Computing Idempotent Elements In 3-Cyclic and 4-Cyclic Refined Neutrosophic Rings of Integers

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Abstract

An element X in a ring R is called idempotent if it equals its square. In this paper, we study the idempotent elements in the 3-cyclic refined neutrosophic ring of integers and 4-cyclic refined neutrosophic ring of integers, where we compute all idempotents in those two rings by solving many different linear Diophantine systems which are generated directly from their the algebraic structure. On the other hand, we use the same Diophantine systems to compute all 2-potent 3-cyclic, and 4-cyclic refined neutrosophic integer elements.

Keywords: 3-cyclic refined neutrosophic ring; 4-cyclic refined neutrosophic ring; idempotent, 2-potent; Diophantine system

1. Introduction and preliminaries

An idempotent element X in the ring R is defined with the following algebraic property $(X)^2 = X$.

The class of idempotent elements is considered as one of a long list of special elements and subsets that arose in ring theory, where they have attracted many authors to write about them, such as invertible elements, Pythagoras' triples and quadruples [15-16], Fermat's triples and nilpotent elements [18].

The concept of an n-cyclic refined neutrosophic ring was proposed for the first time in [9]. The authors tried to combine algebraic rings with neutrosophic algebraic indeterminate elements in one algebraic structure by using a multiplication property which is very similar to the structure of the cyclic abelian group Z_n.

Later, this concept became a fertile material for studying generalizations of algebraic structures associated with it, in which we find several important results related to n-cyclic refined neutrosophic groups [11], n-cyclic refined neutrosophic spaces [10], and even complex numbers [14]. Von Shtawzen et.al [1-4, 6-8, 12] studied the group of units in different types of n-cyclic refined neutrosophic rings, where they showed a close correlation between the classification of the group of units and the nonlinear Diophantine equations [5].

In this work, we are motivated to study the problem of computing idempotent elements and 2-potent elements in two different n-cyclic refined neutrosophic rings (3-cyclic refined neutrosophic ring of integers and 4-cyclic refined neutrosophic ring of integers), where we try to find all possible 3-cyclic and 4-cyclic refined neutrosophic integer idempotents and 2-potents in these rings by solving many different related linear and non-linear Diophantine systems of equations.

The general n-cyclic refined neutrosophic ring is defined as follows:

If R is a ring, the corresponding n-cyclic refined neutrosophic ring is defined as

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$.R_n(I) = \{r_0 + r_1 I_1 + \cdots + r_n I_n; r_i \in R, I_i I_j = I_{i+j\ (mod\ n)}\}$

Addition on $R_n(I)$ is defined as follows:

$(r_0 + r_1 I_1 + \cdots + r_n I_n)$+$(m_0 + m_1 I_1 + \cdots + m_n I_n)$= $(r_0 + m_0) + (r_1 + m_1)I_1 + \cdots + (r_n + m_n)I_n$.

Multiplication on $R_n(I)$ is defined as follows:

$(r_0 + r_1 I_1 + \cdots + r_n I_n).(m_0 + m_1 I_1 + \cdots + m_n I_n)$= $(r_0.m_0) + \left(\sum_{i+j\equiv 1(mod\ n)} r_i.m_j\right)I_1 + \cdots + \left(\sum_{i+j\equiv n(mod\ n)} r_i.m_j\right)I_n$. [9]

is called idempotent if and only if $(r_0 + r_1 I_1 + \cdots + r_n I_n)^2 = r_0 + r_1 I_1 + \cdots + r_0 + r_1 I_1 + \cdots + r_n I_n$ $r_n I_n$, and it is called 2-potent if and only if $(r_0 + r_1 I_1 + \cdots + r_n I_n)^2 = 0$.

Main results:

Definition:

Let $X = x_0 + \sum_{i=1}^{3} x_i I_i \in \mathbb{Z}_3(I)$, then X is called idempotent if and only if: $X^2 = X$.

Remark:

According to [13], $X^2 = X$ is equivalent to:

$$\begin{cases} x_0^2 = x_0 & (1) \\ (\sum_{i=0}^{3} x_i)^2 = \sum_{i=1}^{3} x_i & (2) \\ [x_0 + x_3 - \frac{x_1 + x_2}{2} + i\frac{\sqrt{3}}{2}(x_1 - x_2)]^2 = x_0 + x_3 - \frac{x_1 + x_2}{2} + i\frac{\sqrt{3}}{2}(x_1 - x_2) & (3) \end{cases}$$

Main Results:

Equations (1-2) mean that $x_0 \in \{0.1\}$, $(\sum_{i=0}^{3} x_i) \in \{0.1\}$.

Equation (3) means that:

$$\begin{cases} \left[x_0 + x_3 - \frac{x_1 + x_2}{2}\right]^2 - \frac{3}{4}(x_1 - x_2)^2 = x_0 + x_3 - \frac{x_1 + x_2}{2} & \text{(I)} \\ \left(x_0 + x_3 - \frac{x_1 + x_2}{2}\right)(x_1 - x_2) = \frac{1}{2}(x_1 - x_2) & \text{(II)} \end{cases}$$

From (II), we can see: $(x_1 - x_2)\left(x_0 + x_3 - \frac{x_1 + x_2 + 1}{2}\right) = 0$

Thus: $x_1 = x_2$, or $2(x_0 + x_3) = x_1 + x_2 + 1$

If $x_1 = x_2$, we get from (I):

$(x_0 + x_3 - x_1)^2 = x_0 + x_3 - x_1. hence\ x_0 + x_3 - x_1 \in \{0.1\}$

If $x_0 + x_3 = \frac{1}{2}(x_1 + x_2 + 1)$, we get from (I):

$\frac{1}{4} - \frac{3}{4}(x_1 - x_2)^2 = \frac{1}{2}$ thus: $-\frac{3}{4}(x_1 - x_2)^2 = \frac{1}{4}$,which is not possible.

We discuss the possible cases:

Case (1):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_0 + x_3 - x_1 = 0 \\ x_1 = x_2 \end{cases}$$

Thus $X = 0$.

Case (2):

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$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 = 1 \\ x_0 + x_3 - x_1 = 0 \\ x_1 = x_2 \end{cases}$$

Thus $\begin{cases} 2x_1 + x_3 = 1 \\ x_3 - x_1 = 0 \end{cases} . x_1 = \frac{1}{3}$ which is a contradiction .

Case (3):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_0 + x_3 - x_1 = 1 \\ x_1 = x_2 \end{cases}$$

Thus $\begin{cases} 2x_1 + x_3 = 0 \\ x_3 - x_1 = 1 \end{cases} .\ hence\ x_1 = \frac{-1}{3}$ which is a contradiction .

Case (4):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 = 1 \\ x_0 + x_3 - x_1 = 1 \\ x_1 = x_2 \end{cases} .\text{thus} \begin{cases} 2x_1 + x_3 = 1 \\ x_3 - x_1 = 1 \end{cases} . x_1 = 0. x_3 = 1. x_2 = 0$$

And $X = I_3$.

Case (5):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -1 \\ x_0 + x_3 - x_1 = 0 \\ x_1 = x_2 \end{cases}$$

Thus $\begin{cases} 2x_1 + x_3 = -1 \\ x_3 - x_1 = -1 \end{cases} . x_1 = 0. x_3 = -1. x_2 = 0$

And $X = 1 - I_3$.

Case (6):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ x_0 + x_3 - x_1 = 1 \\ x_1 = x_2 \end{cases}$$

Thus $\begin{cases} 2x_1 + x_3 = 0 \\ x_3 - x_1 = 0 \end{cases} . x_1 = x_3 = x_2 = 0$

And $X = 1$

Case (7):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -1 \\ x_0 + x_3 - x_1 = 1 \\ x_1 = x_2 \end{cases}$$

Thus $\begin{cases} 2x_1 + x_3 = -1 \\ x_3 - x_1 = 0 \end{cases} . x_1 = \frac{-1}{3}$ a contradiction.

Case (8):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ x_0 + x_3 - x_1 = 0 \\ x_1 = x_2 \end{cases} . \begin{cases} 2x_1 + x_3 = 0 \\ x_3 - x_1 = -1 \end{cases} . x_1 = \frac{1}{3}$$ a contradiction.

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So that, the 3-cyclic refined neutrosophic idempotents are: $\{1.I_3.1-I_3\}$

Definition:

$X=x_0+\sum_{i=1}^{3}x_i I_i \in Z_3(I)$ is called 2-potent if and only if $X^2=0$.

Remark:

$X^2=0$ is equivalent to:

$$\begin{cases} x_0^2=0 \Rightarrow x_0=0 \\ (\sum_{i=0}^{3}x_i)^2=0 \Rightarrow x_1+x_2+x_3=0 \\ \left(x_0+x_3-\frac{x_1+x_2}{2}\right)^2-\frac{3}{4}(x_1-x_2)^2=(x_1-x_2)(x_0+x_3-\frac{x_1+x_2}{2})=0 \end{cases}$$

Thus $x_i=0\,;0\le i\le 3\,.and\ X=0\,.$

Definition:

Let $X=x_0+\sum_{i=1}^{4}x_i I_i \in Z_4(I)$, X is called 4-cyclic refined neutrosophic idempotents if and only if: $X^2=X$.

Remark:

According to [ ], $X^2=X$ is equivalent to:

$$\begin{cases} x_0^2=x_0 \Rightarrow x_0\in\{0.1\} \\ \left(\sum_{i=0}^{4}x_i\right)^2=\sum_{i=0}^{4}x_i \Rightarrow \sum_{i=0}^{4}x_i\in\{0.1\} \\ \left(\sum_{i=0}^{4}(-1)^i x_i\right)^2=\sum_{i=0}^{4}(-1)^i x_i \Rightarrow \sum_{i=0}^{4}(-1)^i x_i\in\{0.1\} \\ [x_0+x_4-x_2+i(x_1-x_3)]^2=x_0+x_4-x_2+i(x_1-x_3) \quad \text{(I)} \end{cases}$$

Equation (I) implies:

$$\begin{cases} (x_0+x_4-x_2)^2-(x_1-x_3)^2=x_0+x_4-x_2 & (1) \\ 2(x_0+x_4-x_2)(x_1-x_3)=x_1-x_3 & (2) \end{cases}$$

From (2): $(x_1-x_3)(2x_0+2x_4-2x_2-1)=0$

Thus: $x_1=x_3\ or\ x_0+x_4-x_2=\frac{1}{2}$

If $x_1=x_3$, we get from (1): $x_0+x_4-x_2\in\{0.1\}$

If $x_0+x_4-x_2=\frac{1}{2}$ .then: $-(x_1-x_3)^2=\frac{1}{4}$ a contradiction.

We discuss the possible cases:

Case (1):

$$\begin{cases} x_0=0 \\ x_1+x_2+x_3+x_4=0 \\ -x_1+x_2-x_3+x_4=0 \\ x_0+x_4-x_2=0 \\ x_1=x_3 \end{cases}$$

Thus: $X=0$.

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Case (2):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 1 \\ x_0 + x_4 - x_2 = 0 \\ x_1 = x_3 \end{cases}$$

Thus: $x_2 + x_4 \notin \mathbb{Z}$(a contradiction).

Case (3):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 = x_3 \end{cases}$$

Thus $x_2 + x_4 \notin \mathbb{Z}$(a contradiction).

Case (4):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 1 \\ x_1 = x_3 \end{cases}$$

Thus: $\begin{cases} x_2 + x_4 = 0 \\ x_4 - x_2 = 1 \end{cases} . x_4 = +\frac{1}{2}$(a contradiction).

Case (5):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 1 \\ x_0 + x_4 - x_2 = 1 \\ x_1 = x_3 \end{cases}$$

Thus: $x_2 + x_4 \notin \mathbb{Z}$(a contradiction).

Case (6):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 1 \\ x_1 = x_3 \end{cases}$$

Thus $x_2 + x_4 \notin \mathbb{Z}$(a contradiction).

Case (7):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ -x_1 + x_2 - x_3 + x_4 = 1 \\ x_0 + x_4 - x_2 = 1 \\ x_1 = x_3 \end{cases} . \text{Thus } \begin{cases} x_2 + x_4 = 1 \\ x_4 - x_2 = 1 \end{cases} . \begin{cases} x_4 = 1 \\ x_2 = 0 \end{cases}$$

Also, $x_1 = x_3 = 0$. and $X = I_4$

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Case (8):

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \\ -x_1 + x_2 - x_3 + x_4 = 1 \\ x_0 + x_4 - x_2 = 0 \\ x_1 = x_3 \end{cases} . \text{Thus } \begin{cases} x_2 + x_4 = 1 \\ x_4 - x_2 = 0 \end{cases} . \ \left\{ x_4 = \frac{1}{2} \ \ which\ is\ forbidden. \right.$$

Case (9):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -1 \\ -x_1 + x_2 - x_3 + x_4 = -1 \\ x_0 + x_4 - x_2 = 0 \\ x_1 = x_3 \end{cases} . \text{Thus } \begin{cases} x_2 + x_4 = -1 \\ x_4 - x_2 = -1 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_4 = -1 \end{cases} . x_1 = x_3 = 0$$

And $X = -I_4$.

Case (10):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -1 \\ -x_1 + x_2 - x_3 + x_4 = -1 \\ x_0 + x_4 - x_2 = 1 \\ x_1 = x_3 \end{cases} . \text{Thus } \begin{cases} x_2 + x_4 = -1 \\ x_4 - x_2 = 0 \end{cases} . \left\{ x_4 = \frac{-1}{2} \ \ (\text{ contradiction}). \right.$$

Case (11):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = -1 \\ x_0 + x_4 - x_2 = 0 \\ x_1 = x_3 \end{cases}$$

Thus: $x_2 + x_4 \notin \mathbb{Z}$ (a contradiction).

Case (12):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = -1 \\ x_0 + x_4 - x_2 = 1 \\ x_1 = x_3 \end{cases}$$

$x_2 + x_4 \notin \mathbb{Z}$(a contradiction).

Case (13):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -1 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 = x_3 \end{cases}$$

Thus: $x_2 + x_4 \notin \mathbb{Z}$(impossible).

Case (14):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = -1 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 1 \\ x_1 = x_3 \end{cases}$$

$x_2 + x_4 \notin \mathbb{Z}$

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Case (15):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 0 \\ x_1 = x_3 \end{cases}$$

Thus $\begin{cases} x_2 + x_4 = 0 \\ x_4 - x_2 = 1 \end{cases} . x_4 = \frac{1}{2} \notin \mathbb{Z}$

Case (16):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ x_0 + x_4 - x_2 = 1 \\ x_1 = x_3 \end{cases}$$

$thus \begin{cases} x_2 + x_4 = 0 \\ x_4 - x_2 = 0 \end{cases} . x_4 = x_2 = 0 . x_1 = x_3 = 0$

And $X = 1$

2. Result:

Idempotents in $\mathbb{Z}_4(I)$ $are: \{0.1. I_4. -I_4\}$.

3. Conclusion

In this paper, we have computed the idempotent elements in the 3-cyclic refined neutrosophic ring of integers and 4-cyclic refined neutrosophic ring of integers by solving many different linear Diophantine systems. Also, we used the same Diophantine systems to compute all 2-potent 3-cyclic, and 4-cyclic refined integer elements.

The computation of all idempotents in the n-cyclic refined neutrosophic ring for arbitrary n is still an open problem in general. We hope that our results shown in this study will be helpful in future efforts toward the full and final solution to this interesting problem.

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