On The Real Inner Products and Orthogonality for Plithogenic Vector Spaces of Orders 4 and 5

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Abstract

This paper will study the necessary conditions and sufficient conditions for the problem of orthogonality in 4-plithogenic and 5-plithogenic vector spaces, where we provided a definition of the real scalar products defined above these spaces, and we used the new concept of scalar product that was defined in determining the necessary and sufficient condition for the orthogonality of symbolic 4-plithogenic and 5-plithogenic vectors, in addition to calculating various norms and their metric properties. We have also provided some examples that aim to explain the scientific contribution of this work.

Keywords: plithogenic vector space; inner product; plithogenic norm; orthogonal basis; ortho-normed vectors.

1. Introduction and preliminaries

The study of plithogenic sets and their applications began at the hands of the researcher Smarandache [2], where he first defined the plithogenic set and through it he was able to provide a good visualization of the algebraic structure of the plithogenic set.

This view was later extended to the definition of plithogenic rings [1,4, 19-20], and then these rings were used in the study of modules and plithogenic vector spaces of various orders [3, 28-29], and also in the determination of algebraic bases for them.

Studies have evolved to include the theory of plithogenic numbers [12-16] and the properties of integer numbers associated with them in a similar way to what was done for neutrosophic numbers [6, 9, 17, 22], in addition to many diverse applications in cryptography [23-24], matrix theory [7,8,10-11], and even in solving algebraic equations [5, 25-27].

In this paper, we present a development of the results presented in [30], where we expanded the study to include symbolic 4-plithogenic and 5-plithogenic spaces, where we studied the inner products defined above

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these spaces, as well as various issues of orthogonality and its conditions, in addition to the concept of the orthogonal rule and the organizing rule related to this new type of spaces.

We want to draw the reader's attention that the results from this study are generalizable for higher-order spaces with more complexity in the resulting calculations.

2. Main results and discussion

Symbolic 4-Plithogenic Inner products and orthogonality

Definition:

Let V be R-Vector Space:

Let $4-SP_R=\{l_0+\sum_{i=1}^{4} l_iP_i\,;l_i\in R\}$ be the 4-plithogenic field, $4-SP_V=\{q_0+\sum_{i=1}^{4} q_iP_i\,;q_i\in V\}$ the 4-plithogenic vector space, then:

$\varphi: 4-SP_V\times 4-SP_V\to 4-SP_R$ is called 4-plithogenic inner product if:

1). $\varphi(t,n)=\varphi(n,t)$ where $n,t\in 4-SP_V$.

2). $\varphi(t,t)\ge 0$ for all $t\in 4-SP_V$.

3). $\varphi(u+v,t)=\varphi(u,t)+\varphi(v,t)$.

4). $\varphi(a.t,v)=a\varphi(t,v); t,v\in 4-SP_V, a\in 4-SP_R$.

Theorem.

If $f:V\times V\to R$ is a real product:

For $q=q_0+\sum_{i=1}^{4} q_iP_i\,, n=n_0+\sum_{i=1}^{4} n_iP_i\in 4-SP_V$, we define:

$\varphi(q,n)=f(q_0,n_0)+P_1[f(\sum_{i=0}^{1} q_i\,,\sum_{i=0}^{1} n_i)-f(q_0,n_0)]+P_2[f(\sum_{i=0}^{2} q_i\,,\sum_{i=0}^{2} n_i)-f(\sum_{i=0}^{1} q_i\,,\sum_{i=0}^{1} n_i)]+P_3[f(\sum_{i=0}^{3} q_i\,,\sum_{i=0}^{3} n_i)-f(\sum_{i=0}^{2} q_i\,,\sum_{i=0}^{2} n_i)]+P_4[f(\sum_{i=0}^{4} q_i\,,\sum_{i=0}^{4} n_i)-f(\sum_{i=0}^{3} q_i\,,\sum_{i=0}^{3} n_i)]$.

Then, $\varphi: 4-SP_V\times 4-SP_V\to 4-SP_R$ is 4-plithogenic inner product.

Proof.

Let $f:V\times V\to R$ be an inner product over V.

Suppose $\varphi: 4-SP_V\times 4-SP_V\to 4-SP_R$, where:

$\varphi(q,n)=f(q_0,n_0)+P_1[f(\sum_{i=0}^{1} q_i\,,\sum_{i=0}^{1} n_i)-f(q_0,n_0)]+P_2[f(\sum_{i=0}^{2} q_i\,,\sum_{i=0}^{2} n_i)-f(\sum_{i=0}^{1} q_i\,,\sum_{i=0}^{1} n_i)]+P_3[f(\sum_{i=0}^{3} q_i\,,\sum_{i=0}^{3} n_i)-f(\sum_{i=0}^{2} q_i\,,\sum_{i=0}^{2} n_i)]+P_4[f(\sum_{i=0}^{4} q_i\,,\sum_{i=0}^{4} n_i)-f(\sum_{i=0}^{3} q_i\,,\sum_{i=0}^{3} n_i)]$.

φ is 4-plithogenic real inner product on $4-SP_V$.

$\varphi(q,n)=f(q_0,n_0)+P_1[f(\sum_{i=0}^{1} q_i\,,\sum_{i=0}^{1} n_i)-f(q_0,n_0)]+P_2[f(\sum_{i=0}^{2} q_i\,,\sum_{i=0}^{2} n_i)-f(\sum_{i=0}^{1} q_i\,,\sum_{i=0}^{1} n_i)]+P_3[f(\sum_{i=0}^{3} q_i\,,\sum_{i=0}^{3} n_i)-f(\sum_{i=0}^{2} q_i\,,\sum_{i=0}^{2} n_i)]+P_4[f(\sum_{i=0}^{4} q_i\,,\sum_{i=0}^{4} n_i)-f(\sum_{i=0}^{3} q_i\,,\sum_{i=0}^{3} n_i)]=f(n_0,q_0)+P_1[f(\sum_{i=0}^{1} n_i\,,\sum_{i=0}^{1} q_i)-f(n_0,q_0)]+P_2[f(\sum_{i=0}^{2} n_i\,,\sum_{i=0}^{2} q_i)-f(\sum_{i=0}^{1} n_i\,,\sum_{i=0}^{1} q_i)]+P_3[f(\sum_{i=0}^{3} n_i\,,\sum_{i=0}^{3} q_i)-f(\sum_{i=0}^{2} n_i\,,\sum_{i=0}^{2} q_i)]+P_4[f(\sum_{i=0}^{4} n_i\,,\sum_{i=0}^{4} q_i)-f(\sum_{i=0}^{3} n_i\,,\sum_{i=0}^{3} q_i)]=\varphi(n,q)$.

$\varphi(q,q)=f(q_0,q_0)+P_1[f(\sum_{i=0}^{1} q_i\,,\sum_{i=0}^{1} q_i)-f(q_0,q_0)]+P_2[f(\sum_{i=0}^{2} q_i\,,\sum_{i=0}^{2} q_i)-f(\sum_{i=0}^{1} q_i\,,\sum_{i=0}^{1} q_i)]+P_3[f(\sum_{i=0}^{3} q_i\,,\sum_{i=0}^{3} q_i)-f(\sum_{i=0}^{2} q_i\,,\sum_{i=0}^{2} q_i)]+P_4[f(\sum_{i=0}^{4} q_i\,,\sum_{i=0}^{4} q_i)-f(\sum_{i=0}^{3} q_i\,,\sum_{i=0}^{3} q_i)]=\|q_0\|^2+P_1[\|\sum_{i=0}^{1} q_i\|^2-\|q_0\|^2]+P_2[\|\sum_{i=0}^{2} q_i\|^2-\|\sum_{i=0}^{1} q_i\|^2]+P_3\left[\left\|\sum_{i=0}^{3} q_i\right\|^2-\|\sum_{i=0}^{2} q_i\|^2\right]+P_4\left[\|\sum_{i=0}^{4} q_i\|^2-\left\|\sum_{i=0}^{3} q_i\right\|^2\right]$.

And that is because:

$$\begin{cases} \|q_0\|^2\ge 0 \\ \|q_0\|^2+\left[\left\|\sum_{i=0}^{1} q_i\right\|^2-\|q_0\|^2\right]=\left\|\sum_{i=0}^{1} q_i\right\|^2\ge 0 \\ \|q_0\|^2+\left[\left\|\sum_{i=0}^{1} q_i\right\|^2-\|q_0\|^2\right]+\left[\left\|\sum_{i=0}^{2} q_i\right\|^2-\left\|\sum_{i=0}^{1} q_i\right\|^2\right]=\left\|\sum_{i=0}^{2} q_i\right\|^2\ge 0 \\ \|q_0\|^2+\left[\left\|\sum_{i=0}^{1} q_i\right\|^2-\|q_0\|^2\right]+\left[\left\|\sum_{i=0}^{2} q_i\right\|^2-\left\|\sum_{i=0}^{1} q_i\right\|^2\right]+\left[\left\|\sum_{i=0}^{3} q_i\right\|^2-\left\|\sum_{i=0}^{2} q_i\right\|^2\right]=\left\|\sum_{i=0}^{3} q_i\right\|^2\ge 0 \\ \|q_0\|^2+\left[\left\|\sum_{i=0}^{1} q_i\right\|^2-\|q_0\|^2\right]+\left[\left\|\sum_{i=0}^{2} q_i\right\|^2-\left\|\sum_{i=0}^{1} q_i\right\|^2\right]+\left[\left\|\sum_{i=0}^{3} q_i\right\|^2-\left\|\sum_{i=0}^{2} q_i\right\|^2\right]+\left[\left\|\sum_{i=0}^{4} q_i\right\|^2-\left\|\sum_{i=0}^{3} q_i\right\|^2\right]=\left\|\sum_{i=0}^{4} q_i\right\|^2\ge 0 \end{cases}$$

Now, we let $u=u_0+\sum_{i=1}^{4} u_iP_i\in 4-SP_V$, we have:

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$$\varphi(u+q,n) = f(u_0+q_0,n_0) + P_1\left[f\left(\sum_{i=0}^{1}(q_i+u_i), \sum_{i=0}^{1} n_i\right) - f(u_0+q_0,n_0)\right]$$
$$+ P_2\left[f\left(\sum_{i=0}^{2}(q_i+u_i), \sum_{i=0}^{2} n_i\right) - f\left(\sum_{i=0}^{1}(q_i+u_i), \sum_{i=0}^{1} n_i\right)\right]$$
$$+ P_3\left[f\left(\sum_{i=0}^{3}(q_i+u_i), \sum_{i=0}^{3} n_i\right) - f\left(\sum_{i=0}^{2}(q_i+u_i), \sum_{i=0}^{2} n_i\right)\right]$$
$$+ P_4\left[f\left(\sum_{i=0}^{4}(q_i+u_i), \sum_{i=0}^{4} n_i\right) - f\left(\sum_{i=0}^{3}(q_i+u_i), \sum_{i=0}^{3} n_i\right)\right] = \varphi(u,n) + \varphi(w,n)$$

Let $a = a_0 + \sum_{i=1}^{4} a_i P_i \in 4 - SP_R$, then:

$\varphi(a.q,n) = f(a_0q_0,n_0) + P_1\left[f\left((\sum_{i=0}^{1} a_i)(\sum_{i=0}^{1}(q_i+u_i)), \sum_{i=0}^{1} n_i\right) - f(a_0q_0,n_0)\right] +$
$P_2\left[f\left((\sum_{i=0}^{2} a_i)(\sum_{i=0}^{2}(q_i+u_i)), \sum_{i=0}^{2} n_i\right) - f\left((\sum_{i=0}^{1} a_i)(\sum_{i=0}^{1}(q_i+u_i)), \sum_{i=0}^{1} n_i\right)\right] +$
$P_3\left[f\left((\sum_{i=0}^{3} a_i)(\sum_{i=0}^{3}(q_i+u_i)), \sum_{i=0}^{3} n_i\right) - f\left((\sum_{i=0}^{2} a_i)(\sum_{i=0}^{2}(q_i+u_i)), \sum_{i=0}^{2} n_i\right)\right] +$
$P_4\left[f\left((\sum_{i=0}^{4} a_i)(\sum_{i=0}^{4}(q_i+u_i)), \sum_{i=0}^{4} n_i\right) - f\left((\sum_{i=0}^{3} a_i)(\sum_{i=0}^{3}(q_i+u_i)), \sum_{i=0}^{3} n_i\right)\right] = \big(a_0 +$
$\sum_{i=1}^{4} a_i P_i\big)\varphi(q,n) = a.\varphi(q,n).$

Remark:

Suppose that $\varphi: 4 - SP_V \times 4 - SP_V \to 4 - SP_R$,

Put $f: V \times V \to R$, with $f(q_0,n_0) = \varphi(q_0,n_0)$, where $q_0, n_0 \in V$.

f is a Euclidean inner product clearly.

Example.

Regard the Euclidean real inner product defined on R^2

$f[(\acute{d}_1, d_1''), (\acute{d}_2, d_2'')] = \acute{d}_1\acute{d}_2 + d_1''d_2''$.

The inner product on $4 - SP_V$ is:

$\varphi: 4 - SP_V \times 4 - SP_V \to 4 - SP_R$;

$$\varphi[(d_0,y_0) + (d_1,y_1)P_1 + (d_2,y_2)P_2 + (d_3,y_3)P_3 + (d_4,y_4)P_4, (z_0,s_0) + (z_1,s_1)P_1 + (z_2,s_2)P_2 + (z_3,s_3)P_3 + (z_4,s_4)P_4]$$
$$= f\big((d_0,y_0),(z_0,s_0)\big) + P_1\left[f\left(\left(\sum_{i=0}^{1} d_i, \sum_{i=0}^{1} y_i\right), \left(\sum_{i=0}^{1} z_i, \sum_{i=0}^{1} s_i\right)\right) - f\big((d_0,y_0),(z_0,s_0)\big)\right]$$
$$+ P_2\left[f\left(\left(\sum_{i=0}^{2} d_i, \sum_{i=0}^{2} y_i\right), \left(\sum_{i=0}^{2} z_i, \sum_{i=0}^{2} s_i\right)\right) - f\left(\left(\sum_{i=0}^{1} d_i, \sum_{i=0}^{1} y_i\right), \left(\sum_{i=0}^{1} z_i, \sum_{i=0}^{1} s_i\right)\right)\right]$$
$$+ P_3\left[f\left(\left(\sum_{i=0}^{3} d_i, \sum_{i=0}^{3} y_i\right), \left(\sum_{i=0}^{3} z_i, \sum_{i=0}^{3} s_i\right)\right) - f\left(\left(\sum_{i=0}^{2} d_i, \sum_{i=0}^{2} y_i\right), \left(\sum_{i=0}^{2} z_i, \sum_{i=0}^{2} s_i\right)\right)\right]$$
$$+ P_4\left[f\left(\left(\sum_{i=0}^{4} d_i, \sum_{i=0}^{4} y_i\right), \left(\sum_{i=0}^{4} z_i, \sum_{i=0}^{4} s_i\right)\right) - f\left(\left(\sum_{i=0}^{3} d_i, \sum_{i=0}^{3} y_i\right), \left(\sum_{i=0}^{3} z_i, \sum_{i=0}^{3} s_i\right)\right)\right]$$
$$= d_0z_0 + y_0s_0 + P_1\left[\left(\sum_{i=0}^{1} d_i\right)\left(\sum_{i=0}^{1} z_i\right) + \left(\sum_{i=0}^{1} y_i\right)\left(\sum_{i=0}^{1} s_i\right) - d_0z_0 - y_0s_0\right]$$
$$+ P_2\left[\left(\sum_{i=0}^{2} d_i\right)\left(\sum_{i=0}^{2} z_i\right) + \left(\sum_{i=0}^{2} y_i\right)\left(\sum_{i=0}^{2} s_i\right) - \left(\sum_{i=0}^{1} d_i\right)\left(\sum_{i=0}^{1} z_i\right) - \left(\sum_{i=0}^{1} y_i\right)\left(\sum_{i=0}^{1} s_i\right)\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} d_i\right)\left(\sum_{i=0}^{3} z_i\right) + \left(\sum_{i=0}^{3} y_i\right)\left(\sum_{i=0}^{3} s_i\right) - \left(\sum_{i=0}^{2} d_i\right)\left(\sum_{i=0}^{2} z_i\right) - \left(\sum_{i=0}^{2} y_i\right)\left(\sum_{i=0}^{2} s_i\right)\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} d_i\right)\left(\sum_{i=0}^{4} z_i\right) + \left(\sum_{i=0}^{4} y_i\right)\left(\sum_{i=0}^{4} s_i\right) - \left(\sum_{i=0}^{3} d_i\right)\left(\sum_{i=0}^{3} z_i\right) - \left(\sum_{i=0}^{3} y_i\right)\left(\sum_{i=0}^{3} s_i\right)\right]$$

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Remark.

The norm of $q = q_0 + \sum_{i=1}^{4} q_i P_i$ is defined as follows:

$\|q\| = \sqrt{\varphi(q,q)} = \|q_0\| + P_1[\|\sum_{i=0}^{1} q_i\| - \|q_0\|] + P_2[\|\sum_{i=0}^{2} q_i\| - \|\sum_{i=0}^{1} q_i\|] + P_3[\|\sum_{i=0}^{3} q_i\| - \|\sum_{i=0}^{2} q_i\|] + P_4[\|\sum_{i=0}^{4} q_i\| - \|\sum_{i=0}^{3} q_i\|]$.

Theorem.

Let φ be 4-plithogenic inner product on $4 - SP_V$, then:

1). $\|q\| \geq 0; q \in 4 - SP_V$

2). $\|a.q\| = |a|.\|q\|; a \in 4 - SP_R$

3). $\|q + n\| \leq \|q\| + \|n\|; n \in 4 - SP_V$

4). $|\varphi(q,n)| \leq \|q\|.\|n\|$

5). $q \perp n$ equivalents that $q_0 \perp n_0, \sum_{i=0}^{1} q_i \perp \sum_{i=0}^{1} n_i, \sum_{i=0}^{2} q_i \perp \sum_{i=0}^{2} n_i, \sum_{i=0}^{3} q_i \perp \sum_{i=0}^{3} n_i, \sum_{i=0}^{4} q_i \perp \sum_{i=0}^{4} n_i$.

6). If $q \perp n$, implies $\|q + n\|^2 = \|q\|^2 + \|n\|^2$

Proof.

1). It is an easy to be proven.

2). Let $a = a_0 + \sum_{i=1}^{4} a_i P_i \in 4 - SP_R, q = q_0 + \sum_{i=1}^{4} q_i P_i \in 4 - SP_V$, we have:

$\|a.q\|^2 = \varphi(a.q, a.q) = a^2 \varphi(q,q)$, thus $\|a.q\| = |a|.\|q\|$.

3). $\|q + n\| = \|q_0 + n_0\| + P_1[\|\sum_{i=0}^{1}(q_i + n_i)\| - \|q_0 + n_0\|] + P_2[\|\sum_{i=0}^{2}(q_i + n_i)\| - \|\sum_{i=0}^{1}(q_i + n_i)\|] + P_3[\|\sum_{i=0}^{3}(q_i + n_i)\| - \|\sum_{i=0}^{2}(q_i + n_i)\|] + P_4[\|\sum_{i=0}^{4}(q_i + n_i)\| - \|\sum_{i=0}^{3}(q_i + n_i)\|]$, we have:

$$\begin{cases} \|q_0 + n_0\| \leq \|q_0\| + \|n_0\| \\ \left\|\sum_{i=0}^{1}(q_i + n_i)\right\| \leq \left\|\sum_{i=0}^{1}(q_i)\right\| + \left\|\sum_{i=0}^{1}(n_i)\right\| \\ \left\|\sum_{i=0}^{2}(q_i + n_i)\right\| \leq \left\|\sum_{i=0}^{2}(q_i)\right\| + \left\|\sum_{i=0}^{2}(n_i)\right\| \\ \left\|\sum_{i=0}^{3}(q_i + n_i)\right\| \leq \left\|\sum_{i=0}^{3}(q_i)\right\| + \left\|\sum_{i=0}^{3}(n_i)\right\| \\ \left\|\sum_{i=0}^{4}(q_i + n_i)\right\| \leq \left\|\sum_{i=0}^{4}(q_i)\right\| + \left\|\sum_{i=0}^{4}(n_i)\right\| \end{cases}$$

therefore:

$\|q + n\| \leq \|q\| + \|n\|$.

4).

$|\varphi(q,n)| = |\varphi(q_0, n_0)| + P_1[|\varphi(\sum_{i=0}^{1}(q_i), \sum_{i=0}^{1}(n_i))| - |\varphi(q_0, n_0)|] + P_2[|\varphi(\sum_{i=0}^{2}(q_i), \sum_{i=0}^{2}(n_i))| - |\varphi(\sum_{i=0}^{1}(q_i), \sum_{i=0}^{1}(n_i))|] + P_3[|\varphi(\sum_{i=0}^{3}(q_i), \sum_{i=0}^{3}(n_i))| - |\varphi(\sum_{i=0}^{2}(q_i), \sum_{i=0}^{2}(n_i))|] + P_4[|\varphi(\sum_{i=0}^{4}(q_i), \sum_{i=0}^{4}(n_i))| - |\varphi(\sum_{i=0}^{3}(q_i), \sum_{i=0}^{3}(n_i))|]$.

By using Cauchy-Shwartz inequality, we get:

$$\begin{cases} |f(q_0, n_0)| \leq \|q_0\| + \|n_0\| \\ \left|f\left(\sum_{i=0}^{1}(q_i), \sum_{i=0}^{1}(n_i)\right)\right| \leq \left\|\sum_{i=0}^{1}(q_i)\right\| + \left\|\sum_{i=0}^{1}(n_i)\right\| \\ \left|f\left(\sum_{i=0}^{2}(q_i), \sum_{i=0}^{2}(n_i)\right)\right| \leq \left\|\sum_{i=0}^{2}(q_i)\right\| + \left\|\sum_{i=0}^{2}(n_i)\right\| \\ \left|f\left(\sum_{i=0}^{3}(q_i), \sum_{i=0}^{3}(n_i)\right)\right| \leq \left\|\sum_{i=0}^{3}(q_i)\right\| + \left\|\sum_{i=0}^{3}(n_i)\right\| \\ \left|f\left(\sum_{i=0}^{4}(q_i), \sum_{i=0}^{4}(n_i)\right)\right| \leq \left\|\sum_{i=0}^{4}(q_i)\right\| + \left\|\sum_{i=0}^{4}(n_i)\right\| \end{cases}$$

therefore,

$|\varphi(q,n)| \leq \|q\|.\|n\|$

5). $q \perp n$ equivalents that $\varphi(q,n) = 0$, hence:

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$$\begin{cases} f(q_0,n_0)=0 \Rightarrow q_0 \perp n_0 \\ f\left(\sum_{i=0}^{1}(q_i),\sum_{i=0}^{1}(n_i)\right) \Rightarrow \sum_{i=0}^{1}(q_i) \perp \sum_{i=0}^{1}(n_i) \\ f\left(\sum_{i=0}^{2}(q_i),\sum_{i=0}^{2}(n_i)\right) \Rightarrow \sum_{i=0}^{2}(q_i) \perp \sum_{i=0}^{2}(n_i) \\ \left(\sum_{i=0}^{3}(q_i),\sum_{i=0}^{3}(n_i)\right) \Rightarrow \sum_{i=0}^{3}(q_i) \perp \sum_{i=0}^{3}(n_i) \\ \left(\sum_{i=0}^{4}(q_i),\sum_{i=0}^{4}(n_i)\right) \Rightarrow \sum_{i=0}^{4}(q_i) \perp \sum_{i=0}^{4}(n_i) \end{cases}$$

6). If $q \perp n$, then $\|q+n\|^2 = \varphi(q+n,q+n) = \varphi(q,q)+\varphi(n,n) = \|q\|^2+\|n\|^2$.

Definition:

Let $E=\{O_1,\dots,O_n\}$ be a basis of $4-SP_V$.

E be an orthogonal basis is equivalent to write:

$\varphi(O_t,O_h)=0; i\neq j, 1\le t,h\le 5n$

It is called ortho-normed if:

$\begin{cases}\varphi(O_t,O_h)=0\\ \varphi(O_t,O_h)=1\end{cases}; t\neq h, 1\le t,h\le 5n$

Theorem.

Let $C=\{f_1,f_2,\dots,f_n\}$ be V-ortho-normed basis:

The set $C_P=\{m_i+(n_j-m_i)P_1+(s_k-n_j)P_2+(l_t-s_k)P_3+(h_d-l_t)P_4; m_i,n_j,s_k,h_d,l_t\in C, 1\le i,j,k,t,d\le n\}$ is $4-SP_V$ - ortho-normed basis.

Proof.

Let

$M_1=m_i+(n_j-m_i)P_1+(s_k-n_j)P_2+(l_t-s_k)P_3+(h_d-l_t)P_4\in C, M_2=\acute{m}_i+(\acute{n}_j-\acute{m}_i)P_1+(\acute{s}_k-\acute{n}_j)P_2+(l_t{}'-\acute{s}_k)P_3+(h_d{}'-l_t{}')P_4\in C$, we have:

$\varphi(M_1,M_2)=f(m_i,\acute{m}_i)+P_1[f(n_j,\acute{n}_j)-f(m_i,\acute{m}_i)]+P_2[f(s_k,\acute{s}_k)-f(n_j,\acute{n}_j)]+P_3[f(l_t,l_t{}')-f(s_k,\acute{s}_k)]+P_4[f(h_d,h_d{}')-f(l_t,l_t{}')]=0$, thus $M_1\perp M_2$.

On the other hand,

$\|M_1\|=\|m_i\|+P_1[\|n_j\|-\|m_i\|]+P_2[\|s_k\|-\|n_j\|]+P_3[\|l_t\|-\|s_k\|]+P_4[\|h_d\|-\|l_t\|]=1$, so that C_P is ortho-normed basis.

3. Symbolic 5-Plithogenic Inner products and orthogonality

Definition:

Let V be an R- vector space.

Let $5-SP_R=\{l_0+\sum_{i=1}^{5}l_iP_i; l_i\in R\}$ be the 5-plithogenic field, $5-SP_V=\{q_0+\sum_{i=1}^{5}q_iP_i; q_i\in V\}$ the 5-plithogenic vector space, then:

$\varphi: 5-SP_V\times 5-SP_V\to 5-SP_R$ is called 5-plithogenic inner product if:

1). $\varphi(t,n)=\varphi(n,t)$ for all $n,t\in 5-SP_V$.

2). $\varphi(t,t)\ge 0$ for all $\in 5-SP_V$.

3). $\varphi(u+v,t)=\varphi(u,t)+\varphi(v,t)$.

4). $\varphi(a.t,v)=a\varphi(t,v); t,v\in 5-SP_V, a\in 5-SP_R$.

Theorem.

If $f:V\times V\to R$ with:

For $q=q_0+\sum_{i=1}^{5}q_iP_i, n=n_0+\sum_{i=1}^{5}n_iP_i\in 5-SP_V$, we define:

$\varphi(q,n)=f(q_0,n_0)+P_1[f(\sum_{i=0}^{1}q_i,\sum_{i=0}^{1}n_i)-f(q_0,n_0)]+P_2[f(\sum_{i=0}^{2}q_i,\sum_{i=0}^{2}n_i)-f(\sum_{i=0}^{1}q_i,\sum_{i=0}^{1}n_i)]+P_3[f(\sum_{i=0}^{3}q_i,\sum_{i=0}^{3}n_i)-f(\sum_{i=0}^{2}q_i,\sum_{i=0}^{2}n_i)]+P_4[f(\sum_{i=0}^{4}q_i,\sum_{i=0}^{4}n_i)-f(\sum_{i=0}^{3}q_i,\sum_{i=0}^{3}n_i)]+P_5[f(\sum_{i=0}^{5}q_i,\sum_{i=0}^{5}n_i)-f(\sum_{i=0}^{4}q_i,\sum_{i=0}^{4}n_i)]$..

Then, $\varphi: 5-SP_V\times 5-SP_V\to 5-SP_R$ is 5-plithogenic inner product.

Proof.

We put $\varphi: 5-SP_V\times 5-SP_V\to 5-SP_R$, where:

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$\varphi(q,n) = f(q_0,n_0) + P_1[f(\sum_{i=0}^{1} q_i , \sum_{i=0}^{1} n_i) - f(q_0,n_0)] + P_2[f(\sum_{i=0}^{2} q_i , \sum_{i=0}^{2} n_i) - f(\sum_{i=0}^{1} q_i , \sum_{i=0}^{1} n_i)] + P_3[f(\sum_{i=0}^{3} q_i , \sum_{i=0}^{3} n_i) - f(\sum_{i=0}^{2} q_i , \sum_{i=0}^{2} n_i)] + P_4[f(\sum_{i=0}^{4} q_i , \sum_{i=0}^{4} n_i) - f(\sum_{i=0}^{3} q_i , \sum_{i=0}^{3} n_i)] + P_5[f(\sum_{i=0}^{5} q_i , \sum_{i=0}^{5} n_i) - f(\sum_{i=0}^{4} q_i , \sum_{i=0}^{4} n_i)]$.

$We\ have$:

$\varphi(q,n) = f(q_0,n_0) + P_1[f(\sum_{i=0}^{1} q_i , \sum_{i=0}^{1} n_i) - f(q_0,n_0)] + P_2[f(\sum_{i=0}^{2} q_i , \sum_{i=0}^{2} n_i) - f(\sum_{i=0}^{1} q_i , \sum_{i=0}^{1} n_i)] + P_3[f(\sum_{i=0}^{3} q_i , \sum_{i=0}^{3} n_i) - f(\sum_{i=0}^{2} q_i , \sum_{i=0}^{2} n_i)] + P_4[f(\sum_{i=0}^{4} q_i , \sum_{i=0}^{4} n_i) - f(\sum_{i=0}^{3} q_i , \sum_{i=0}^{3} n_i)] + P_5[f(\sum_{i=0}^{5} q_i , \sum_{i=0}^{5} n_i) - f(\sum_{i=0}^{4} q_i , \sum_{i=0}^{4} n_i)] = f(n_0,q_0) + P_1[f(\sum_{i=0}^{1} n_i , \sum_{i=0}^{1} q_i) - f(n_0,q_0)] + P_2[f(\sum_{i=0}^{2} n_i , \sum_{i=0}^{2} q_i) - f(\sum_{i=0}^{1} n_i , \sum_{i=0}^{1} q_i)] + P_3[f(\sum_{i=0}^{3} n_i , \sum_{i=0}^{3} q_i) - f(\sum_{i=0}^{2} n_i , \sum_{i=0}^{2} q_i)] + P_4[f(\sum_{i=0}^{4} n_i , \sum_{i=0}^{4} q_i) - f(\sum_{i=0}^{3} n_i , \sum_{i=0}^{3} q_i)] + P_5[f(\sum_{i=0}^{5} n_i , \sum_{i=0}^{5} q_i) - f(\sum_{i=0}^{4} n_i , \sum_{i=0}^{4} q_i)] = \varphi(n,q)$.

$\varphi(q,q) = f(q_0,q_0) + P_1[f(\sum_{i=0}^{1} q_i , \sum_{i=0}^{1} q_i) - f(q_0,q_0)] + P_2[f(\sum_{i=0}^{2} q_i , \sum_{i=0}^{2} q_i) - f(\sum_{i=0}^{1} q_i , \sum_{i=0}^{1} q_i)] + P_3[f(\sum_{i=0}^{3} q_i , \sum_{i=0}^{3} q_i) - f(\sum_{i=0}^{2} q_i , \sum_{i=0}^{2} q_i)] + P_4[f(\sum_{i=0}^{4} q_i , \sum_{i=0}^{4} q_i) - f(\sum_{i=0}^{3} q_i , \sum_{i=0}^{3} q_i)] + P_5[f(\sum_{i=0}^{5} q_i , \sum_{i=0}^{5} q_i) - f(\sum_{i=0}^{4} q_i , \sum_{i=0}^{4} q_i)] = \|q_0\|^2 + P_1[\|\sum_{i=0}^{1} q_i\|^2 - \|q_0\|^2] + P_2[\|\sum_{i=0}^{2} q_i\|^2 - \|\sum_{i=0}^{1} q_i\|^2] + P_3\left[\left\|\sum_{i=0}^{3} q_i\right\|^2 - \|\sum_{i=0}^{2} q_i\|^2\right] + P_4\left[\|\sum_{i=0}^{4} q_i\|^2 - \left\|\sum_{i=0}^{3} q_i\right\|^2\right] + P_5\left[\left\|\sum_{i=0}^{5} q_i\right\|^2 - \|\sum_{i=0}^{4} q_i\|^2\right]$.

And that is because:

$$\|q_0\|^2 \geq 0$$

$$\|q_0\|^2 + \left[\left\|\sum_{i=0}^{1} q_i\right\|^2 - \|q_0\|^2\right] = \left\|\sum_{i=0}^{1} q_i\right\|^2 \geq 0$$

$$\|q_0\|^2 + \left[\left\|\sum_{i=0}^{1} q_i\right\|^2 - \|q_0\|^2\right] + \left[\left\|\sum_{i=0}^{2} q_i\right\|^2 - \left\|\sum_{i=0}^{1} q_i\right\|^2\right] = \left\|\sum_{i=0}^{2} q_i\right\|^2 \geq 0$$

$$\|q_0\|^2 + \left[\left\|\sum_{i=0}^{1} q_i\right\|^2 - \|q_0\|^2\right] + \left[\left\|\sum_{i=0}^{2} q_i\right\|^2 - \left\|\sum_{i=0}^{1} q_i\right\|^2\right] + \left[\left\|\sum_{i=0}^{3} q_i\right\|^2 - \left\|\sum_{i=0}^{2} q_i\right\|^2\right] = \left\|\sum_{i=0}^{3} q_i\right\|^2 \geq 0$$

$$\|q_0\|^2 + \left[\left\|\sum_{i=0}^{1} q_i\right\|^2 - \|q_0\|^2\right] + \left[\left\|\sum_{i=0}^{2} q_i\right\|^2 - \left\|\sum_{i=0}^{1} q_i\right\|^2\right] + \left[\left\|\sum_{i=0}^{3} q_i\right\|^2 - \left\|\sum_{i=0}^{2} q_i\right\|^2\right] + \left[\left\|\sum_{i=0}^{4} q_i\right\|^2 - \left\|\sum_{i=0}^{3} q_i\right\|^2\right] = \left\|\sum_{i=0}^{4} q_i\right\|^2 \geq 0$$

$$\|q_0\|^2 + \left[\left\|\sum_{i=0}^{1} q_i\right\|^2 - \|q_0\|^2\right] + \left[\left\|\sum_{i=0}^{2} q_i\right\|^2 - \left\|\sum_{i=0}^{1} q_i\right\|^2\right] + \left[\left\|\sum_{i=0}^{3} q_i\right\|^2 - \left\|\sum_{i=0}^{2} q_i\right\|^2\right] + \left[\left\|\sum_{i=0}^{4} q_i\right\|^2 - \left\|\sum_{i=0}^{3} q_i\right\|^2\right] + \left[\left\|\sum_{i=0}^{5} q_i\right\|^2 - \left\|\sum_{i=0}^{4} q_i\right\|^2\right] = \left\|\sum_{i=0}^{5} q_i\right\|^2 \geq 0$$

Now, we let $u = u_0 + \sum_{i=1}^{5} u_i P_i \in 5 - SP_V$, we have:

$$\begin{aligned}\varphi(u+q,n) = f(u_0+q_0,n_0) + P_1\left[f\left(\sum_{i=0}^{1}(q_i+u_i), \sum_{i=0}^{1} n_i\right) - f(u_0+q_0,n_0)\right] \\ + P_2\left[f\left(\sum_{i=0}^{2}(q_i+u_i), \sum_{i=0}^{2} n_i\right) - f\left(\sum_{i=0}^{1}(q_i+u_i), \sum_{i=0}^{1} n_i\right)\right] \\ + P_3\left[f\left(\sum_{i=0}^{3}(q_i+u_i), \sum_{i=0}^{3} n_i\right) - f\left(\sum_{i=0}^{2}(q_i+u_i), \sum_{i=0}^{2} n_i\right)\right] \\ + P_4\left[f\left(\sum_{i=0}^{4}(q_i+u_i), \sum_{i=0}^{4} n_i\right) - f\left(\sum_{i=0}^{3}(q_i+u_i), \sum_{i=0}^{3} n_i\right)\right] \\ + P_5\left[f\left(\sum_{i=0}^{5}(q_i+u_i), \sum_{i=0}^{5} n_i\right) - f\left(\sum_{i=0}^{4}(q_i+u_i), \sum_{i=0}^{4} n_i\right)\right] = \varphi(u,n) + \varphi(w,n)\end{aligned}$$

Let $a = a_0 + \sum_{i=1}^{5} a_i P_i \in 5 - SP_R$, then:

$\varphi(a.q,n) = f(a_0 q_0, n_0) + P_1\left[f\left((\sum_{i=0}^{1} a_i)(\sum_{i=0}^{1}(q_i+u_i)), \sum_{i=0}^{1} n_i\right) - f(a_0 q_0, n_0)\right] + P_2\left[f\left((\sum_{i=0}^{2} a_i)(\sum_{i=0}^{2}(q_i+u_i)), \sum_{i=0}^{2} n_i\right) - f\left((\sum_{i=0}^{1} a_i)(\sum_{i=0}^{1}(q_i+u_i)), \sum_{i=0}^{1} n_i\right)\right] + P_3\left[f\left((\sum_{i=0}^{3} a_i)(\sum_{i=0}^{3}(q_i+u_i)), \sum_{i=0}^{3} n_i\right) - f\left((\sum_{i=0}^{2} a_i)(\sum_{i=0}^{2}(q_i+u_i)), \sum_{i=0}^{2} n_i\right)\right] + P_4\left[f\left((\sum_{i=0}^{4} a_i)(\sum_{i=0}^{4}(q_i+u_i)), \sum_{i=0}^{4} n_i\right) - f\left((\sum_{i=0}^{3} a_i)(\sum_{i=0}^{3}(q_i+u_i)), \sum_{i=0}^{3} n_i\right)\right] +$

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$P_5\left[f\left((\sum_{i=0}^{5} a_i)(\sum_{i=0}^{5}(q_i+u_i)),\sum_{i=0}^{5} n_i\right) - f\left((\sum_{i=0}^{4} a_i)(\sum_{i=0}^{4}(q_i+u_i)),\sum_{i=0}^{4} n_i\right)\right] = (a_0 + \sum_{i=1}^{5} a_i P_i)\varphi(q,n) = a.\varphi(q,n).$

Example.

Regard the Euclidean real inner product defined on R^2

$f[(\acute{d}_1, d_1''),(\acute{d}_2, d_2'')] = \acute{d}_1\acute{d}_2 + d_1'' d_2''.$

The inner product on $5-SP_V$ is:

$\varphi: 5-SP_V \times 5-SP_V \to 5-SP_R;$

$$\begin{aligned}
&\varphi[(d_0,y_0)+(d_1,y_1)P_1+(d_2,y_2)P_2+(d_3,y_3)P_3+(d_4,y_4)P_4,(z_0,s_0)+(z_1,s_1)P_1+(z_2,s_2)P_2 \\
&\quad +(z_3,s_3)P_3+(z_4,s_4)P_4+(z_5,s_5)P_5] \\
&= f((d_0,y_0),(z_0,s_0)) + P_1\left[f\left(\left(\sum_{i=0}^{1} d_i,\sum_{i=0}^{1} y_i\right),\left(\sum_{i=0}^{1} z_i,\sum_{i=0}^{1} s_i\right)\right) - f((d_0,y_0),(z_0,s_0))\right] \\
&+ P_2\left[f\left(\left(\sum_{i=0}^{2} d_i,\sum_{i=0}^{2} y_i\right),\left(\sum_{i=0}^{2} z_i,\sum_{i=0}^{2} s_i\right)\right) - f\left(\left(\sum_{i=0}^{1} d_i,\sum_{i=0}^{1} y_i\right),\left(\sum_{i=0}^{1} z_i,\sum_{i=0}^{1} s_i\right)\right)\right] \\
&+ P_3\left[f\left(\left(\sum_{i=0}^{3} d_i,\sum_{i=0}^{3} y_i\right),\left(\sum_{i=0}^{3} z_i,\sum_{i=0}^{3} s_i\right)\right) - f\left(\left(\sum_{i=0}^{2} d_i,\sum_{i=0}^{2} y_i\right),\left(\sum_{i=0}^{2} z_i,\sum_{i=0}^{2} s_i\right)\right)\right] \\
&+ P_4\left[f\left(\left(\sum_{i=0}^{4} d_i,\sum_{i=0}^{4} y_i\right),\left(\sum_{i=0}^{4} z_i,\sum_{i=0}^{4} s_i\right)\right) - f\left(\left(\sum_{i=0}^{3} d_i,\sum_{i=0}^{3} y_i\right),\left(\sum_{i=0}^{3} z_i,\sum_{i=0}^{3} s_i\right)\right)\right] \\
&+ P_5\left[f\left(\left(\sum_{i=0}^{5} d_i,\sum_{i=0}^{5} y_i\right),\left(\sum_{i=0}^{5} z_i,\sum_{i=0}^{5} s_i\right)\right) - f\left(\left(\sum_{i=0}^{4} d_i,\sum_{i=0}^{4} y_i\right),\left(\sum_{i=0}^{4} z_i,\sum_{i=0}^{4} s_i\right)\right)\right] \\
&= d_0 z_0 + y_0 s_0 + P_1\left[\left(\sum_{i=0}^{1} d_i\right)\left(\sum_{i=0}^{1} z_i\right) + \left(\sum_{i=0}^{1} y_i\right)\left(\sum_{i=0}^{1} s_i\right) - d_0 z_0 - y_0 s_0\right] \\
&+ P_2\left[\left(\sum_{i=0}^{2} d_i\right)\left(\sum_{i=0}^{2} z_i\right) + \left(\sum_{i=0}^{2} y_i\right)\left(\sum_{i=0}^{2} s_i\right) - \left(\sum_{i=0}^{1} d_i\right)\left(\sum_{i=0}^{1} z_i\right) - \left(\sum_{i=0}^{1} y_i\right)\left(\sum_{i=0}^{1} s_i\right)\right] \\
&+ P_3\left[\left(\sum_{i=0}^{3} d_i\right)\left(\sum_{i=0}^{3} z_i\right) + \left(\sum_{i=0}^{3} y_i\right)\left(\sum_{i=0}^{3} s_i\right) - \left(\sum_{i=0}^{2} d_i\right)\left(\sum_{i=0}^{2} z_i\right) - \left(\sum_{i=0}^{2} y_i\right)\left(\sum_{i=0}^{2} s_i\right)\right] \\
&+ P_4\left[\left(\sum_{i=0}^{4} d_i\right)\left(\sum_{i=0}^{4} z_i\right) + \left(\sum_{i=0}^{4} y_i\right)\left(\sum_{i=0}^{4} s_i\right) - \left(\sum_{i=0}^{3} d_i\right)\left(\sum_{i=0}^{3} z_i\right) - \left(\sum_{i=0}^{3} y_i\right)\left(\sum_{i=0}^{3} s_i\right)\right] \\
&+ P_5\left[\left(\sum_{i=0}^{5} d_i\right)\left(\sum_{i=0}^{5} z_i\right) + \left(\sum_{i=0}^{5} y_i\right)\left(\sum_{i=0}^{5} s_i\right) - \left(\sum_{i=0}^{4} d_i\right)\left(\sum_{i=0}^{4} z_i\right) - \left(\sum_{i=0}^{4} y_i\right)\left(\sum_{i=0}^{4} s_i\right)\right]
\end{aligned}$$

Remark.

The norm of $q = q_0 + \sum_{i=1}^{5} q_i P_i$ is defined as follows:

$\|q\| = \sqrt{\varphi(q,q)} = \|q_0\| + P_1[\|\sum_{i=0}^{1} q_i\| - \|q_0\|] + P_2[\|\sum_{i=0}^{2} q_i\| - \|\sum_{i=0}^{1} q_i\|] + P_3[\|\sum_{i=0}^{3} q_i\| - \|\sum_{i=0}^{2} q_i\|] + P_4[\|\sum_{i=0}^{4} q_i\| - \|\sum_{i=0}^{3} q_i\|] + P_5[\|\sum_{i=0}^{5} q_i\| - \|\sum_{i=0}^{4} q_i\|].$

Theorem.

Let φ be 5-plithogenic inner product on $5-SP_V$, then:

1). $\|q\| \geq 0; q \in 5-SP_V$

2). $\|a.q\| = |a|.\|q\|; a \in 5-SP_R$

3). $\|q+n\| \leq \|q\| + \|n\|; n \in 5-SP_V$

4). $|\varphi(q,n)| \leq \|q\|.\|n\|$

5). $q \perp n$ eqivalents $q_0 \perp n_0, \sum_{i=0}^{1} q_i \perp \sum_{i=0}^{1} n_i, \sum_{i=0}^{2} q_i \perp \sum_{i=0}^{2} n_i, \sum_{i=0}^{3} q_i \perp \sum_{i=0}^{3} n_i, \sum_{i=0}^{4} q_i \perp \sum_{i=0}^{4} n_i, \sum_{i=0}^{5} q_i \perp \sum_{i=0}^{5} n_i.$

6). If $q \perp n$, implies $\|q+n\|^2 = \|q\|^2 + \|n\|^2$

Proof.

1). Easy to be proven.

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2). Let $a = a_0 + \sum_{i=1}^{5} a_i P_i \in 5 - SP_R, q = q_0 + \sum_{i=1}^{5} q_i P_i \in 5 - SP_V$, therefor:
$\|a.q\|^2 = \varphi(a.q, a.q) = a^2\varphi(q,q)$, thus $\|a.q\| = |a|.\|q\|$.

3). $\|q+n\| = \|q_0+n_0\| + P_1[\|\sum_{i=0}^{1}(q_i+n_i)\| - \|q_0+n_0\|] + P_2[\|\sum_{i=0}^{2}(q_i+n_i)\| - \|\sum_{i=0}^{1}(q_i+n_i)\|] + P_3[\|\sum_{i=0}^{3}(q_i+n_i)\| - \|\sum_{i=0}^{2}(q_i+n_i)\|] + P_4[\|\sum_{i=0}^{4}(q_i+n_i)\| - \|\sum_{i=0}^{3}(q_i+n_i)\|] + P_5[\|\sum_{i=0}^{5}(q_i+n_i)\| - \|\sum_{i=0}^{4}(q_i+n_i)\|]$, therefor:

$$\begin{cases} \|q_0+n_0\| \le \|q_0\| + \|n_0\| \\ \left\|\sum_{i=0}^{1}(q_i+n_i)\right\| \le \left\|\sum_{i=0}^{1}(q_i)\right\| + \left\|\sum_{i=0}^{1}(n_i)\right\| \\ \left\|\sum_{i=0}^{2}(q_i+n_i)\right\| \le \left\|\sum_{i=0}^{2}(q_i)\right\| + \left\|\sum_{i=0}^{2}(n_i)\right\| \\ \left\|\sum_{i=0}^{3}(q_i+n_i)\right\| \le \left\|\sum_{i=0}^{3}(q_i)\right\| + \left\|\sum_{i=0}^{3}(n_i)\right\| \\ \left\|\sum_{i=0}^{4}(q_i+n_i)\right\| \le \left\|\sum_{i=0}^{4}(q_i)\right\| + \left\|\sum_{i=0}^{4}(n_i)\right\| \\ \left\|\sum_{i=0}^{5}(q_i+n_i)\right\| \le \left\|\sum_{i=0}^{5}(q_i)\right\| + \left\|\sum_{i=0}^{5}(n_i)\right\| \end{cases}$$

therefor: $\|q+n\| \le \|q\| + \|n\|$.

4). $|\varphi(q,n)| = |\varphi(q_0,n_0)| + P_1[|\varphi(\sum_{i=0}^{1}(q_i), \sum_{i=0}^{1}(n_i))| - |\varphi(q_0,n_0)|] + P_2[|\varphi(\sum_{i=0}^{2}(q_i), \sum_{i=0}^{2}(n_i))| - |\varphi(\sum_{i=0}^{1}(q_i), \sum_{i=0}^{1}(n_i))|] + P_3[|\varphi(\sum_{i=0}^{3}(q_i), \sum_{i=0}^{3}(n_i))| - |\varphi(\sum_{i=0}^{2}(q_i), \sum_{i=0}^{2}(n_i))|] + P_4[|\varphi(\sum_{i=0}^{4}(q_i), \sum_{i=0}^{4}(n_i))| - |\varphi(\sum_{i=0}^{3}(q_i), \sum_{i=0}^{3}(n_i))|] + P_5[|\varphi(\sum_{i=0}^{5}(q_i), \sum_{i=0}^{5}(n_i))| - |\varphi(\sum_{i=0}^{4}(q_i), \sum_{i=0}^{4}(n_i))|]$.

By using Cauchy-Shwartz inequality, we get:

$$\begin{cases} |f(q_0,n_0)| \le \|q_0\| + \|n_0\| \\ \left|f\left(\sum_{i=0}^{1}(q_i), \sum_{i=0}^{1}(n_i)\right)\right| \le \left\|\sum_{i=0}^{1}(q_i)\right\| + \left\|\sum_{i=0}^{1}(n_i)\right\| \\ \left|f\left(\sum_{i=0}^{2}(q_i), \sum_{i=0}^{2}(n_i)\right)\right| \le \left\|\sum_{i=0}^{2}(q_i)\right\| + \left\|\sum_{i=0}^{2}(n_i)\right\| \\ \left|f\left(\sum_{i=0}^{3}(q_i), \sum_{i=0}^{3}(n_i)\right)\right| \le \left\|\sum_{i=0}^{3}(q_i)\right\| + \left\|\sum_{i=0}^{3}(n_i)\right\| \\ \left|f\left(\sum_{i=0}^{4}(q_i), \sum_{i=0}^{4}(n_i)\right)\right| \le \left\|\sum_{i=0}^{4}(q_i)\right\| + \left\|\sum_{i=0}^{4}(n_i)\right\| \\ \left|f\left(\sum_{i=0}^{5}(q_i), \sum_{i=0}^{5}(n_i)\right)\right| \le \left\|\sum_{i=0}^{5}(q_i)\right\| + \left\|\sum_{i=0}^{5}(n_i)\right\| \end{cases}$$

therefor:,
$|\varphi(q,n)| \le \|q\|.\|n\|$

5). $q \perp n$ equivalents $\varphi(q,n) = 0$, hence:

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$$\begin{cases} f(q_0, n_0) = 0 \Rightarrow q_0 \perp n_0 \\ f\left(\sum_{i=0}^{1}(q_i), \sum_{i=0}^{1}(n_i)\right) \Rightarrow \sum_{i=0}^{1}(q_i) \perp \sum_{i=0}^{1}(n_i) \\ f\left(\sum_{i=0}^{2}(q_i), \sum_{i=0}^{2}(n_i)\right) \Rightarrow \sum_{i=0}^{2}(q_i) \perp \sum_{i=0}^{2}(n_i) \\ \left(\sum_{i=0}^{3}(q_i), \sum_{i=0}^{3}(n_i)\right) \Rightarrow \sum_{i=0}^{3}(q_i) \perp \sum_{i=0}^{3}(n_i) \\ \left(\sum_{i=0}^{4}(q_i), \sum_{i=0}^{4}(n_i)\right) \Rightarrow \sum_{i=0}^{4}(q_i) \perp \sum_{i=0}^{4}(n_i) \\ \left(\sum_{i=0}^{5}(q_i), \sum_{i=0}^{5}(n_i)\right) \Rightarrow \sum_{i=0}^{5}(q_i) \perp \sum_{i=0}^{5}(n_i) \end{cases}$$

6). if $q \perp n$, implies $\|q + n\|^2 = \varphi(q + n, q + n) = \varphi(q, q) + \varphi(n, n) = \|q\|^2 + \|n\|^2$.

4. Conclusion

In this paper, we presented a development of the results presented in [30], where we expanded the study to include symbolic 4-plithogenic and 5-plithogenic spaces, where we studied the inner products defined above these spaces, as well as various issues of orthogonality and its conditions, in addition to the concept of the orthogonal rule and the organizing rule related to this new type of spaces. In future studies, we aim to extend our results to the case of n-plithogenic vector spaces.

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