Intuitionistic Possibility Fermatean Fuzzy Soft Sets

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Abstract

In this study, we introduce a new concept by making Possibility Fermatean fuzzy soft sets into a more general concept, namely Intuitionistic Possibility Fermatean fuzzy soft sets. We present examples of the application of this theory to a decision-making problem. From a theoretical point of view, we review the basic properties of this model and define the operations essential to its framework. Comprehensive definitions of complement, union, and intersection, as well as AND and OR operations are meticulously presented. As a transition from theory to practical application within this innovative context, we present an algorithm for solving decision-making problems, contributing to the practical implementation of this extended concept. This research aims to improve our understanding of the intuitionistic possibility of Fermatean fuzzy soft sets and to bridge the gap between theoretical advances and their real-world utility in decision-making problems.

Keywords: Fuzzy Set, Soft Set; Fuzzy Soft Set; Fermatean Fuzzy Set; Fermatean Fuzzy Soft Set; Intuitionistic Fuzzy Soft Set; Possibility Fuzzy Soft Set; Intuitionistic Possibility Fermatean Fuzzy Soft Set.

1. Introduction

Zadeh [1] 1965 presented the fuzzy set (FS) concept as a mathematical tool to handle vagueness and ambiguity in real-world data where the concept of decision-making (DM) problems under uncertainty was introduced by Bellman and Zadeh [2] in 1970. Atanassov [3] as a generalization of Zadeh's concept, to address complexity and uncertainty by adding the non-membership to the fuzzy set introduced the concept of intuitionistic fuzzy sets (IFSs). Many researchers studied and applied this idea in the field of decision-making (DM) problems. For more information about FS, and IFS, see [4-7]. As an extension of intuitionistic fuzzy sets (IFS) theory Yager [8] introduced Pythagorean fuzzy set (PyFS) by characterizing membership value and non-membership value, with the condition that the square sum of these values is less than or equal to 1.

Molodtsov [9] 1999 introduced a soft set as a novel approach, wherein distinct parameters were assigned preferences for each alternative, effectively addressing the expressed limitations. After that, fuzzy soft sets [10] and intuitionistic fuzzy soft sets (IFSS) [11] were formulated, and their diverse properties and applications were investigated [12]. As a natural generalization of IFSS Pythagorean fuzzy soft set (PyFSS) defined by Peng et al. [13], which represents a parameterized family of Pythagorean Fuzzy Sets (PFSs). Smarandache [14] introduced the concept of neutrosophic which is a generalization of intuitionistic set and gives many applications of this concept in many fields in real-life problems. Many applications of PyFSS were applied in medical diagnosis, DM, for more information see [15-26]. Majumdar [27] in 2010, introduced the concept of generalized fuzzy soft sets (GFSS) and studied some of its properties, and applied this concept to decision-making problems and medical diagnosis.

Baesho [28] in the same year, used the Majumdar concept to introduce the concept of generalized intuitionistic fuzzy soft sets, where degrees are linked to the parameterization of fuzzy sets in the definition of an intuitionistic fuzzy soft set. The novel concept possibility of fuzzy soft sets (PFSS) introduced by Alkhazaleh et al [29] in 2011, which is a more realistic concept of the generalized fuzzy soft set where degrees are linked to each element in the universe during the parameterization of fuzzy sets within the definition of a fuzzy soft set. Bashir et al [30] 2012 expanded this concept to the possibility of intuitionistic fuzzy soft sets. They assigned a possibility value to each

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element in the universe during the parameterization of fuzzy sets within the definition of an intuitionistic fuzzy soft set. Also, they applied this concept in addressing decision-making problems and medical diagnosis. Fermatean fuzzy set (FFS) was introduced by Senapati and Yager [31].

In this paper, they compared Fermatean fuzzy sets with Pythagorean fuzzy sets and intuitionistic fuzzy sets. They also defined a complement operator of Fermatean fuzzy sets and introduced its fundamental set of operations and defined a score function, accuracy function, and Euclidean distance between two Fermatean fuzzy sets. A Fermatean fuzzy TOPSIS method was established to fix multiple criteria decision-making problems. Sivadas et al. [32] presented the concept of Fermatean fuzzy soft sets as a hybrid structure which includes the characteristics of Fermatean fuzzy set along with the parameterization of soft set. They defined a few basic operations such as union, intersection, and complement. An algorithm to solve the decision-making problem is developed using the aggregation operators defined for this structure. In 2023, Dliouah et al. [33] introduced a possible Fermatean fuzzy soft set (PFFSS) as a new multiple attribute decision-making model which is a combination of the possibility fuzzy soft sets by Alkhazaleh et al and Fermatean fuzzy sets by Senapati and Yager. They studied some operations and properties of this model, including complement, restricted union, and extended intersection are discussed. This model was applied for multiple attribute decision-making and solved with the help of a newly launched algorithm.

2. Preliminary

In this section, we will cover certain definitions and features of Intuitionistic Possibility Fermatean fuzzy soft sets.

Definition 1: Fuzzy Set

Let X be a non-empty set. A fuzzy set A in X is characterized by its membership function $\mu_A : X \to [0,1]$ and $\mu_A(X)$ is interpreted as the degree of membership of element X in fuzzy set A, for each $x \in X$. It is clear that A is completely determined by the set of tuples $A = \{(x, \mu_A(x)) : x \in X\}$.

Definition 2: Intuitionistic fuzzy sets

The intuitionistic fuzzy sets defined on a non-empty set X as objects having the form $A = \{\langle x, \alpha_A(\mathrm{x}), \beta_A(x)\rangle : x \in X\}$, where the functions $\alpha_A : X \to [0,1]$ and $\beta_A : X \to [0,1]$, denote the degree of membership and the degree of non-membership of each element to the set A respectively, and $0 \le \alpha_A(x) + \beta_A(x) \le 2$, for all $x \in X$. Clearly, when $\beta_A(x) = 1 - \alpha_A(x)$, for every $x \in X$, the set A becomes a fuzzy set.

Definition 3: Soft set

Let U be the universal set and E be the set of attributes with respect to U. Let $P(U)$ be the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U and its mapping is given as $F : A \to P(U)$. It is also defined as $(F, A) = \{F(e) \in P(U) : e \in E, F(e) = \varnothing \; if \; e \notin A\}$.

Definition 4: Fuzzy Soft Set

Let U be the initial universal set and let E be the set of parameters. Let I^U denote the collection of all fuzzy subsets of U. Let $A \subseteq E$. A pair (F, E) is called a fuzzy soft set over U where F is a mapping given by: $F : A \to I^U$.

Definition 5: Generalized Fuzzy Soft Set

Let $U = \{u_1, u_2, ..., u_n\}$ be a universal set of elements and $E = \{e_1, e_2, ..., e_m\}$ be a set of parameters. The pair (U, E) is called a soft universe. Let $F : E \to I^U$ and μ be a fuzzy subset of E, *i.e,* $\mu : E \to I = [0,1]$, where I^U is the collection of all Fuzzy subset of U. Let F_μ be the mapping $F : E \to I^U \times I$ be a function defined as follows: $F_\mu(e) = (F(e), \mu(e))$, where $F_e \in I^U$. Then F_μ is called a Generalized fuzzy soft set

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(GFSS in short) over the soft universe (U,E). Here for each parameter $e_i, F_\mu(e_i)=(F(e_i),\mu(e_i))$, indicates not only the degree of belongingness of the elements of U in $F(e_i)$, also the degree of possibility of such belongingness which is represented by μ_i. So, we can write this as follows:

$$F_\mu(e_i)=\left\{\left(\left(\frac{u_1}{F(e_i)(u_1)},\frac{u_2}{F(e_i)(u_2)},.....,\frac{u_n}{F(e_i)(u_n)}\right),\mu(e_i)\right)\right\},\forall u\in U,e\in E$$

Definition 6: Intuitionistic Fuzzy soft set

Consider U and E as universal sets and a set of parameters, respectively. Let $P(U)$ denotes the set of all Intuitionistic Fuzzy sets of U. Let $A\subseteq E$.A pair (F,A) is an intuitionistic fuzzy soft set over U. where F is a mapping given by $F:A\to P(U)$.

Definition 7: Possibility Fuzzy Soft Set

Let $U=\{u_1,u_2,...,u_n\}$ be the universal set of elements and $E=\{e_1,e_2,...,e_m\}$ be the universal set of parameters. The pair (U,E) is called a soft universe. Let $F:E\to I^U$ and μ be a fuzzy subset of E, *i.e*, $\mu:E\to I^U=[0,1]$, where I^U is the collection of all Fuzzy subset of U. Let F_μ be the mapping $F:E\to I^U\times I^U$ be a function defined as follows: $F_\mu(e)=(F(e)(u),\mu(e)(u))$, $\forall u\in U$. Then F_μ is called a Possibility fuzzy soft set (PFSS in short) over the soft universe (U,E). Here for each parameter $e_i,F_\mu(e_i)=(F(e_i),\mu(e_i))$, indicates not only the degree of belongingness of the elements of U in $F(e_i)$,but also the degree of possibility of such belongingness which is represented by μ_i. So, we can write this as follows:

$$F_\mu=\left\{\left(e,\left(\frac{u}{F(e)(u)},\mu(e)(u)\right)\right)\right\},\forall u\in U,e\in E\cdot$$

Definition 8: Fermatean Fuzzy set

Fermatean fuzzy set F in the universe set U is an object with the type $F=\{(u,\alpha F(u),\beta F(u)):u\in U\}$ where $\alpha F:U\to[0,1]$ and $\beta F:U\to[0,1]$, with the condition $0\le(\alpha F(u))^3+(\beta F(u))^3\le 1$ for all $u\in U$.

Definition 9: Fermatean fuzzy soft set

Let E be any set of deferent parameters, and let U be the universe, $A\subseteq E$ a Fermatean fuzzy soft set (FFSS) on U is defined as the pair (F,W) where F is mapping given by $F:W\to FFS(U)$, where $FFS(U)$ is the set of all Fermatean fuzzy sets over U. Here for any parameter $e\in A, F(e)$ is the Fermatean fuzzy set given as $F(e)=\{(u,\alpha F(e)(u),\ \beta F(e)(u)):u\in U\}$ where $\alpha F(e)(u)$ and $\beta F(e)(u)$ are corresponding degrees of membership and non-membership $0\le(\alpha F(e)(u))^3+(\beta F(e)(u))^3\le 1$. Hence

$$(F,A)=\{(e,\{(u,\alpha F(e)(u),\beta F(e)(u)\}):e\in A,u\in U\}$$

Definition 10: Possibility Fermatean Fuzzy soft set

Let U be a universal set and E be the set of parameters. For (U,E) being the soft universe, consider the maps $G:E\to FF^U$ and $P:E\to FS^U$. Then, for an arbitrary $B\subset E$, a pair (G_P,B) is referred to as a

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possibility Fermatean fuzzy soft set (or PFFSS) over soft universe (U,E), where a function $G_P : B \to FF^U \times FS^U$ is defined as below:

$$G_P(e) = \left\{ \left(e, \left(\frac{q}{G(e)(q)}, p(e)(q) \right) \right) \right\}, \forall q \in U, e \in E .$$

3. Fundamental of Intuitionistic Possibility Fermatean Neutrosophic Soft Set

In this section, we introduce a new concept by using the Possibility Fermatean fuzzy soft set as introduced by Dliouah et al. (2023), to the more general concept which is Intuitionistic Possibility Fermatean fuzzy soft sets.

Definition 11: Intuitionistic Possibility Fermatean Neutrosophic Soft Set

Let $U = \{c_1, c_2, ..., c_n\}$ be a set of universes and $E = \{e_1, \mathrm{e}_2, ..., e_m\}$ be a set of parameters. Define $F : E \to FE(\mathrm{U})$ where $FE(\mathrm{U})$ is the collection of all Fermatean fuzzy subsets of U. Let μ be an intuitionistic fuzzy subset of U, that is, $\mu : E \to IN(U)$ where $IN(U)$ is the collection of all intuitionistic fuzzy subsets of U and let $F_\mu : E \to FE(U) \times IN(U)$ be a function defined as follows:

$F_\mu(e) = (F(\mathrm{e})(\mathrm{x}), \mu(\mathrm{e})(\mathrm{x}))$, where $F(\mathrm{e})(\mathrm{x}) = (\alpha(x), \beta(x))$ and $\mu(\mathrm{e})(\mathrm{x}) = (\lambda(x), \nu(x))$ $\forall x \in U$.

Including the conditions: $0 \le (\alpha(x))^3 + (\beta(x))^3 \le 1$ and $0 \le (\lambda(x)) + (\nu(x)) \le 2$.

Then F_μ is called an Intuitionistic Possibility Fermatean fuzzy soft set (IPFFSS in short) over the soft universe (U,E). For each $e \in E$, $F_\mu(e) = (F(\mathrm{e})(\mathrm{x}), \mu(\mathrm{e})(\mathrm{x}))$ indicates not only the degree of belongingness of the elements of U in $F(e_i)$, but also the degree of possibility of belongingness of the elements U of in $F(e_i)$, which is represented by $\mu(e_i)$. So we can write $F_\mu(e_i)$ as follows:

$$F_\mu(e_i)(x) = \left\{ \left(\frac{x}{F(e_i)(x)}, \mu(e_i)(x) \right), \forall x \in U \right\} .$$

Example 1: Let $U = \{c_1, c_2, c_3\}$ be a universe set. Let $E = \{e_1, e_2, e_3\}$ be a set of parameters, and let $\mu : E \to IN(U)$. We define a function $F_\mu : E \to FE(U) \times IN(U)$ as follows:

$$F_\mu(e_1) = \left\{ \left(\frac{c_1}{(0.5, 0.3)}, (0.7, 0.1) \right), \left(\frac{c_2}{(0.8, 0.1)}, (0.5, 0.2) \right), \left(\frac{c_3}{(0.6, 0.2)}, (0.6, 0.3) \right) \right\},$$

$$F_\mu(e_2) = \left\{ \left(\frac{c_1}{(0.7, 0.2)}, (0.5, 0.3) \right), \left(\frac{c_2}{(0.6, 0.2)}, (0.6, 0) \right), \left(\frac{c_3}{(0.5, 0.1)}, (0.8, 0.1) \right) \right\},$$

$$F_\mu(e_3) = \left\{ \left(\frac{c_1}{(0.9, 0)}, (0.6, 0.2) \right), \left(\frac{c_2}{(0.8, 0.1)}, (0.7, 0.1) \right), \left(\frac{c_3}{(0.7, 0.2)}, (0.5, 0.1) \right) \right\}.$$

Then F_μ is an IPFFSS over (U,E). In matrix notation we write

$$F_\mu = \begin{pmatrix} (0.5, 0.3), (0.7, 0.1) & (0.8, 0.1), (0.5, 0.2) & (0.6, 0.2), (0.6, 0.3) \\ (0.7, 0.2), (0.5, 0.3) & (0.6, 0.2), (0.6, 0) & (0.5, 0.1), (0.8, 0.1) \\ (0.9, 0), (0.6, 0.2) & (0.8, 0.1), (0.7, 0.1) & (0.7, 0.2), (0.5, 0.1) \end{pmatrix} .$$

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Definition 12: Let F_μ and G_ρ be two IPFFSSs over (U,E). F_μ is said to be an intuitionistic possibility Fermatean fuzzy soft subset (IPFFS subset) of G_ρ and one writes $F_\mu \subseteq G_\rho$ if:

i. $\mu(e)$ is an intuitionistic fuzzy subset of $\rho(e)$, for all $e \in E$,
ii. $F(e)$ is a Fermatean fuzzy soft subset of $G(e)$, for all $e \in E$.

Example 2: Let $U=\{x_1,x_2,x_3\}$ be a universe set and $E=\{e_1,e_2,e_3\}$ be a set of parameters. Let F_δ and G_ρ be functions defined as follows:

$$F_\delta(e_1)=\left\{\left(\frac{x_1}{(0.5,0.3)},(0.7,0.1)\right),\left(\frac{x_2}{(0.8,0.1)},(0.5,0.2)\right),\left(\frac{x_3}{(0.6,0.2)},(0.6,0.3)\right)\right\},$$
$$F_\delta(e_2)=\left\{\left(\frac{x_1}{(0.7,0.2)},(0.5,0.3)\right),\left(\frac{x_2}{(0.6,0.2)},(0.6,0.1)\right),\left(\frac{x_3}{(0.5,0.1)},(0.8,0.2)\right)\right\},$$
$$F_\delta(e_3)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.4,0.2)\right),\left(\frac{x_2}{(0.4,0.3)},(0.7,0.1)\right),\left(\frac{x_3}{(0.7,0.2)},(0.5,0.3)\right)\right\}.$$
$$G_\rho(e_1)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.8,0)\right),\left(\frac{x_2}{(0.9,0)},(0.6,0.1)\right),\left(\frac{x_3}{(0.7,0.1)},(0.7,0.2)\right)\right\},$$
$$G_\rho(e_2)=\left\{\left(\frac{x_1}{(0.8,0.1)},(0.6,0.2)\right),\left(\frac{x_2}{(0.7,0.1)},(0.7,0)\right),\left(\frac{x_3}{(0.6,0)},(0.9,0.1)\right)\right\},$$
$$G_\rho(e_3)=\left\{\left(\frac{x_1}{(0.7,0.1)},(0.5,0.1)\right),\left(\frac{x_2}{(0.5,0.2)},(0.8,0)\right),\left(\frac{x_3}{(0.8,0.1)},(0.6,0.2)\right)\right\}.$$

It is clear that F_δ is an IPFFS subset of G_ρ .

Definition 13: Let F_μ and G_ρ be two IPFFSSs over (U,E). F_μ is said to be equal and one writes $F_\mu = G_\rho$ if :

i. $\mu(e)$ is equal to $\rho(e)$, for all $e \in E$,
ii. $F(e)$ is equal to $G(e)$, for all $e \in E$.

Definition 14: An IPFFSS is said to be a null intuitionistic possibility Fermatean fuzzy soft set, denoted by N, if $\mathrm{N}: E \to FE(U)\times IN(U)$ such that: $\mathrm{N}(e)=\left(F(e)(x),\mu(e)(x)\right)$, $\forall e \in E$, Where $F(\mathrm{e})=(0,1)$, and $\mu(\mathrm{e})=(0,1)$, $\forall e \in E$.

Example 3: Let $U=\{c_1,c_2,c_3\}$ be a universe set. Let $E=\{e_1,e_2,e_3\}$ be a set of parameters. Let F_μ defined as follows:

$$F_\mu(e_1)=\left\{\left(\frac{c_1}{(0,1)},(0,1)\right),\left(\frac{c_2}{(0,1)},(0,1)\right),\left(\frac{c_3}{(0,1)},(0,1)\right)\right\},$$
$$F_\mu(e_2)=\left\{\left(\frac{c_1}{(0,1)},(0,1)\right),\left(\frac{c_2}{(0,1)},(0,1)\right),\left(\frac{c_3}{(0,1)},(0,1)\right)\right\},$$
$$F_\mu(e_3)=\left\{\left(\frac{c_1}{(0,1)},(0,1)\right),\left(\frac{c_2}{(0,1)},(0,1)\right),\left(\frac{c_3}{(0,1)},(0,1)\right)\right\}.$$

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Then F_μ is a null intuitionistic possibility Fermatean fuzzy soft set.

Definition 15: An IPFFSS is said to be an absolute intuitionistic possibility Fermatean fuzzy soft set, denoted by A, if $\mathrm{A}: E \to FE(U)\times IN(U)$ such that: $\mathrm{A}(e)=\left(F(e)(x),\mu(e)(x)\right)$, $\forall e\in E$, Where $F(\mathrm{e})=\left(1,0\right)$, and $\mu(\mathrm{e})=\left(1,0\right)$, $\forall e\in E$.

Example 4: Consider Example 2 then by using an absolute intuitionistic possibility Fermatean fuzzy soft set, we have F_μ as follows:

$$F_\mu(e_1)=\left\{\left(\frac{c_1}{(1,0)},(1,0)\right),\left(\frac{c_2}{(1,0)},(1,0)\right),\left(\frac{c_3}{(1,0)},(1,0)\right)\right\},$$
$$F_\mu(e_2)=\left\{\left(\frac{c_1}{(1,0)},(1,0)\right),\left(\frac{c_2}{(1,0)},(1,0)\right),\left(\frac{c_3}{(1,0)},(1,0)\right)\right\},$$
$$F_\mu(e_3)=\left\{\left(\frac{c_1}{(1,0)},(1,0)\right),\left(\frac{c_2}{(1,0)},(1,0)\right),\left(\frac{c_3}{(1,0)},(1,0)\right)\right\}.$$

Definition 16: Let F_δ be an IPFFSS over $\left(U,E\right)$. Then the complement of F_δ, denoted by F_δ^c, is defined by $F_\delta^c=G_\rho$ such that $\rho(e)=(\delta(\mathrm{e}))^c$ and $G(e)=(F(e))^c$, $\forall e\in E$, where c is an intuitionistic fuzzy complement and $G(e)$ be the Fermatean fuzzy soft complement.

Example 5: Consider Example 2 then by using the basic intuitionistic fuzzy complement and Fermatean fuzzy soft complement, we have $F_\mu^c=G_\rho$ as follows

$$G_\rho=\begin{pmatrix}(0.3,0.5),(0.1,0.7) & (0.1,0.8),(0.2,0.5) & (0.2,0.6),(0.3,0.6)\\ (0.2,0.7),(0.3,0.5) & (0.2,0.6),(0,0.6) & (0.1,0.5),(0.1,0.8)\\ (0,0.9),(0.2,0.6) & (0.1,0.8),(0.1,0.7) & (0.2,0.7),(0.1,0.5)\end{pmatrix}.$$

4. Union and Intersection of IPFFSSs

In this section, we study the operations on IPFFSSs by defining the concepts of union and intersection of IPFFSSs and derive some of their properties, and give some examples.

Definition 17: Suppose F_δ and G_ρ are two IPFFSSs over *U*. The union of F_δ and G_ρ, denoted by $F_\delta\cup G_\rho$, is an IPFFSS $H_\theta: E\to FE(U)\times IN(U)$ defined by:

$H_\theta(e)=((\mathrm{H(e)}(x)),(\theta(e)(x))$, $\forall e\in E$, such that $H(e)=(F(e)\cup G(e))$ and $\theta(e)=S(\mu(e),\rho(e))$, where S is an intuitionistic fuzzy S-norm and $\cup$ is a Fermatean fuzzy soft union

$$F\cup G=\left\{\left(x,\max\left(T_F(x),T_G(x)\right),\frac{I_F(x)+I_G(x)}{2},\min\left(F_F(x),F_G(x)\right)\right\}.$$

Example 6: Suppose $U=\{x_1,x_2,x_3\}$ is a set of universes and $E=\{e_1,e_2,e_3\}$ a set of parameters.

Defined F_δ and G_ρ as follows:

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$$F_{\delta}(e_1)=\left\{\left(\frac{x_1}{(0.5,0.3)},(0.7,0.2)\right),\left(\frac{x_2}{(0.8,0.1)},(0.5,0.2)\right),\left(\frac{x_3}{(0.6,0.2)},(0.6,0.3)\right)\right\},$$

$$F_{\delta}(e_2)=\left\{\left(\frac{x_1}{(0.7,0.2)},(0.5,0.3)\right),\left(\frac{x_2}{(0.6,0.2)},(0.6,0.1)\right),\left(\frac{x_3}{(0.5,0.1)},(0.8,0.2)\right)\right\},$$

$$F_{\delta}(e_3)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.4,0.2)\right),\left(\frac{x_2}{(0.4,0.3)},(0.7,0.1)\right),\left(\frac{x_3}{(0.7,0.2)},(0.5,0.3)\right)\right\}.$$

$$\mathrm{G}_{\rho}(e_1)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.8,0.1)\right),\left(\frac{x_2}{(0.9,0)},(0.6,0.1)\right),\left(\frac{x_3}{(0.7,0.1)},(0.7,0.2)\right)\right\},$$

$$\mathrm{G}_{\rho}(e_2)=\left\{\left(\frac{x_1}{(0.5,0.1)},(0.6,0.2)\right),\left(\frac{x_2}{(0.7,0.3)},(0.5,0.3)\right),\left(\frac{x_3}{(0.4,0.5)},(0.9,0.1)\right)\right\},$$

$$\mathrm{G}_{\rho}(e_3)=\left\{\left(\frac{x_1}{(0.4,0.3)},(0.5,0.1)\right),\left(\frac{x_2}{(0.7,0.2)},(0.6,0.2)\right),\left(\frac{x_3}{(0.8,0.1)},(0.3,0.5)\right)\right\}.$$

By using the basic intuitionistic fuzzy union and Fermatean fuzzy soft union we have

$$\mathrm{H}_{\theta}(e_1)=\left\{\begin{array}{l}\left(\frac{x_1}{\max(0.5,0.6),\min(0.3,0.2)},\max(0.7,0.8),\min(0.2,0.1)\right),\\ \left(\frac{x_2}{\max(0.8,0.9),\min(0.1,0)},\max(0.5,0.6),\min(0.2,0.1)\right),\\ \left(\frac{x_3}{\max(0.6,0.7),\min(0.2,0.1)},\max(0.6,0.7)\min(0.3,0.2)\right)\end{array}\right\}$$

$$\mathrm{H}_{\theta}(e_1)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.8,0.1)\right),\left(\frac{x_2}{(0.9,0)},(0.6,0.1)\right),\left(\frac{x_3}{(0.7,0.1)},(0.7,0.2)\right)\right\}$$

Similarly, we get:

$$\mathrm{H}_{\theta}(e_2)=\left\{\left(\frac{x_1}{(0.7,0.1)},(0.6,0.2)\right),\left(\frac{x_2}{(0.7,0.2)},(0.6,0.1)\right),\left(\frac{x_3}{(0.5,0.1)},(0.9,0.1)\right)\right\},$$

$$\mathrm{H}_{\theta}(e_3)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.5,0.1)\right),\left(\frac{x_2}{(0.7,0.2)},(0.7,0.1)\right),\left(\frac{x_3}{(0.8,0.1)},(0.5,0.3)\right)\right\}.$$

Definition 17: Suppose F_{μ} and G_{ρ} are two IPFFSSs over *U*. The intersection of F_{μ} and G_{ρ}, denoted by $F_{\mu}\cap G_{\rho}$, is an IPFFSS $\mathrm{H}_{\theta}:E\rightarrow FE(U)\times \mathrm{IN}(U)$ defined by: $H_{\theta}(e)=((\mathrm{H(e)}(x)),(\theta(e)(x))$, $\forall e\in E$, such that $H(e)=(F(e)\cap G(e))$ and $\theta(e)=T(\mu(e),\rho(e))$,where T is an intuitionistic fuzzy T-norm and $\cap$ is a Fermatean fuzzy soft intersection.

Example 7 : Suppose $U=\{c_1,c_2,c_3\}$ is a set of universes and $E=\{e_1,e_2,e_3\}$ a set of parameters. Let F_{μ} and G_{ρ} be functions defined as follows:

$$F_{\mu}(e_1)=\left\{\left(\frac{c_1}{(0.5,0.3)},(0.7,0.2)\right),\left(\frac{c_2}{(0.8,0.1)},(0.5,0.2)\right),\left(\frac{c_3}{(0.6,0.2)},(0.6,0.3)\right)\right\},$$

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$$F_{\mu}(e_2)=\left\{\left(\frac{c_1}{(0.7,0.2)},(0.5,0.3)\right),\left(\frac{c_2}{(0.6,0.2)},(0.6,0.1)\right),\left(\frac{c_3}{(0.5,0.1)},(0.8,0.2)\right)\right\},$$

$$F_{\mu}(e_3)=\left\{\left(\frac{c_1}{(0.6,0.2)},(0.4,0.2)\right),\left(\frac{c_2}{(0.4,0.3)},(0.7,0.1)\right),\left(\frac{c_3}{(0.7,0.2)},(0.5,0.3)\right)\right\}.$$

$$G_{\rho}(e_1)=\left\{\left(\frac{c_1}{(0.6,0.2)},(0.8,0.1)\right),\left(\frac{c_2}{(0.9,0)},(0.6,0.1)\right),\left(\frac{c_3}{(0.7,0.1)},(0.7,0.2)\right)\right\},$$

$$G_{\rho}(e_2)=\left\{\left(\frac{c_1}{(0.5,0.1)},(0.6,0.2)\right),\left(\frac{c_2}{(0.7,0.3)},(0.5,0.3)\right),\left(\frac{c_3}{(0.4,0.5)},(0.9,0.1)\right)\right\},$$

$$G_{\rho}(e_3)=\left\{\left(\frac{c_1}{(0.4,0.3)},(0.5,0.1)\right),\left(\frac{c_2}{(0.7,0.2)},(0.6,0.2)\right),\left(\frac{c_3}{(0.8,0.1)},(0.3,0.5)\right)\right\}.$$

By using the basic intuitionistic fuzzy set intersection and Fermatean fuzzy soft intersection we can calculate $H_{\theta}(e_1)$ as follows

$$\left(\frac{c_1}{\min(0.5,0.6),\max(0.3,0.2)},\min(0.7,0.8),\max(0.2,0.1)\right),$$

$$\left(\frac{c_2}{\min(0.8,0.9),\max(0.1,0)},\min(0.5,0.6),\max(0.2,0.1)\right),$$

$$\left(\frac{c_3}{\min(0.6,0.7),\max(0.2,0.1)},\min(0.6,0.7)\max(0.3,0.2)\right)$$

$$H_{\theta}(e_1)=\left\{\left(\frac{c_1}{(0.5,0.3)},(0.7,0.2)\right),\left(\frac{c_2}{(0.8,0.1)},(0.5,0.2)\right),\left(\frac{c_3}{(0.6,0.2)},(0.6,0.3)\right)\right\}$$

By using same method we have:

$$H_{\theta}(e_2)=\left\{\left(\frac{c_1}{(0.5,0.2)},(0.5,0.3)\right),\left(\frac{c_2}{(0.6,0.3)},(0.5,0.3)\right),\left(\frac{c_3}{(0.4,0.5)},(0.8,0.2)\right)\right\},$$

$$H_{\theta}(e_3)=\left\{\left(\frac{c_1}{(0.4,0.3)},(0.4,0.2)\right),\left(\frac{c_2}{(0.4,0.3)},(0.6,0.2)\right),\left(\frac{c_3}{(0.7,0.2)},(0.3,0.5)\right)\right\}.$$

Proposition 1: Let F_{δ} , G_{ρ} and H_{θ} be any three IPFFSSs over (U,E), then the following hold:

i. $F_{\delta}\cap G_{\rho}=G_{\rho}\cap F_{\delta}$

ii. $F_{\delta}\cup G_{\rho}=G_{\rho}\cup F_{\delta}$

iii. $F_{\delta}\cap(G_{\rho}\cap H_{\theta})=(F_{\delta}\cap G_{\rho})\cap H_{\theta}$

iv. $F_{\delta}\cup(G_{\rho}\cup H_{\theta})=(F_{\delta}\cup G_{\rho})\cup H_{\theta}$

Proof:

i. $F_{\delta}\cap G_{\rho}=\left(\min\left\{\phi_{F_{\delta}},\phi_{G_{\rho}}\right\},\max\left\{\varphi_{F_{\delta}},\varphi_{G_{\rho}}\right\}\right)$

$$=\left(\min\left\{\phi_{G_{\rho}},\phi_{F_{\delta}}\right\},\max\left\{\varphi_{G_{\rho}},\varphi_{F_{\delta}}\right\}\right)$$

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$$= G_\rho \cap F_\delta .$$

ii. $$F_\delta \cup G_\rho = \left(\max\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\}, \min\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\} \right)$$
$$= \left(\max\left\{ \phi_{G_\rho}, \phi_{F_\delta} \right\}, \min\left\{ \varphi_{G_\rho}, \varphi_{F_\delta} \right\} \right)$$
$$= G_\rho \cup F_\delta .$$

iii. $$F_\delta \cap (G_\rho \cap H_\theta) = \left(\phi_{F_\delta}, \varphi_{F_\delta} \right) \cap \left(\min\left\{ \phi_{G_\rho}, \phi_{H_\theta} \right\}, \max\left\{ \varphi_{G_\rho}, \varphi_{H_\theta} \right\} \right)$$
$$= \left(\min\left\{ \phi_{F_\delta}, \min\left\{ \phi_{G_\rho}, \phi_{H_\theta} \right\} \right\}, \max\left\{ \varphi_{F_\delta}, \max\left\{ \varphi_{G_\rho}, \varphi_{H_\theta} \right\} \right\} \right)$$
$$= \left(\min\left\{ \min\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\}, \phi_{H_\theta} \right\}, \max\left\{ \max\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\}, \varphi_{H_\theta} \right\} \right)$$
$$= \left(\min\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\}, \max\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\} \right) \cap \left(\phi_{H_\theta}, \varphi_{H_\theta} \right)$$
$$= (F_\delta \cap G_\rho) \cap H_\theta .$$

iv. $$F_\delta \cup (G_\rho \cup H_\theta) = \left(\phi_{F_\delta}, \varphi_{F_\delta} \right) \cup \left(\max\left\{ \phi_{G_\rho}, \phi_{H_\theta} \right\}, \min\left\{ \varphi_{G_\rho}, \varphi_{H_\theta} \right\} \right)$$
$$= \left(\max\left\{ \phi_{F_\delta}, \max\left\{ \phi_{G_\rho}, \phi_{H_\theta} \right\} \right\}, \min\left\{ \varphi_{F_\delta}, \min\left\{ \varphi_{G_\rho}, \varphi_{H_\theta} \right\} \right\} \right)$$
$$= \left(\max\left\{ \max\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\}, \phi_{H_\theta} \right\}, \min\left\{ \min\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\}, \varphi_{H_\theta} \right\} \right)$$
$$= \left(\max\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\}, \min\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\} \right) \cup \left(\phi_{H_\theta}, \varphi_{H_\theta} \right)$$

Proposition 2: Let F_δ , G_ρ are two IPFFSSs over $\left(U, E\right)$, then the following holds:

I. $(F_\delta \cap G_\rho)^c = \left(F_\delta\right)^c \cup \left(G_\rho\right)^c$

II. $(F_\delta \cup G_\rho)^c = \left(F_\delta\right)^c \cap \left(G_\rho\right)^c$

Proof:

I. $$(F_\delta \cap G_\rho)^c = \left(\left(\min\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\}, \max\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\} \right), \mathrm{T}\left(\delta, \rho\right) \right)^c$$
$$= \left(\max\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\}, \min\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\} \right), \left(T\left(\delta, \rho\right) \right)^c$$
$$= \left(\varphi_{F_\delta}, \phi_{F_\delta} \right) \cup \left(\varphi_{G_\rho}, \phi_{G_\rho} \right), \mathrm{S}\left(\delta, \rho\right)$$
$$= \left(F_\delta\right)^c \cup \left(G_\rho\right)^c .$$

II. $$(F_\delta \cup G_\rho)^c = \left(\left(\max\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\}, \min\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\} \right), \mathrm{S}\left(\delta, \rho\right) \right)^c$$
$$= \left(\min\left\{ \varphi_{F_\delta}, \varphi_{G_\rho} \right\}, \max\left\{ \phi_{F_\delta}, \phi_{G_\rho} \right\} \right), \left(S\left(\delta, \rho\right) \right)^c$$

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$$=\left(\varphi_{F_\delta},\phi_{F_\delta}\right)\cap\left(\varphi_{G_\rho},\phi_{G_\rho}\right),\mathrm{T}\left(\delta,\rho\right)$$

$$=\left(F_\delta\right)^c\cap\left(G_\rho\right)^c.$$

5. AND and OR Operations on IPFFSSs with Applications in Decision Making

In this section, we introduce the definitions of AND and OR operations on Intuitionistic possibility Fermatean fuzzy soft sets. Applications of the Intuitionistic possibility Fermatean fuzzy soft sets in DM problem are given.

Definition 18: If $\left(F_\delta,R\right)$ and $\left(G_\rho,M\right)$ are two IPFFSSs then $\left(F_\delta,R\right)$ AND $\left(G_\rho,M\right)$, denoted by $\left(F_\delta,R\right)\wedge\left(G_\rho,M\right)$ is defined by $\left(F_\delta,R\right)\wedge\left(G_\rho,\mathrm{M}\right)=\left(H_\theta,R\times M\right)$, Where $H_\theta(\varepsilon,\omega)=\left(H(\varepsilon,\omega)(\mathrm{x}),\theta(\varepsilon,\omega)(\mathrm{x})\right)$, $\forall(\varepsilon,\omega)\in R\times M$, such that $H(\varepsilon,\omega)=(F(\varepsilon)\bigcap\mathrm{G}(\omega))$ and $\theta(\varepsilon,\omega)=T(\delta(\varepsilon),\rho(\omega))$, $\forall(\varepsilon,\omega)\in R\times M$.

Example 8: Suppose $U=\{x_1,x_2,x_3\}$ is a set of cars and there are three parameters $E=\{e_1,e_2,e_3\}$ which describe their performances. Suppose a company wants to buy one such car depending on any two of the parameters only. Let there be two observations F_δ and G_ρ by two experts defined as follows:

$$F_\delta(e_1)=\left\{\left(\frac{x_1}{(0.5,0.3)},(0.7,0.2)\right),\left(\frac{x_2}{(0.8,0.1)},(0.5,0.2)\right),\left(\frac{x_3}{(0.6,0.2)},(0.6,0.3)\right)\right\},$$

$$F_\delta(e_2)=\left\{\left(\frac{x_1}{(0.7,0.2)},(0.5,0.3)\right),\left(\frac{x_2}{(0.6,0.2)},(0.6,0.1)\right),\left(\frac{x_3}{(0.5,0.1)},(0.8,0.2)\right)\right\},$$

$$F_\delta(e_3)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.4,0.2)\right),\left(\frac{x_2}{(0.4,0.3)},(0.7,0.1)\right),\left(\frac{x_3}{(0.7,0.2)},(0.5,0.3)\right)\right\}$$

$$\mathrm{G}_\rho(e_1)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.8,0.1)\right),\left(\frac{x_2}{(0.9,0)},(0.6,0.1)\right),\left(\frac{x_3}{(0.7,0.1)},(0.7,0.2)\right)\right\},$$

$$\mathrm{G}_\rho(e_2)=\left\{\left(\frac{x_1}{(0.5,0.1)},(0.6,0.2)\right),\left(\frac{x_2}{(0.7,0.3)},(0.5,0.3)\right),\left(\frac{x_3}{(0.4,0.5)},(0.9,0.1)\right)\right\},$$

$$\mathrm{G}_\rho(e_3)=\left\{\left(\frac{x_1}{(0.4,0.3)},(0.5,0.1)\right),\left(\frac{x_2}{(0.7,0.2)},(0.6,0.2)\right),\left(\frac{x_3}{(0.8,0.1)},(0.7,0.2)\right)\right\}.$$

Then $\left(F_\delta,R\right)\wedge\left(G_\rho,\mathrm{M}\right)=\left(H_\theta,R\times M\right)$ where,

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$$H_{\theta}(e_1,e_1)=\left\{\begin{array}{l}\left(\dfrac{x_1}{\min(0.5,0.6),\max(0.3,0.2)},\min(0.7,0.8),\max(0.2,0.1)\right),\\ \left(\dfrac{x_2}{\min(0.8,0.9),\max(0.1,0)},\min(0.5,0.6),\max(0.2,0.1)\right),\\ \left(\dfrac{x_3}{\min(0.6,0.7),\max(0.2,0.1)},\min(0.6,0.7)\max(0.3,0.2)\right)\end{array}\right\}$$

$$H_{\theta}(e_1,e_1)=\left\{\left(\frac{x_1}{(0.5,0.3)},(0.7,0.2)\right),\left(\frac{x_2}{(0.8,0.1)},(0.5,0.2)\right),\left(\frac{x_3}{(0.6,0.2)},(0.6,0.3)\right)\right\}$$

Similarly, we get:

$$H_{\theta}(e_1,e_2)=\left\{\left(\frac{x_1}{(0.5,0.3)},(0.6,0.2)\right),\left(\frac{x_2}{(0.7,0.3)},(0.5,0.3)\right),\left(\frac{x_3}{(0.4,0.5)},(0.6,0.3)\right)\right\},$$

$$H_{\theta}(e_1,e_3)=\left\{\left(\frac{x_1}{(0.4,0.3)},(0.5,0.2)\right),\left(\frac{x_2}{(0.7,0.2)},(0.5,0.2)\right),\left(\frac{x_3}{(0.6,0.2)},(0.6,0.2)\right)\right\},$$

$$H_{\theta}(e_2,e_1)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.5,0.3)\right),\left(\frac{x_2}{(0.6,0.2)},(0.6,0.1)\right),\left(\frac{x_3}{(0.5,0.1)},(0.7,0.2)\right)\right\},$$

$$H_{\theta}(e_2,e_2)=\left\{\left(\frac{x_1}{(0.5,0.2)},(0.5,0.3)\right),\left(\frac{x_2}{(0.6,0.3)},(0.5,0.3)\right),\left(\frac{x_3}{(0.4,0.5)},(0.8,0.2)\right)\right\},$$

$$H_{\theta}(e_2,e_3)=\left\{\left(\frac{x_1}{(0.4,0.3)},(0.5,0.3)\right),\left(\frac{x_2}{(0.6,0.2)},(0.6,0.2)\right),\left(\frac{x_3}{(0.5,0.1)},(0.7,0.2)\right)\right\},$$

$$H_{\theta}(e_3,e_1)=\left\{\left(\frac{x_1}{(0.6,0.2)},(0.4,0.2)\right),\left(\frac{x_2}{(0.4,0.3)},(0.6,0.1)\right),\left(\frac{x_3}{(0.7,0.2)},(0.5,0.3)\right)\right\},$$

$$H_{\theta}(e_3,e_2)=\left\{\left(\frac{x_1}{(0.5,0.2)},(0.4,0.2)\right),\left(\frac{x_2}{(0.4,0.3)},(0.5,0.3)\right),\left(\frac{x_3}{(0.4,0.5)},(0.5,0.3)\right)\right\},$$

$$H_{\theta}(e_3,e_3)=\left\{\left(\frac{x_1}{(0.4,0.3)},(0.4,0.2)\right),\left(\frac{x_2}{(0.4,0.3)},(0.6,0.2)\right),\left(\frac{x_3}{(0.7,0.2)},(0.5,0.3)\right)\right\}.$$

In matrix notation we have:

$$(F_{\delta},R)\wedge(G_{\rho},M)=\begin{pmatrix}((0.5,0.3),(0.7,0.2)) & ((0.8,0.1),(0.5,0.2)) & ((0.6,0.2),(0.6,0.3))\\ ((0.5,0.3),(0.6,0.2)) & ((0.7,0.3),(0.5,0.3)) & ((0.4,0.5),(0.6,0.3))\\ ((0.4,0.3),(0.5,0.2)) & ((0.7,0.2),(0.5,0.2)) & ((0.6,0.2),(0.6,0.2))\\ ((0.6,0.2),(0.5,0.3)) & ((0.6,0.2),(0.6,0.1)) & ((0.5,0.1),(0.7,0.2))\\ ((0.5,0.2),(0.5,0.3)) & ((0.6,0.3),(0.5,0.3)) & ((0.4,0.5),(0.8,0.2))\\ ((0.4,0.3),(0.5,0.3)) & ((0.6,0.2),(0.6,0.2)) & ((0.5,0.1),(0.7,0.2))\\ ((0.6,0.2),(0.4,0.2)) & ((0.4,0.3),(0.6,0.1)) & ((0.7,0.2),(0.5,0.3))\\ ((0.5,0.2),(0.4,0.2)) & ((0.4,0.3),(0.5,0.3)) & ((0.4,0.5),(0.5,0.3))\\ ((0.4,0.3),(0.4,0.2)) & ((0.4,0.3),(0.6,0.2)) & ((0.7,0.2),(0.5,0.3))\end{pmatrix}$$

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Now to identify the best car we first calculate the difference between the membership and non-membership values and then we mark the highest numerical grade indicated in parenthesis in each row. The matrix displaying various numerical grades is presented below:

$$\begin{pmatrix} (0.2,0.5) & (0.\underline{7},0.3) & (0.4,0.3) \\ (0.2,0.4) & (0.\underline{4},0.2) & (-0.1,0.3) \\ (0.1,0.3) & (0.\underline{5},0.3) & (0.4,0.4) \\ (0.\underline{4},0.2) & (0.\underline{4},0.5) & (0.\underline{4},0.5) \\ (0.\underline{3},0.2) & (0.\underline{3},0.2) & (-0.1,0.6) \\ (0.1,0.2) & (0.\underline{4},0.4) & (0.\underline{4},0.5) \\ (0.4,0.2) & (0.1,0.5) & (0.\underline{5},0.2) \\ (0.\underline{3},0.2) & (0.1,0.2) & (-0.1,0.2) \\ (0.1,0.2) & (0.1,0.4) & (0.\underline{5},0.2) \end{pmatrix}$$

Now the score for each car is determined by summing the products of the distinct numerical grades with their corresponding value of θ. The desired car is identified as the one with the highest score. We do not consider the different numerical grades of the car against the pairs $(e_i, e_i), i = 1,2,3$, as both parameters are the same, (see Table 1).

Table 1: Grade table

H	x_i	highest numerical grade	θ_i
(e_1,e_1)	x_2	x	x
(e_1,e_2)	x_2	0.4	0.2
(e_1,e_3)	x_2	0.5	0.3
(e_2,e_1)	x_1, x_2, x_3	0.4	0.2,0.5,0.5
(e_2,e_2)	x_1, x_2	x	x
(e_2,e_3)	x_2, x_3	0.4	0.4,0.5
(e_3,e_1)	x_3	0.5	0.2
(e_3,e_2)	x_1	0.3	0.2
(e_3,e_3)	x_3	x	x

Score$(x_1) = (0.4\times0.2)+(0.3\times0.2) = 0.14$

Score$(x_2) = (0.4\times0.2)+(0.5\times0.3)+(0.4\times0.5)+(0.4\times0.4) = 0.59$

Score$(x_3) = (0.4\times0.5)+(0.4\times0.2)+(0.5\times0.2) = 0.38$

Then the company will select the car with the highest score. Hence, they will buy car x_2.

Definition 19: If (F_δ, R) and (G_ρ, M) are two IPFFSSs then (F_δ, R) OR (G_ρ, M), denoted by $(F_\delta, R)\vee(G_\rho, M)$ is defined by $(F_\delta, R)\vee(G_\rho, \mathrm{M}) = (H_\theta, R\times M)$,Where $H_\theta(\varepsilon,\omega) = (H(\varepsilon,\omega)(\mathrm{x}), \theta(\varepsilon,\omega)(\mathrm{x}))$, $\forall(\varepsilon,\omega)\in R\times M$, such that $H(\varepsilon,\omega) = (F(\varepsilon)\bigcup \mathrm{G}(\omega))$ and $\theta(\varepsilon,\omega) = S(\delta(\varepsilon), \rho(\omega))$, $\forall(\varepsilon,\omega)\in R\times M$.

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Example 9: Consider Example 8, and suppose the company wants to buy a car depending on any one of two parameters.

Then $\left(F_{\delta}, R\right) \vee \left(G_{\rho}, \mathrm{M}\right) = \left(H_{\theta}, R \times M\right)$ where,

$$\mathrm{H}_{\theta}(e_1, e_1) = \left\{ \begin{array}{l} \left(\dfrac{x_1}{\max(0.5,0.6), \min(0.3,0.2)}, \max(0.7,0.8), \min(0.2,0.1) \right), \\ \left(\dfrac{x_2}{\max(0.8,0.9), \min(0.1,0)}, \max(0.5,0.6), \min(0.2,0.1) \right), \\ \left(\dfrac{x_3}{\max(0.6,0.7), \min(0.2,0.1)}, \max(0.6,0.7) \min(0.3,0.2) \right) \end{array} \right\}$$

$$\mathrm{H}_{\theta}(e_1, e_1) = \left\{ \left(\frac{x_1}{(0.6,0.2)}, (0.8,0.1) \right), \left(\frac{x_2}{(0.9,0)}, (0.6,0.1) \right), \left(\frac{x_3}{(0.7,0.1)}, (0.7,0.2) \right) \right\}$$

Similarly we get:

$$\mathrm{H}_{\theta}(e_1, e_2) = \left\{ \left(\frac{x_1}{(0.5,0.1)}, (0.7,0.2) \right), \left(\frac{x_2}{(0.8,0.1)}, (0.5,0.2) \right), \left(\frac{x_3}{(0.6,0.2)}, (0.9,0.1) \right) \right\},$$

$$\mathrm{H}_{\theta}(e_1, e_3) = \left\{ \left(\frac{x_1}{(0.5,0.3)}, (0.7,0.1) \right), \left(\frac{x_2}{(0.8,0.1)}, (0.6,0.2) \right), \left(\frac{x_3}{(0.8,0.1)}, (0.7,0.2) \right) \right\},$$

$$\mathrm{H}_{\theta}(e_2, e_1) = \left\{ \left(\frac{x_1}{(0.7,0.2)}, (0.8,0.1) \right), \left(\frac{x_2}{(0.9,0)}, (0.6,0.1) \right), \left(\frac{x_3}{(0.7,0.1)}, (0.8,0.2) \right) \right\},$$

$$\mathrm{H}_{\theta}(e_2, e_2) = \left\{ \left(\frac{x_1}{(0.7,0.1)}, (0.6,0.2) \right), \left(\frac{x_2}{(0.7,0.2)}, (0.6,0.1) \right), \left(\frac{x_3}{(0.5,0.1)}, (0.9,0.1) \right) \right\},$$

$$\mathrm{H}_{\theta}(e_2, e_3) = \left\{ \left(\frac{x_1}{(0.7,0.2)}, (0.5,0.1) \right), \left(\frac{x_2}{(0.7,0.2)}, (0.6,0.1) \right), \left(\frac{x_3}{(0.8,0.1)}, (0.8,0.2) \right) \right\},$$

$$\mathrm{H}_{\theta}(e_3, e_1) = \left\{ \left(\frac{x_1}{(0.6,0.2)}, (0.8,0.1) \right), \left(\frac{x_2}{(0.9,0)}, (0.7,0.1) \right), \left(\frac{x_3}{(0.7,0.1)}, (0.7,0.2) \right) \right\},$$

$$\mathrm{H}_{\theta}(e_3, e_2) = \left\{ \left(\frac{x_1}{(0.6,0.1)}, (0.6,0.2) \right), \left(\frac{x_2}{(0.7,0.3)}, (0.7,0.1) \right), \left(\frac{x_3}{(0.7,0.2)}, (0.9,0.1) \right) \right\},$$

$$\mathrm{H}_{\theta}(e_3, e_3) = \left\{ \left(\frac{x_1}{(0.6,0.2)}, (0.5,0.1) \right), \left(\frac{x_2}{(0.7,0.2)}, (0.7,0.1) \right), \left(\frac{x_3}{(0.8,0.1)}, (0.7,0.2) \right) \right\}$$

In matrix notation we have:

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$$(F_\delta, R) \vee (G_\rho, \mathrm{M}) = \begin{pmatrix} ((0.6,0.2),(0.8,0.1)) & ((0.9,0),(0.6,0.1)) & ((0.7,0.1),(0.7,0.2)) \\ ((0.5,0.1),(0.7,0.2)) & ((0.8,0.1),(0.5,0.2)) & ((0.6,0.2),(0.9,0.1)) \\ ((0.5,0.3),(0.7,0.1)) & ((0.8,0.1),(0.6,0.2)) & ((0.8,0.1),(0.7,0.2)) \\ ((0.7,0.2),(0.8,0.1)) & ((0.9,0),(0.6,0.1)) & ((0.7,0.1),(0.8,0.2)) \\ ((0.7,0.1),(0.6,0.2)) & ((0.7,0.2),(0.6,0.1)) & ((0.5,0.1),(0.9,0.1)) \\ ((0.7,0.2),(0.5,0.1)) & ((0.7,0.2),(0.6,0.1)) & ((0.8,0.1),(0.8,0.2)) \\ ((0.6,0.2),(0.8,0.1)) & ((0.9,0),(0.7,0.1)) & ((0.7,0.1),(0.7,0.2)) \\ ((0.6,0.1),(0.6,0.2)) & ((0.7,0.3),(0.7,0.1)) & ((0.7,0.2),(0.9,0.1)) \\ ((0.6,0.2),(0.5,0.1)) & ((0.7,0.2),(0.7,0.1)) & ((0.8,0.1),(0.7,0.2)) \end{pmatrix}$$

Now to identify the best car we first calculate the difference between the membership and non-membership values and then we mark the highest numerical grade indicated in parenthesis in each row. The matrix displaying various numerical grades is presented below:

$$\begin{pmatrix} (0.4,0.7) & (0.\underline{9},0.5) & (0.6,0.5) \\ (0.4,0.5) & (0.\underline{7},0.3) & (0.4,0.8) \\ (0.2,0.6) & (0.\underline{7},0.4) & (0.\underline{7},0.5) \\ (0.5,0.7) & (0.\underline{9},0.5) & (0.6,0.6) \\ (0.\underline{6},0.4) & (0.5,0.5) & (0.4,0.8) \\ (0.5,0.4) & (0.5,0.5) & (0.\underline{7},0.6) \\ (0.4,0.7) & (0.\underline{9},0.6) & (0.6,0.5) \\ (0.\underline{5},0.4) & (0.4,0.6) & (0.\underline{5},0.8) \\ (0.4,0.4) & (0.5,0.6) & (0.\underline{7},0.5) \end{pmatrix}$$

Now the score for each car is determined by summing the products of the distinct numerical grades with their corresponding values of θ. The desired car is identified as the one with the highest score. We do not consider the different numerical grades of the car against the pairs $(e_i, e_i), i = 1,2,3$, as both parameters are the same, (see Table 2).

Table 2: Grade table.

H	x_i	highest numerical grade	θ_i
(e_1,e_1)	x_2	x	x
(e_1,e_2)	x_2	0.7	0.3
(e_1,e_3)	x_2, x_3	0.7	0.4,0.5
(e_2,e_1)	x_2	0.9	0.5
(e_2,e_2)	x_1	x	x
(e_2,e_3)	x_3	0.7	0.6
(e_3,e_1)	x_2	0.9	0.6
(e_3,e_2)	x_1, x_3	0.5	0.4,0.8
(e_3,e_3)	x_3	x	x

$\text{Score}(x_1) = (0.5 \times 0.4) = 0.2$

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$$\text{Score}\left(x_2\right) = (0.7\times 0.3) + (0.7\times 0.4) + (0.9\times 0.5) + (0.9\times 0.6) = 1.48$$

$$\text{Score}\left(x_3\right) = (0.7\times 0.5) + (0.7\times 0.6) + (0.5\times 0.8) = 1.17$$

Then the company will select the car with the highest score. Hence, they will buy car x_2 .

6. Conclusion

In this paper, we have presented the concept of intuitionistic possibility Fermatean Fuzzy Soft Sets and studied some of its properties. The operations of complement, union, and intersection have been established for the intuitionistic possibility Fermatean Fuzzy Soft Sets. Applications of this theory have been given to solve a decision-making problem.

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