



On The Diophantine 3-Cyclic Refined Neutrosophic Roots of Unity

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Abstract

The 3-cyclic refined neutrosophic roots of unity are exactly the solutions of the Diophantine equation $X^n = 1$ in the 3-cyclic refined neutrosophic ring of integers $Z_3(I)$. This paper is dedicated to finding all 3-cyclic refined neutrosophic Diophantine roots of unity, where it proves that there exist only three solutions for the case of odd order (n), and twelve different solutions for the case of even order (n). On the other hand, the group generated from all solutions will be classified as a finite abelian group with direct products of finite cyclic groups.

Keywords: 3-cyclic refined neutrosophic rings; 3-cyclic refined Diophantine root of unity; Von Shtawzen's group of units; neutrosophic unit

1. Introduction and basic concepts

The concept of an n-cyclic refined neutrosophic ring was proposed for the first time in [9]. The authors tried to combine algebraic rings with neutrosophic algebraic indeterminate elements in one algebraic structure by using a multiplication property which is very similar to the structure of the cyclic abelian group Z_n .

Later, this concept became a fertile material for studying generalizations of algebraic structures associated with it, in which we find several important results related to n-cyclic refined neutrosophic groups [11], n-cyclic refined neutrosophic spaces [10], and even complex numbers [14]. Von Shtawzen et.al [1-4, 12] studied the group of units in different types of n-cyclic refined neutrosophic rings, where they showed a close correlation between the classification of the group of units and the nonlinear Diophantine equations [5].

In [6-8], Sankari et.al have proved Von Shtawzen's conjectures and found all solutions for the related Diophantine equations. Laterally, Salman and others, have presented an important result that clarifies how 3-cyclic refined neutrosophic integers can be represented by using complex roots of unity [13].

This has prompted us to study for the first time the 3-cyclic refined neutrosophic integer solutions for the equation $X^n = 1$, where we show that it has three different solutions if n is odd, and 12 different solutions if n is even.

We point out here, that the general state of the matter for any value of n is still an open research question.

Definition [9]

If R is a ring, the corresponding n-cyclic refined neutrosophic ring is defined as

$$.R_n(I) = \{r_0 + r_1I_1 + \dots + r_nI_n; r_i \in R, I_iI_j = I_{i+j \pmod n}\}$$

Addition on $R_n(I)$ is defined as follows:

$$(r_0 + r_1I_1 + \dots + r_nI_n) + (m_0 + m_1I_1 + \dots + m_nI_n) = (r_0 + m_0) + (r_1 + m_1)I_1 + \dots + (r_n + m_n)I_n.$$

Multiplication on $R_n(I)$ is defined as follows:

$$(r_0 + r_1I_1 + \dots + r_nI_n) \cdot (m_0 + m_1I_1 + \dots + m_nI_n) = (r_0 \cdot m_0) + (\sum_{i+j \equiv 1 \pmod n} r_i \cdot m_j)I_1 + \dots + (\sum_{i+j \equiv n \pmod n} r_i \cdot m_j)I_n.$$

Remark:

The 3-cyclic refined neutrosophic roots of unity are exactly the solutions of the Diophantine equation $X^n = 1$ in the 3-cyclic refined neutrosophic ring of integers $Z_3(I)$

Main results

Definition:

Let $X = x_0 + \sum_{i=1}^3 x_i I_i \in Z_3(I)$, we say that X is a Diophantine 3-cyclic refined neutrosophic root of unity if and only if: $X^n = 1$.

Remark:

According to [13] by using the isomorphism, the equation $X^n = 1$ is equivalent to:

$$\begin{cases} x_0^n = 1 & (1) \\ (\sum_{i=0}^3 x_i)^n = 1 & (2) \\ (x_0 + x_3 + x_1\alpha + x_2\alpha^2)^n = 1 & (3) \end{cases} ; \alpha = e^{\frac{2\pi i}{3}}, x_i \in \mathbb{Z}$$

Discussion:

Equation (1) implies that: $x_0 \in \{-1, 1\}$ if n is even, and $x_0 = 1$ if n is odd.

Equation (2) implies that: $\sum_{i=0}^3 x_i \in \{-1, 1\}$ if n is even, and $\sum_{i=0}^3 x_i = 1$ if n is odd.

For $(x_0 + x_3 + x_1\alpha + x_2\alpha^2)$, we can write:

$$\begin{aligned} x_0 + x_3 + x_1\alpha + x_2\alpha^2 &= \left(x_0 + x_3 - \frac{x_1}{2} - \frac{x_2}{2}\right) + i\left(\frac{\sqrt{3}}{2}x_1 - \frac{\sqrt{3}}{2}x_2\right) \\ &= \left(x_0 + x_3 - \frac{x_1 + x_2}{2}\right) + \frac{\sqrt{3}}{2}i(x_1 - x_2) \end{aligned}$$

Equation (3), implies that: $(x_0 + x_3 + x_1\alpha + x_2\alpha^2) = e^{\frac{2\pi k i}{n}}; 1 \leq k \leq n$.

This is equivalent to: $|x_0 + x_3 + x_1\alpha + x_2\alpha^2| = 1$, there for: $\left(x_0 + x_3 - \frac{x_1 + x_2}{2}\right)^2 + \frac{3}{4}(x_1 - x_2)^2 = 1$

By multiplying both sides with 4, we get:

$$(2x_0 + 2x_3 - x_1 - x_2)^2 + 3(x_1 - x_2)^2 = 4$$

$$\text{Put: } \begin{cases} T = 2x_0 + 2x_3 - x_1 - x_2 \\ R = x_1 - x_2 \end{cases}$$

We get the following quadratic Diophantine equation:

$$T^2 + 3R^2 = 4$$

The possible solutions are:

$$\begin{cases} T = R = 1 \\ T = R = -1 \\ T = 1, R = -1 \\ T = -1, R = 1 \\ T = 2, R = 0 \\ T = -2, R = 0 \end{cases}$$

We must discuss all cases:

For n is odd, we have:

Case (1):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ T = R = 1 \end{cases}$$

So that: $\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = -1 \\ x_1 - x_2 = 1 \end{cases}$, implies $x_3 = -\frac{1}{3}$ which is a contradiction.

Case (2):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ T = R = -1 \end{cases}$$

So that: $\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = -3 \\ x_1 - x_2 = -1 \end{cases}$, thus $x_3 = -1, x_2 = 1, x_1 = 0$ and $X = 1 + I_2 - I_3$

Case (3):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ T = 1, R = -1 \end{cases}$$

So that: $\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = -1 \\ x_1 - x_2 = -1 \end{cases}$, thus $x_3 = -\frac{1}{3}$ which is a contradiction.

Case (4):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ T = -1, R = +1 \end{cases}$$

So that: $\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = -3 \\ x_1 - x_2 = 1 \end{cases}$, thus $x_3 = -1, x_2 = 0, x_1 = 1$

Hence $X = 1 + I_1 - I_3$

Case (5):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ T = 2, R = 0 \end{cases}$$

So that: $\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$, thus $x_3 = 0, x_1 = 0, x_2 = 0$

Case (6):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ T = -2, R = 0 \end{cases}$$

$$\text{So that: } \begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = -4 \\ x_1 - x_2 = 0 \end{cases}, \text{ so that } x_3 = \frac{-4}{3} \text{ which is a contradiction.}$$

For even values of n, we have:

Case (1):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ R = T = 1 \end{cases}, \text{ we get a contradiction.}$$

Case (2):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ R = T = -1 \end{cases}, \text{ we get } X = 1 + I_2 - I_3$$

Case (3):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ R = 1, T = -1 \end{cases}, \text{ we get } X = 1 + I_1 - I_3$$

Case (4):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ R = -1, T = 1 \end{cases}, \text{ we get a contradiction.}$$

Case (5):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ R = 2, T = 0 \end{cases}, \text{ we get } X = 1$$

Case (6):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = 0 \\ R = -2, T = 0 \end{cases}, \text{ we get a contradiction.}$$

Case (7):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ T = R = 1 \end{cases}$$

$$\text{So that: } \begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = 3 \\ x_1 - x_2 = 1 \end{cases}, \text{ hence } x_3 = 1, x_2 = -1, x_1 = 0$$

$$\text{And } X = -1 - I_2 + I_3$$

Case (8):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ T = R = -1 \end{cases}$$

So that:
$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = 1 \\ x_1 - x_2 = -1 \end{cases}$$
, we get a contradiction.

Case (9):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ T = 1, R = -1 \end{cases}$$

So that:
$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = 3 \\ x_1 - x_2 = -1 \end{cases}$$
, thus $x_3 = 1, x_2 = 0, x_1 = -1$ and $X = -1 - I_1 + I_3$

Case (10):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ T = -1, R = 1 \end{cases}$$

So that:
$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = 1 \\ x_1 - x_2 = 1 \end{cases}$$
, we get a contradiction.

Case (11):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ T = 2, R = 0 \end{cases}$$

So that:
$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = 4 \\ x_1 - x_2 = 0 \end{cases}$$
, we get a contradiction

Case (12):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ T = -2, R = 0 \end{cases}$$

implies
$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 + 2x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$$
, thus $x_0 = -1, x_3 = 0, x_1 = x_2 = 0$, and $X = -1$

Case (13):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ T = R = 1 \end{cases}$$

implies
$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ -x_1 - x_2 + 2x_3 = -4 \\ x_1 - x_2 = 1 \end{cases}$$
, thus $x_3 = -2, x_2 = -\frac{1}{2}$, which is a contradiction.

Case (14):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ T = R = -1 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ -x_1 - x_2 + 2x_3 = -3 \\ x_1 - x_2 = -1 \end{cases}, \text{ we get a contradiction.}$$

Case (15):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ T = 1, R = -1 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ -x_1 - x_2 + 2x_3 = -1 \\ x_1 - x_2 = -1 \end{cases}, x_3 = -1, x_2 = 0, x_1 = -1 \text{ and } X = 1 - I_1 - I_3$$

Case (16):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ T = -1, R = 1 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ -x_1 - x_2 + 2x_3 = -3 \\ x_1 - x_2 = 0 \end{cases}, \text{ it is a contradiction.}$$

Case (17):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ T = 2, R = 0 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ -x_1 - x_2 + 2x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}, \text{ we get a contradiction.}$$

Case (18):

$$\begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ T = -2, R = 0 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = 1 \\ x_1 + x_2 + x_3 = -2 \\ -x_1 - x_2 + 2x_3 = -4 \\ x_1 - x_2 = 0 \end{cases}, x_3 = -2, x_2 = 0, x_1 = 0 \text{ and } X = 1 - 2I_3$$

Case (19):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ T = R = 1 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ -x_1 - x_2 + 2x_3 = 3 \\ x_1 - x_2 = 1 \end{cases}, \text{ we get a contradiction.}$$

Case (20):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ T = R = -1 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ -x_1 - x_2 + 2x_3 = 1 \\ x_1 - x_2 = -1 \end{cases}, x_3 = 1, x_2 = 1, x_1 = 0 \text{ and } X = -1 + I_2 + I_3$$

Case (21):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ T = 1, R = -1 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ -x_1 - x_2 + 2x_3 = 3 \\ x_1 - x_2 = -1 \end{cases}, \text{ it is a contradiction}$$

Case (22):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ T = -1, R = 1 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ -x_1 - x_2 + 2x_3 = 1 \\ x_1 - x_2 = 1 \end{cases}, x_3 = 1, x_2 = 0, x_1 = 1 \text{ and } X = -1 + I_1 + I_3$$

Case (23):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ T = 2, R = 0 \end{cases}$$

$$\text{implies } \begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ -x_1 - x_2 + 2x_3 = 4 \\ x_1 - x_2 = 0 \end{cases}, x_3 = 2, x_2 = 0, x_1 = 0 \text{ and } X = -1 + 2I_3$$

Case (24):

$$\begin{cases} x_0 = -1 \\ x_1 + x_2 + x_3 = 2 \\ T = -2, R = 0 \end{cases}$$

Implies a contradiction.

Result:

If n is odd, the solutions of $X^n = 1$ are:

$\{1, 1 + I_2 - I_3, 1 + I_1 - I_3\}$. This set forms an abelian group under multiplication, and it is isomorphic to Z_3 .

If n is even, the solutions are:

$\{1, -1, 1 + I_2 - I_3, 1 + I_1 - I_3, 1 + I_1 - I_3, -1 - I_2 + I_3, -1 - I_1 + I_3, 1 - 2I_3, -1 + I_2 + I_3, 1 - I_1 - I_3, -1 + I_1 + I_3, -1 + 2I_3\}$. This set forms an abelian group under multiplication, and it is isomorphic to $Z_2 \times Z_2 \times Z_3$.

2. Conclusion

In this paper, we computed all 3-cyclic refined neutrosophic Diophantine roots of unity, where we proved that there exist only three solutions for the case of odd order (n), and twelve different solutions

for the case of even order (n). On the other hand, we classified the group generated from all solutions as a direct product of finite cyclic groups.

In the future, we aim to extend our results for the general case of the n -cyclic refined neutrosophic ring of integers.

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