b_3}{b_4-b_3}) | x-y=z), & a_3 \leq x \leq a_4, -b_3 \leq y \leq -b_2 \\ \sup(\min(w_2 - w_2 \frac{x-a_4}{a_5-a_4}, w'_2 - w'_2 \frac{y-b_4}{b_5-b_4}) | x-y=z), & a_4 \leq x \leq a_5, -b_2 \leq y \leq -b_1 \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Note that the membership function of  $\tilde{C}$  given in 7 is indeed equal to:

$$\mu_{\tilde{C}}(z) = \begin{cases} w''_1 \frac{z-a_1+b_5}{a_2-b_4-a_1+b_5}, & a_1-b_5 \leq z \leq a_2-b_4 \\ w''_1 + (1-w''_1) \frac{z-a_2+b_4}{a_3-b_3-a_2+b_4}, & a_2-b_4 \leq z \leq a_3-b_3 \\ 1 + (w''_2-1) \frac{z-a_3+b_3}{a_4-b_2-a_3+b_3}, & a_3-b_3 \leq z \leq a_4-b_2 \\ w''_2 - w''_2 \frac{z-a_4+b_2}{a_5-b_1-a_4+b_2}, & a_4-b_2 \leq z \leq a_5-b_1 \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

It should be noted that the subtraction of two NGPtFN's  $\tilde{A}$  and  $\tilde{B}$  is another NGPtFN  $\tilde{C} = (a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1; w''_1, w''_2)$  with the membership function given in 8.

[xtick=data, axis x line=center, xticklabels= $a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1$ , ytick=0,0.5,1,0.2,0, yticklabels=0, $w''_1, 1, w''_2$ , axis y line=left, enlargelimits=0.05] coordinates (1.5,0) (2.5,0.5) (3.5,1) (4.8,0.2) (6,0);

Figure 5: Rough sketch for  $\tilde{A}(-)\tilde{B}$

### 4.3.2 Substraction of two NGPtFN's based on function principle

Under the function principle the subtraction of NGPtF numbers  $\tilde{A}$  and  $\tilde{B}$  is another NGPtFN  $\tilde{A} \ominus \tilde{B} = \tilde{C} = (c_1, c_2, c_3, c_4, c_5; w''_1, w''_2)$ .

Here we first construct:

$$l_A = \min\{x | \mu_{\tilde{A}}(x) \geq w''_1\}$$

$$l_B = \min\{x | \mu_{\tilde{B}}(x) \geq w''_1\}$$

$$r_A = \max\{x | \mu_{\tilde{A}}(x) \geq w''_2\}$$

$$r_B = \max\{x | \mu_{\tilde{B}}(x) \geq w''_2\}, \text{ and } w''_1 = \min\{w_1, w'_1\}, w''_2 = \min\{w_2, w'_2\}. \text{ Then:}$$

$$T_1 = \{a_1 - b_1, a_1 - b_5, a_5 - b_1, a_5 - b_5\}$$

$$T_2 = \{l_A - l_B, l_A - r_B, r_A - l_A, r_A - r_B\}$$

$$T_3 = a_3 - b_3. \text{ Finally:}$$

$$c_1 = \min T_1, c_2 = \min T_2, c_3 = T_3, c_4 = \max T_2, \text{ and } c_5 = \max T_1.$$

### 4.4 Multiplication of two NGPtFN's

#### 4.4.1 Multiplication of two NGPtFN's based on extension principle

Let  $\tilde{A}(\cdot)\tilde{B} = \tilde{C}$  where  $\mu_{\tilde{C}}(z) = \sup(\min(\mu_{\tilde{A}_w}(x), \mu_{\tilde{B}_w}(y)) | x.y=z)$ .

Let  $w''_1 = \min(w_1, w'_1)$ , and  $w''_2 = \min(w_2, w'_2)$ . Then the membership function  $\mu_{\tilde{C}}$  for  $\tilde{C}$  is defined as following:

$$\mu_{\tilde{C}}(z) = \begin{cases} w_1'' \frac{z - a_1 - b_1}{a_2 + b_2 - a_1 - b_1}, & a_1 \leq x \leq a_2, a_1.b_1 \leq z \leq a_2.b_2 \\ w_1'' + (1 - w_1'') \frac{z - a_2 - b_2}{a_3 + b_3 - a_2 - b_2}, & a_2 \leq x \leq a_3, a_2.b_2 \leq z \leq a_3.b_3 \\ 1 + (w_2'' - 1) \frac{z - a_3 - b_3}{a_4 + b_4 - a_3 - b_3}, & a_3 \leq x \leq a_4, a_3.b_3 \leq z \leq a_4.b_4 \\ w_2'' - w_2'' \frac{z - a_4 - b_4}{a_5 + b_5 - a_4 - b_4}, & a_4 \leq x \leq a_5, a_4.b_4 \leq z \leq a_5.b_5 \\ 0, & otherwise \end{cases} \tag{9}$$

Note that the membership function of  $\tilde{C}$  given in 9 is indeed equal to:

$$\mu_{\tilde{C}}(z) = \begin{cases} \sup(\min(w_1 \frac{x - a_1}{a_2 - a_1}, w_1' \frac{y - b_1}{b_2 - b_1}) | x.y = z), & a_1 \leq x \leq a_2, b_1 \leq y \leq b_2 \\ \sup(\min(w_1 + (1 - w_1) \frac{x - a_2}{a_3 - a_2}, w_1' + (1 - w_1') \frac{y - b_2}{b_3 - b_2}) | x.y = z), & a_2 \leq x \leq a_3, b_2 \leq y \leq b_3 \\ \sup(\min(1 + (w_2 - 1) \frac{x - a_3}{a_4 - a_3}, 1 + (w_2' - 1) \frac{y - b_3}{b_4 - b_3}) | x.y = z), & a_3 \leq x \leq a_4, b_3 \leq y \leq b_4 \\ \sup(\min(w_2 - w_2 \frac{x - a_4}{a_5 - a_4}, w_2' - w_2' \frac{y - b_4}{b_5 - b_4}) | x.y = z), & a_4 \leq x \leq a_5, b_4 \leq y \leq b_5 \\ 0, & otherwise \end{cases} \tag{10}$$

And it should be noted that this result is not a NGPtFN anymore, but a fuzzy number with drum-like shaped discrete membership function.

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[ xtick=data, axis x line=center, xticklabels=c<sub>1</sub>,c<sub>2</sub>,c<sub>3</sub>,c<sub>4</sub>,c<sub>5</sub>, ytick=0,0.7,1,0.3,0, yticklabels= 0,w<sub>1</sub>'',1,w<sub>2</sub>'', axis y line=left, enlargelimits=0.05] +[smooth]coordinates (1,0) (2,0.7) (3,7,1) (4,5,0.3) (6,5,0);

Figure 6: Rough sketch for  $\tilde{A}(\cdot)\tilde{B}$

#### 4.4.2 Multiplication of two NGPtFN's based on function principle

Under the function principle the multiplication of NGPtF numbers  $\tilde{A}$  and  $\tilde{B}$  is another NGPtFN  $\tilde{A} \otimes \tilde{B} = \tilde{C} = (c_1, c_2, c_3, c_4, c_5; w_1'', w_2'')$ . Here we first construct:

$$l_A = \min\{x | \mu_{\tilde{A}}(x) \geq w_1''\}$$

$$l_B = \min\{x | \mu_{\tilde{B}}(x) \geq w_1''\}$$

$$r_A = \max\{x | \mu_{\tilde{A}}(x) \geq w_2''\}$$

$$r_B = \max\{x | \mu_{\tilde{B}}(x) \geq w_2''\}, \text{ and } w_1' = \min\{w_1, w_1'\}, w_2' = \min\{w_2, w_2'\}. \text{ Then:}$$

$$T_1 = \{a_1.b_1, a_1.b_5, a_5.b_1, a_5.b_5\}$$

$$T_2 = \{l_A.l_B, l_A.r_B, r_A.l_B, r_A.r_B\}$$

$$T_3 = a_3.b_3. \text{ Finally:}$$

$$c_1 = \min T_1, c_2 = \min T_2, c_3 = T_3, c_4 = \max T_2, \text{ and } c_5 = \max T_1.$$

#### 4.5 Division of two NGPtFN's

##### 4.5.1 Division of two NGPtFN's based on extension principle

Let  $\tilde{A}(\div)\tilde{B} = \tilde{C}$  where  $\mu_{\tilde{C}}(z) = \sup(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) | x \div y = z)$ .

Let  $w_1' = \min(w_1, w_1')$ , and  $w_2' = \min(w_2, w_2')$ . Then the membership function  $\mu_{\tilde{C}}$  for  $\tilde{C}$  is defined as following:

[xtick=data, axis x line=center, xticklabels= $c_1, c_2, c_3, c_4, c_5$ , ytick=0,0.4,1,0.8,0, yticklabels=0,  $w_1'', 1, w_2''$ , axis y line=center, enlargelimits=0.05] +[smooth]coordinates (1,0) (2.5,0.4) (3.7,1) (4.5,0.8) (6,0);

Figure 7: Rough sketch for  $\tilde{A}(\div)\tilde{B}$

$$\mu_{\tilde{C}}(z) = \begin{cases} w_1'' \frac{z - \frac{a_1}{b_5}}{\frac{a_2}{b_4} - \frac{a_1}{b_5}}, & a_1 \leq x \leq a_2, \frac{a_1}{b_5} \leq z \leq \frac{a_2}{b_4} \\ w_1'' + (1 - w_1'') \frac{z - \frac{a_2}{b_4}}{\frac{a_3}{b_3} - \frac{a_2}{b_4}}, & a_2 \leq x \leq a_3, \frac{a_2}{b_4} \leq z \leq \frac{a_3}{b_3} \\ 1 + (w_2'' - 1) \frac{z - \frac{a_3}{b_3}}{\frac{a_4}{b_2} - \frac{a_3}{b_3}}, & a_3 \leq x \leq a_4, \frac{a_3}{b_3} \leq z \leq \frac{a_4}{b_2} \\ w_2'' - w_2'' \frac{z - \frac{a_4}{b_2}}{\frac{a_5}{b_1} - \frac{a_4}{b_2}}, & a_4 \leq x \leq a_5, \frac{a_4}{b_2} \leq z \leq \frac{a_5}{b_1} \\ 0, & otherwise \end{cases} \tag{11}$$

Note that the membership function of  $\tilde{C}$  given in 11 is indeed equal to:

$$\mu_{\tilde{C}}(z) = \begin{cases} \sup(\min(w_1 \frac{x - a_1}{a_2 - a_1}, w_2' - w_2' \frac{y - b_4}{b_5 - b_4}) | \frac{x}{y} = z), & a_1 \leq x \leq a_2, b_4 \leq y \leq b_5 \\ \sup(\min(w_1 + (1 - w_1) \frac{x - a_2}{a_3 - a_2}, 1 + (w_2' - 1) \frac{y - b_3}{b_4 - b_3}) | \frac{x}{y} = z), & a_2 \leq x \leq a_3, b_3 \leq y \leq b_4 \\ \sup(\min(1 + (w_2 - 1) \frac{x - a_3}{a_4 - a_3}, w_1' + (1 - w_1') \frac{y - b_2}{b_3 - b_2}) | \frac{x}{y} = z), & a_3 \leq x \leq a_4, b_2 \leq y \leq b_3 \\ \sup(\min(w_2 - w_2 \frac{x - a_4}{a_5 - a_4}, w_1' \frac{y - b_1}{b_2 - b_1}) | \frac{x}{y} = z), & a_4 \leq x \leq a_5, b_1 \leq y \leq b_2 \\ 0, & otherwise \end{cases} \tag{12}$$

And it should be noted that this result is not a NGPtFN anymore, but a fuzzy number with drum-like shaped discrete membership function.

**4.5.2 Division of two NGPtFN's based on function principle**

Under the function principle the division of NGPtF numbers  $\tilde{A}$  and  $\tilde{B}$  is another NGPtFN  $\tilde{A} \oslash \tilde{B} = \tilde{C} = (c_1, c_2, c_3, c_4, c_5; w_1'', w_2'')$ . Here we first construct:

$$\begin{aligned} l_A &= \min\{x \mid \mu_{\tilde{A}}(x) \geq w_1''\} \\ l_B &= \min\{x \mid \mu_{\tilde{B}}(x) \geq w_1''\} \\ r_A &= \max\{x \mid \mu_{\tilde{A}}(x) \geq w_2''\} \\ r_B &= \max\{x \mid \mu_{\tilde{B}}(x) \geq w_2''\}, \text{ and } w_1' = \min\{w_1, w_1'\}, w_2' = \min\{w_2, w_2'\}. \text{ Then:} \\ T_1 &= \left\{ \frac{a_1}{b_1}, \frac{a_1}{b_5}, \frac{a_5}{b_1}, \frac{a_5}{b_5} \right\} \\ T_2 &= \left\{ \frac{l_A}{l_B}, \frac{l_A}{r_B}, \frac{r_A}{l_B}, \frac{r_A}{r_B} \right\} \\ T_3 &= \frac{a_3}{b_3}. \text{ Finally:} \\ c_1 &= \min T_1, c_2 = \min T_2, c_3 = T_3, c_4 = \max T_2, \text{ and } c_5 = \max T_1. \end{aligned}$$

**5 Remarks and Conclusion**

In this study we proposed GPtFN's and arithmetic operations for NGPtFN's based on both extension and function principles and while defining the operations we became curious of the main differences of these two principles.

First of all, we can conclude that fuzzy arithmetic operations of NGPtFN's are easier and more simple under function principle than the extension principle. Moreover the multiplication of two NGPtFN's membership function, under the extension principle has a new membership function with a two-sided parabolic drum-like shaped graph(see Figure 6), whereas the multiplication of two NGPtFN's, under the function principle is again a NGPtFN. And a similar conclusion can be make for the division of two NGPtFN's(see Figure 7). Therefore

we can conclude that in different fields of social sciences and in several engineering disciplines, we can use more complex fuzzy numbers such as NGPtFN's in order to evaluate utilities and other attributes, and the operations defined in this study will give the decision makers much convenience.

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