On Some Novel Results About Fuzzy n-Standard Number Theoretical Systems and Fuzzy Pythagoras Triples

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Abstract
The main goal of this work is to study the solutions of linear congruencies in the standard and n-standard fuzzy number theoretical systems, where we present necessary and sufficient conditions for fuzzy congruence to be solvable in these systems. Also, we provide the conditions of fuzzy nilpotency and fuzzy invertibility modulo integers with many illustrated examples. On the other hand, we suggest an algorithm to generate fuzzy Pythagoras triples and fuzzy Pythagoras quadruples in the standard fuzzy number theoretical system.

Keywords: standard fuzzy number theoretical system; n-standard system; fuzzy Pythagoras triple; fuzzy nilpotent element; fuzzy invertible element

1. Introduction and preliminaries

Fuzzy logic is a revolutionary style of generalization of classical logic, in which the concept of a membership function is used to express the degree to which an element belongs to a specific set [4].
Fuzzy logic and its generalizations have been widely used in many different areas of knowledge, such as decision-making theory [1], algebraic structures [2-3], and even engineering [5].
The fuzzy membership function began to be used in the study of number theory as a new branch of theoretical mathematics in [6], where for the first time the concept of a fuzzy number theoretical system, fuzzy Diophantine equations, and even fuzzy congruencies and division was formulated.
Recently, these previous results were built upon in the study of a new type of commutative algebras, which are called two-fold fuzzy algebras [7-8].
In this research paper, we aim to expand the results of previous studies, as we present an algorithm for calculating fuzzy Pythagorean triples and quadruples, as well as solving fuzzy linear congruencies in the standard fuzzy number theoretical system, and n-standard number theoretical system.
First, we recall some basic concepts:

Definition [6]:
The n-standard fuzzy number theoretical system is defined as (Z,μ), with:

$$\mu:\mathbb{Z}\to]0,1];\ \mu(x)=\begin{cases}\frac{1}{|x|^n}\ ;\ x\neq 0\\ 1\ ;\ x=0\end{cases}.$$

The standard fuzzy number theoretical system is defined as (Z,μ), with:

$$\mu:\mathbb{Z}\to]0,1];\ \mu(x)=\begin{cases}\frac{1}{|x|}\ ;\ x\neq 0\\ 1\ ;\ x=0\end{cases}$$

Definition [6]:
Let $(\mathbb{Z},\mu)$ be a fuzzy number theoretical system (FNTS), then for $a,b\in\mathbb{Z}$, we say that $a|b$ if and only if $\frac{\mu(b)}{\mu(a)}\in\mathbb{Z}^+$.

Definition:
1] Let $a,b\in(\mathbb{Z},\mu)$, we say (a,b) are relatively prime if and only if they not have a common divisor.

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2] $a \in \mathbb{Z}$ is called a fuzzy prime element if and only if it has not any divisor different from itself.

Definition [6]:

Let $(\mathbb{Z}, \mu)$ be a (FNTS), and $a, b, c \in \mathbb{Z}$, we define:

if and only if:$a \equiv b(mod\ c)$

$\frac{|\mu(a)-\mu(b)|}{\mu(c)} \in \mathbb{Z}^+$

Definition [6]:

Let $(\mathbb{Z}, \mu)$ be a (FNTS), we define:

1] The linear Diophantine fuzzy equation in one variable $a.\mu(x) = \mu(b)$; $a \in \mathbb{Z}$, $x, b \in \mathbb{Z}$.

2] The linear fuzzy Diophantine equation in two variables:

; $a, b, l, c, x, y \in \mathbb{Z}. a.\mu(x) + b.\mu(y) = l.\mu(c)$

3] The fuzzy Pythagoras triple $(x\ , y, z)$:

$$(\mu(x))^2 + (\mu(y))^2 = (\mu(z))^2$$

4] The fuzzy Pythagoras quadruple (x, y, z, t):

$$(\mu(x))^2 + (\mu(y))^2 + (\mu(z))^2 = (\mu(t))^2$$

5] The fuzzy Fermat's triples (x, y, z):

$$(\mu(x))^n + (\mu(y))^n = (\mu(z))^n; \quad n \geq 3$$

Main Discussion

Theorem:

Let (x, y, z) be a fuzzy Pythagoras triple in the standard fuzzy number theoretical system (Z, μ), then (zx, zy, xy) is a classical Pythagoras triple in $\mathbb{Z}$.

Proof:

$(\mu(x))^2 + (\mu(y))^2 = (\mu(z))^2$, thus $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}$, hence:

$z^2x^2 + z^2y^2 = x^2y^2$, so that $(zx)^2 + (zy)^2 = (xy)^2$.

Theorem:

Let (a, b, c) be a classical Pythagoras triple in $\mathbb{Z}$, then there exists a fuzzy Pythagoras triple in the standard system $(\mathbb{Z}, \mu)$ generated by (a, b, c).

Proof:

Since (a, b, c) is a Pythagoras triple in $\mathbb{Z}$, we can write $a^2 + b^2 = c^2$.

Put: $\begin{cases} x = ac \\ y = bc \\ z = ab \end{cases}$

$(\mu(x))^2 + (\mu(y))^2 = \frac{1}{x^2} + \frac{1}{y^2} = \frac{x^2+y^2}{x^2y^2} = \frac{c^2(a^2+b^2)}{a^2b^2c^4} = \frac{a^2+b^2}{a^2b^2c^2} = \frac{c^2}{a^2b^2c^2} = \frac{1}{a^2b^2} = \frac{1}{z^2} = (\mu(z))^2$, which implies the proof.

Example:

Consider the classical Pythagoras triple $(a, b, c) = (3,4,5)$, hence $(x, y, z) = (15,20,12)$ is a fuzzy Pythagoras triple in the standard system $(\mathbb{Z}, \mu)$.

Theorem:

Let (x, y, z, t) be a Pythagoras fuzzy quadruple in the standard system $(\mathbb{Z}, \mu)$, then (tyz, txz, txy, xyz) is a classical Pythagoras quadruple.

Proof:

We have: $(\mu(x))^2 + (\mu(y))^2 + (\mu(z))^2 = (\mu(t))^2$, hence:

$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{t^2}$, thus

$(tyz)^2 + (txz)^2 + (txy)^2 = (xyz)^2$.

Theorem:

Let (a, b, c, d) be a classical Pythagoras quadruple in $\mathbb{Z}$, then there exists a fuzzy Pythagoras quadruple in $(\mathbb{Z}, \mu)$ generated by (a, b, c, d).

Proof:

It can be held directly by putting: $\begin{cases} x = bcd \\ y = acd \\ z = abd \\ t = abc \end{cases}$

Theorem:

Let $(\mathbb{Z}, \mu)$ be the n-standard fuzzy number theoretical system with $n \geq 2$, the $(\mathbb{Z}, \mu)$ has no Pythagoras triples.

Proof:

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Assume that (x,y,z) is a fuzzy Pythagoras triple in $(\mathbb{Z}, \mu)$, hence:
$(\mu(x))^2 + (\mu(y))^2 = (\mu(z))^2$, hence:
$\frac{1}{|x|^{2n}} + \frac{1}{|y|^{2n}} = \frac{1}{|z|^{2n}} \Rightarrow |z|^{2n}.(|x|^{2n} + |y|^{2n}) = |xy|^{2n} \Rightarrow |z|.\sqrt[2n]{|x|^{2n} + |y|^{2n}} = |xy| \in \mathbb{Z}^+$,
thus: $\sqrt[2n]{|x|^{2n} + |y|^{2n}} = t \in \mathbb{Z}^+$
This means that:
$|x|^{2n} + |y|^{2n} = t^{2n}$.
This means that $(|x|, |y|, t)$ is a solution for Fermat's Diophantine equation
$|x|^k + |y|^k = t^k; k \geq 4$,
which contradicts Fermat's last theorem, and the proof is complete.

Theorem:

Let $(\mathbb{Z}, \mu)$ be the standard fuzzy number theoretical system, then $(\mathbb{Z}, \mu)$ has no Fermat's triples.
Proof:
Assume that (x, y, z) is a fuzzy Fermat's triple, then:
$(\mu(x))^k + (\mu(y))^k = (\mu(z))^k$, hence:
$\frac{1}{|x|^k} + \frac{1}{|y|^k} = \frac{1}{|z|^k}$,
thus: $|z^k|(|x^k| + |y^k|) = |xy|^k$,
so that $\sqrt[k]{|x|^k + |y|^k} \in \mathbb{Z}^+$, hence
$|x|^k + |y|^k = |t|^k$,
which means that (x, y, t) is a solution of Fermat's last equation, where
$t \in \mathbb{Z}$, which is a contradiction, and our proof is complete.

Remark:

By a similar argument, we can prove that the n-standard system has no Fermat's triples

Theorem:

Let $(\mathbb{Z}, \mu)$ be the standard system, then:
1] a is idempotent modulo c if and only if $a^2 \big| |c|(|a| - 1)$.
2] a is nilpotent modulo c with order n if and only if $a^n \big| |c|^n (a^{n-1} - 1)$ and $a^k \big| |c|^k (a^{k-1} - 1)$; k < n.
Proof:
1] a is idempotent modulo c if and only if
$a \equiv a^2 (mod c)$, hence $\frac{|\mu(a) - \mu(a^2)|}{\mu(c)} \in \mathbb{Z}^+$,
thus $\frac{\frac{1}{|a|} - \frac{1}{|a^2|}}{\frac{1}{|c|}} \in \mathbb{Z}^+$, so that
$|c|.\frac{(|a|-1)}{a^2} \in \mathbb{Z}^+$, and $a^2 \big| |c|.(|a| - 1)$.
2] a is nilpotent modulo c with order n if and only if $a \equiv a^n (mod c)$, and $a \not\equiv a^k (mod c)$ for all k < n.
This is equivalent to: $\frac{|\mu(a) - \mu(a^n)|}{\mu(c)} \in \mathbb{Z}^+$, hence
$\frac{|c|^n.(|a^{n-1}|-1)}{|a^n|} \in \mathbb{Z}^+$, thus
$|a^n| \big| |c|^n.(|a^{n-1}| - 1)$ and $|a^k| \big| |c|^k.(|a^{k-1}| - 1)$.

Example:

Consider *a=3, c=18*, hence $a^2 = 9|c.(a-1) = 36$, thus *a=3* is idempotent modulo *c=18*.

Definition:

Let $a, c \in \mathbb{Z}$, hence a is called invertible modulo(c) if and only if there exists $x \in \mathbb{Z}$ such that $ax \equiv 1(mod\ c)$.

Theorem:

Let $(\mathbb{Z}, \mu)$ be the standard system, hence (a) is invertible modulo(c) if and only if there exists $x \in \mathbb{Z}$ such that:
$|ax| \big| |c|$.
Proof:
The integer *(a)* is fuzzy invertible modulo *(c)* if and only if there exists
$x \in \mathbb{Z}$ such that:
$ax \equiv 1(mod\ c) \Rightarrow \frac{\left|\frac{1}{|ax|} - 1\right|}{\frac{1}{|c|}} \in \mathbb{Z}^+$,
thus $|c| - \frac{|c|}{|ax|} = t \in \mathbb{Z}^+$, which is equivalent to:
$|ax| \big| |c|$

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Example:
Take c=5 , a=3, hence a=3 is not invertible modulo 5 for c=6 , a=3, hence a=3 is invertible modulo 6, that is because there exists $x = 1$ or $x = 2$ such that $ax|c$.
Remark:
In the standard system $(\mathbb{Z}, \mu)$, we can see:
1] If c is prime in $\mathbb{Z}$, hence c is only invertible modulo itself.
2] If $a|c$ in $\mathbb{Z}$, then a is invertible modulo c.
Definition:
Let $(\mathbb{Z}, \mu)$ be any fuzzy number theoretical system, and $a, c \in \mathbb{Z}$, then the set $\{x \in \mathbb{Z};\ ax \equiv_F 1(mod\ c)\}$ is called the set of (a) inverses, we denote it by a^c_{inv}.
Remark:
In the standard system $(\mathbb{Z}, \mu)$, for $a, c \in \mathbb{Z}$, we have
$a^c_{inv} = \{x \in \mathbb{Z};\ |ax|\big||c|\}$.
Remark:
In the n-standard system $(\mathbb{Z}, \mu)$, for $a, c \in \mathbb{Z}$, we have
$a^c_{inv} = \{x \in \mathbb{Z};\ |ax|^n\big||c|^n\}$.
Example:
In the standard system $(\mathbb{Z}, \mu)$, for $a = 3, c = 6$, we have: $3^6_{inv} = \{1,2,-1,-2\}$
Remark:
In the standard system $(\mathbb{Z}, \mu)$, we have:
$a^c_{inv} = \{x \in \mathbb{Z};\ \exists q \in \mathbb{Z}^+; x = \frac{|c|}{|q:a|}\}$.
In the n-standard system $(\mathbb{Z}, \mu)$, we have:
$a^c_{inv} = \{x \in \mathbb{Z};\ \exists q \in \mathbb{Z}^+; x = \frac{|c|}{\sqrt[n]{q}.|a|}\}$.
Solving the linear fuzzy congruence in the n-standard system:
Definition:
Let $(\mathbb{Z}, \mu)$ be any fuzzy number theoretical system, we define the linear congruence in one variable as follows:
$ax \equiv_F b(mod\ c); a, b, c \in \mathbb{Z}$.
Theorem:
Let $(\mathbb{Z}, \mu)$ be the standard system, the solutions of $ax \equiv_F b(mod\ c)$ are:
$|x| = \frac{|bc|}{|ab|t+|ac|}$ or
$|x| = \frac{|bc|}{|ac|-t|ab|}$
under the conditions:
$|ab|t + |ac|\big||bc|$ or $\frac{|bc|}{|ac|-t|ab|}$; $t \in \mathbb{Z}^+$
Proof:
$ax \equiv_F b(mod\ c)$ is equivalent to:
$\frac{|\mu(c)-\mu(b)|}{\mu(c)} = t \in \mathbb{Z}^+ \Rightarrow \left|\frac{1}{|ax|} - \frac{1}{|b|}\right| = \frac{t}{|c|}$ (1)
We have two possible cases:
Case(1):
If $|ax| \le |b|$, hence equation (1) implies:

$$\frac{1}{|ax|} = \frac{t}{|c|} + \frac{1}{|b|} = \frac{t|b| + |c|}{|bc|} \Rightarrow |x| = \frac{|bc|}{t|ab| + |ac|}$$

Under the condition $t|ab| + |ac|\big||bc|$.
Case (2):
If $|ax| > |b|$, hence (1) implies:

$$\frac{1}{|ax|} = \frac{1}{|b|} - \frac{t}{|c|} = \frac{|c| - t|b|}{|bc|} \Rightarrow |x| = \frac{|bc|}{|ac| - t|ab|}$$

Under the condition $|ac| - t|ab|\big||bc|$.
Example:
Consider $3x \equiv_F 4(mod\ 9)$, where a=3, b=4, c=9.

$$\begin{cases} |bc| = 36 \\ |ac| = 27 \\ |ab| = 12 \end{cases}$$

$|ac| - t|ab| = 27 - 12t|36$ if: $t \in \{2,3\}$.

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For t=2, we get $|x| = \frac{36}{3} = 12$, and for t=3, we get $|x| = \frac{36}{-9} = -4$, which is impossible, thus $x \in \{12, -12\}$.
$|ac| + t|ab| = 27 + 12t|36$ if: $t \in \{-2, -3\}$, which is a contradiction, since t is positive.

Theorem:

Let $(\mathbb{Z}, \mu)$ be the n-standard system, then the solutions of $ax \equiv_F b(mod\ c)$ are:

$|x| = \frac{|bc|}{(|ac|^n - t|ab|^n)^{\frac{1}{n}}}$ or

$|x| = \frac{|bc|}{(|ac|^n + t|ab|^n)^{\frac{1}{n}}}$ under the conditions:

$(|ac|^n - t|ab|^n)\big||bc|$ or $(|ac|^n + t|ab|^n)\big||bc|$ and

$|ac|^n - t|ab|^n = S_1{}^n$ or $|ac|^n + t|ab|^n = S_2{}^n$; $S_1, S_2 \in \mathbb{Z}$

Proof:

Assume that $ax \equiv_F b(mod\ c)$, hence:

$$\left|\frac{1}{|ax|^n} - \frac{1}{|b|^n}\right| = \frac{t}{|c|^n}\ ; t \in \mathbb{Z}^+$$

If $|ax| \le |b|$, then

$\frac{1}{|ax|^n} = \frac{t}{|c|^n} + \frac{1}{|b|^n}$, hence:

$|x^n| = \frac{|bc|^n}{|ab|^n t + |ac|^n}$

under the condition

$|ab|^n t + |ac|^n||bc|^n$.

If $|ax| > |b|$, then

$\frac{-1}{|ax|^n} = \frac{t}{|c|^n} - \frac{1}{|b|^n}$, hence:

$|x^n| = \frac{|bc|^n}{|ac|^n - t|ab|^n}$

under the condition

$|ac|^n - t|ab|^n||bc|^n$.

Which implies the proof.

2. Conclusion

In this paper, we have studied the solutions of linear congruencies in the standard and n-standard fuzzy number theoretical systems, where we presented necessary and sufficient conditions for fuzzy congruence to be solvable in these systems. Also, we provided the conditions of fuzzy nilpotency and fuzzy Invertibility modulo integers with many illustrated examples.

On the other hand, we suggested an algorithm to generate fuzzy Pythagoras triples and fuzzy Pythagoras quadruples in the standard fuzzy number theoretical system.

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