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Multipolar neutrosophic subalgebras/ideals of UP-algebras

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Abstract

The notion of neutrosophic m-polar fuzzy sets is much wider than the notion of m-polar fuzzy sets. In this paper, we apply the theory of neutrosophic m-polar fuzzy set on UP-algebras. We introduce the concepts of neutrosophic m-polar fuzzy subalgebras, neutrosophic m-polar fuzzy ideals and neutrosophic m-polar fuzzy strong ideals and some essential properties are discussed. We characterize neutrosophic m-polar fuzzy subalgebras in terms of fuzzy subalgebras and subalgebras of UP-algebras.

Keywords: Neutrosophic m-polar fuzzy sets; neutrosophic m-polar fuzzy subalgebras; neutrosophic m-polar fuzzy ideals

1 Introduction

Zadeh[7] introduced fuzzy set theory as an alternative to probability theory to deal with uncertainty, recognizing that uncertainty is an inherent attribute of information. Fuzzy set theory provides a more general framework for handling uncertainty, allowing for more flexible and nuanced representations of vague or imprecise information. UP-algebras, introduced by Iampan,[3] are a class of logical algebras that have connections with posets (partially ordered sets). They have been extensively studied and applied across various branches of mathematics, including group theory, functional analysis, probability theory, topology, and fuzzy set theory. Neutrosophic m-polar fuzzy sets extend fuzzy set theory to handle indeterminacy, ambiguity, and inconsistency more comprehensively. In this paper, we are applying the theory of neutrosophic m-polar fuzzy sets to UP-algebras, introducing concepts such as neutrosophic m-polar fuzzy subalgebras, ideals, and strong ideals. This paper discusses the characterization of neutrosophic m-polar fuzzy subalgebras in terms of fuzzy subalgebras and subalgebras of UP-algebras. This exploration of properties and relationships between different types of subalgebras within the context of neutrosophic m-polar fuzzy sets provides insights into the interplay between algebraic structures and fuzzy set theory.

2 Preliminaries

Definition 2.1. [3] An algebra $X = (X, *, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following conditions.

$$(\forall x, y, z \in X)((y * z) * ((x * y) * (x * z)) = 0), \tag{1}$$

$$(\forall x \in X)(0 * x = x), \tag{2}$$

$$(\forall x \in X)(x * 0 = 0), \tag{3}$$

$$(\forall x, y \in X)(x * y = 0 = y * x \;\Rightarrow\; x = y). \tag{4}$$

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We define a binary relation " $\leq$ " on a UP-algebra X as follows:

$$(\forall x, y \in X)\left(x \leq y \Leftrightarrow x * y = 0 \right). \tag{5}$$

In a UP-algebra X, the following assertions are valid (see[3]).

$$(\forall x \in X)(x * x = 0), \tag{6}$$
$$(\forall x, y, z \in X)(x * y = 0,\ y * z = 0 \Rightarrow x * z = 0), \tag{7}$$
$$(\forall x, y, z \in X)(x * y = 0 \Rightarrow (z * x) * (z * y) = 0), \tag{8}$$
$$(\forall x, y, z \in X)(x * y = 0 \Rightarrow (y * z) * (x * z) = 0), \tag{9}$$
$$(\forall x, y \in X)(x * (y * x) = 0), \tag{10}$$
$$(\forall x, y \in X)((y * x) * x = 0 \Leftrightarrow x = y * x), \tag{11}$$
$$(\forall x, y \in X)(x * (y * y) = 0). \tag{12}$$

Definition 2.2. [2–4] A nonempty subset A of a UP-algebra $X = (X, *, 0)$ is called

1. a UP-subalgebra of X if it satisfies the following condition:

$$(\forall x, y \in A)(x * y \in A). \tag{13}$$

2. a UP-ideal of X if it satisfies the following conditions:

$$0 \in A \tag{14}$$
$$(\forall x, y, z \in X)(x * (y * z) \in A, y \in A \Rightarrow x * z \in A). \tag{15}$$

3. a strong UP-ideal of X if it satisfies the condition (14) and the following condition:

$$(\forall x, y, z \in X)((z * y) * (z * x) \in A, y \in A \Rightarrow x \in A). \tag{16}$$

Definition 2.3. [1] An m-polar fuzzy set Φ on a nonempty set X is a mapping $\Phi : X \to [0,1]^m$. The membership value of every element $x \in X$ is denoted by $\Phi(x) = (\pi_1 \circ \varphi(x), \pi_2 \circ \varphi(x) \cdots, \pi_m \circ \varphi(x))$, where $\pi_i \circ \varphi : [0,1]^m \to [0,1]$ is defined the i-th projection mapping.

Given an m-polar fuzzy set on a set X, we consider the sets, $U(\Phi, \zeta) = \{x \in X : \Phi(x) \geq \zeta\}$ and $L(\Phi, \xi) = \{x \in X : \Phi(x) \leq \xi\}$, where $\zeta = (\zeta_1, \zeta_2, \zeta_3 \cdots, \zeta_m)$, $\xi = (\xi_1, \xi_2, \xi_3 \cdots, \xi_m)$, that is,

$$U(\Phi, \zeta) = \{x \in X : (\pi_1 \circ \Phi)(x) \geq \zeta_1, (\pi_2 \circ \Phi)(x) \geq \zeta_2, \cdots, (\pi_m \circ \Phi)(x) \geq \zeta_m\},$$

$$L(\Phi, \xi) = \{x \in X : (\pi_1 \circ \Phi)(x) \leq \xi_1, (\pi_2 \circ \Phi)(x) \leq \xi_2, \cdots, (\pi_m \circ \Phi)(x) \leq \xi_m\},$$

which are called an m-upper (lower) set of Φ.

3 Multipolar neutrosophic subalgebras

Definition 3.1. Let X be a UP-algebra. An m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ in X is called an

1. m-polar neutrosophic subalgebra of X if it satisfies the following condition:

$$(\forall x, y \in X)\begin{pmatrix} \Phi(x * y) \geq \inf\{\Phi(x), \Phi(y)\} \\ \Upsilon(x * y) \geq \inf\{\Upsilon(x), \Upsilon(y)\} \\ \Psi(x * y) \leq \sup\{\Psi(x), \Psi(y)\} \end{pmatrix}. \tag{17}$$

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2. m-polar neutrosophic UP-ideal of X if it satisfies the following conditions:

$$(\forall\, x \in X) \begin{pmatrix} \Phi(0) \geq \Phi(x) \\ \Upsilon(0) \geq \Upsilon(x) \\ \Psi(0) \leq \Psi(x) \end{pmatrix}, \tag{18}$$

$$(\forall\, x, y, z \in X) \begin{pmatrix} \Phi(x * z) \geq \inf\{\Phi(x * (y * z)), \Phi(y)\} \\ \Upsilon(x * z) \geq \inf\{\Upsilon(x * (y * z)), \Upsilon(y)\} \\ \Psi(x * z) \leq \sup\{\Psi(x * (y * z)), \Psi(y)\} \end{pmatrix}. \tag{19}$$

3. m-polar neutrosophic strong UP-ideal of X if it satisfies the following conditions: (18) and

$$(\forall\, x, y, z \in X) \begin{pmatrix} \Phi(x) \geq \inf\{\Phi((z * y) * (z * x)), \Phi(y)\} \\ \Upsilon(x) \geq \inf\{\Upsilon((z * y) * (z * x)), \Upsilon(y)\} \\ \Psi(x) \leq \sup\{\Psi((z * y) * (z * x)), \Psi(y)\} \end{pmatrix}. \tag{20}$$

Proposition 3.2. *Every m-polar neutrosophic subalgebra $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ of a UP-algebra X satisfies*

$$(\forall x \in X) \left(\ \Phi(0) \geq \Phi(x), \Upsilon(0) \geq \Upsilon(x), \Psi(0) \leq \Psi(x) \ \right) \tag{21}$$

Proof. For any $x \in X$, we have

$$\begin{gathered} \pi_i \circ \varphi(0) = \pi_i \circ \varphi(x * x) \geq \inf\{\pi_i \circ \varphi(x), \pi_i \circ \varphi(x)\} = \pi_i \circ \varphi(x) \\ \pi_i \circ \gamma(0) = \pi_i \circ \gamma(x * x) \geq \inf\{\pi_i \circ \gamma(x), \pi_i \circ \gamma(x)\} = \pi_i \circ \gamma(x) \\ \pi_i \circ \psi(0) = \pi_i \circ \psi(x * x) \leq \sup\{\pi_i \circ \psi(x), \pi_i \circ \psi(x)\} = \pi_i \circ \psi(x). \end{gathered}$$

□

Proposition 3.3. *If $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-subalgebra of a UP-algebra X, then*

$$(\forall x \in X) \left(\ \Phi(0 * x) \geq \Phi(x), \Upsilon(0 * x) \geq \Upsilon(x), \Psi(0 * x) \leq \Psi(x) \ \right). \tag{22}$$

Proof. For any $x \in X$, we have

$$\begin{aligned} \pi_i \circ \varphi(0 * x) &\geq \inf\{\pi_i \circ \varphi(0), \pi_i \circ \varphi(x)\} \\ &= \inf\{\pi_i \circ \varphi(x * x), \pi_i \circ \varphi(x)\} \\ &\geq \inf\{\inf\{\pi_i \circ \varphi(x), \pi_i \circ \varphi(x)\}, \pi_i \circ \varphi(x)\} \\ &= \pi_i \circ \varphi(x), \end{aligned}$$

$$\begin{aligned} \pi_i \circ \gamma(0 * x) &\geq \inf\{\pi_i \circ \gamma(0), \pi_i \circ \gamma(x)\} \\ &= \inf\{\pi_i \circ \gamma(x * x), \pi_i \circ \gamma(x)\} \\ &\geq \inf\{\inf\{\pi_i \circ \gamma(x), \pi_i \circ \gamma(x)\}, \pi_i \circ \gamma(x)\} \\ &= \pi_i \circ \gamma(x), \end{aligned}$$

$$\begin{aligned} \pi_i \circ \psi(0 * x) &\leq \sup\{\pi_i \circ \psi(0), \pi_i \circ \psi(x)\} \\ &= \sup\{\pi_i \circ \psi(x * x), \pi_i \circ \psi(x)\} \\ &\leq \sup\{\pi_i \circ \psi(x), \pi_i \circ \psi(x), \pi_i \circ \psi(x)\} \\ &= \pi_i \circ \psi(x). \end{aligned}$$

□

Theorem 3.4. *A m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic strongly UP-ideal of X if and only if it is constant.*

Proof. Assume that $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is constant for all $x \in U$. Then we have $\pi_i \circ \varphi(x) = \pi_i \circ \varphi(0)$, $\pi_i \circ \gamma(x) = \pi_i \circ \gamma(0)$ and $\pi_i \circ \psi(x) = \pi_i \circ \psi(0)$. Then

$$\begin{aligned} \pi_i \circ \varphi(x) &\geq \pi_i \circ \varphi(0) \\ &= \inf\{\pi_i \circ \varphi(0), \pi_i \circ \varphi(0)\} \\ &= \inf\{\pi_i \circ \varphi((z * y) * (z * x)), \pi_i \circ \varphi(y)\}, \\ \pi_i \circ \gamma(x) &\geq \pi_i \circ \gamma(0) \\ &= \inf\{\pi_i \circ \gamma(0), \pi_i \circ \gamma(0)\} \\ &= \inf\{\pi_i \circ \gamma((z * y) * (z * x)), \pi_i \circ \gamma(y)\}, \\ \pi_i \circ \psi(x) &\leq \pi_i \circ \psi(0) \\ &= \sup\{\pi_i \circ \psi(0), \pi_i \circ \psi(0)\} \\ &= \sup\{\pi_i \circ \psi((z * y) * (z * x)), \pi_i \circ \psi(y)\}. \end{aligned}$$

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Hence $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic strongly UP-ideal of X. Conversely, assume that $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic strongly UP-ideal of X. Then for any $x \in X$,

$$\begin{array}{rcl}
\pi_i \circ \varphi(x) & \geq & \inf\{\pi_i \circ \varphi((x * 0) * (x * x)), \pi_i \circ \varphi(0)\} \\
& = & \inf\{\pi_i \circ \varphi(0 * (x * x)), \pi_i \circ \varphi(0)\} \\
& = & \inf\{\pi_i \circ \varphi(x * x), \pi_i \circ \varphi(0)\} \\
& = & \inf\{\pi_i \circ \varphi(0), \pi_i \circ \varphi(0)\} \\
& = & \pi_i \circ \varphi(0),
\end{array}$$

$$\begin{array}{rcl}
\pi_i \circ \gamma(x) & \geq & \inf\{\pi_i \circ \gamma((x * 0) * (x * x)), \pi_i \circ \gamma(0)\} \\
& = & \inf\{\pi_i \circ \gamma(0 * (x * x)), \pi_i \circ \gamma(0)\} \\
& = & \inf\{\pi_i \circ \gamma(x * x), \pi_i \circ \gamma(0)\} \\
& = & \inf\{\pi_i \circ \gamma(0), \pi_i \circ \gamma(0)\} \\
& = & \pi_i \circ \gamma(0),
\end{array}$$

$$\begin{array}{rcl}
\pi_i \circ \psi(x) & \leq & \sup\{\pi_i \circ \psi((x * 0) * (x * x)), \pi_i \circ \psi(0)\} \\
& = & \sup\{\pi_i \circ \psi(0 * (x * x)), \pi_i \circ \psi(0)\} \\
& = & \sup\{\pi_i \circ \psi(x * x), \pi_i \circ \psi(0)\} \\
& = & \sup\{\pi_i \circ \psi(0), \pi_i \circ \psi(0)\} \\
& = & \pi_i \circ \psi(0).
\end{array}$$

□

Theorem 3.5. *Every m-polar neutrosophic strong UP-ideal of X is a m-polar neutrosophic UP-ideal of X.*

Proof. Let $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ be a m-polar neutrosophic strong UP-ideal of X, and let $x, y \in X$. Then $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ satisfies (19). By Theorem 3.4, we have $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ constant. Next, let $x, y, z \in X$. Then

$$\begin{array}{rcl}
\pi_i \circ \varphi(x * z) & \geq & \inf\{\pi_i \circ \varphi((z * y) * (z * (x * z))), \pi_i \circ \varphi(y)\} \\
& = & \inf\{\pi_i \circ \varphi((z * y) * 0), \pi_i \circ \varphi(y)\} \\
& = & \inf\{\pi_i \circ \varphi(0), \pi_i \circ \varphi(y)\} = \pi_i \circ \varphi(y) \\
& \geq & \inf\{\pi_i \circ \varphi(x * (y * z)), \pi_i \circ \varphi(y)\} \\
\pi_i \circ \gamma(x * z) & \geq & \inf\{\pi_i \circ \gamma((z * y) * (z * (x * z))), \pi_i \circ \gamma(y)\} \\
& = & \inf\{\pi_i \circ \gamma((z * y) * 0), \pi_i \circ \gamma(y)\} \\
& = & \inf\{\pi_i \circ \gamma(0), \pi_i \circ \gamma(y)\} = \pi_i \circ \gamma(y) \\
& \geq & \inf\{\pi_i \circ \gamma(x * (y * z)), \pi_i \circ \gamma(y)\} \\
\pi_i \circ \psi(x * z) & \leq & \sup\{\pi_i \circ \psi((z * y) * (z * (x * z))), \pi_i \circ \psi(y)\} \\
& = & \sup\{\pi_i \circ \psi((z * y) * 0), \pi_i \circ \psi(y)\} \\
& = & \sup\{\pi_i \circ \psi(0), \pi_i \circ \psi(y)\} = \pi_i \circ \psi(y) \\
& \leq & \sup\{\pi_i \circ \psi(x * (y * z)), \pi_i \circ \psi(y)\}.
\end{array}$$

Hence $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-ideal of X. □

Remark 3.6. Let $(X, *, 0)$ be a UP-algebra where $X = \{0, 1, 2, 3, 4\}$ and the operation $*$ is given by table 1.

Table 1

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	2	4
3	0	0	0	0	4
4	0	1	2	3	0

Now define 5-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ on X given by table 2.
Then $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ be a 5-polar neutrosophic UP-ideal but not a 5-polar neutrosophic strong UP-ideal of X.

Lemma 3.7. *If $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-ideal of X, then we have the following*

$$(\forall\, x, y \in X) \left(\quad y \leq x \;\Rightarrow \left\{ \begin{array}{l} \Phi(y) \leq \Phi(x) \\ \Upsilon(y) \leq \Upsilon(x) \\ \Psi(x) \leq \Psi(y) \end{array} \right. \right). \tag{23}$$

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Table 2: Table for membership values

X	Φ	Υ	Ψ
0	$(1.0, 0.9, 0.7, 0.9, 0.6)$	$(1.0, 0.6, 0.8, 0.9, 0.7)$	$(0.0, 0.2, 0.1, 0.3, 0.0)$
1	$(0.7, 0.6, 0.4, 0.8, 0.5)$	$(0.8, 0.4, 0.6, 0.7, 0.5)$	$(0.3, 0.5, 0.4, 0.6, 0.3)$
2	$(0.6, 0.5, 0.3, 0.7, 0.4)$	$(0.7, 0.3, 0.5, 0.6, 0.4)$	$(0.5, 0.7, 0.6, 0.8, 0.5)$
3	$(0.6, 0.5, 0.3, 0.7, 0.4)$	$(0.7, 0.3, 0.5, 0.6, 0.4)$	$(0.5, 0.7, 0.6, 0.8, 0.5)$
4	$(0.4, 0.3, 0.1, 0.5, 0.2)$	$(0.5, 0.1, 0.3, 0.4, 0.2)$	$(0.7, 0.9, 0.8, 1.0, 0.7)$

Proof. Let $x, y \in X$ be such that $x \le y$. Then $x * y = 0$ and so

$$\begin{aligned}
\pi_i \circ \varphi(x) &= \pi_i \circ \varphi(0 * x) \\
&\ge \inf\{\pi_i \circ \varphi((0 * (y * x)), \pi_i \circ \varphi(y)\} \\
&= \inf\{\pi_i \circ \varphi(y * x), \pi_i \circ \varphi(y)\} \\
&= \inf\{\pi_i \circ \varphi(0), \pi_i \circ \varphi(y)\} \\
&= \pi_i \circ \varphi(y),
\end{aligned}$$

$$\begin{aligned}
\pi_i \circ \gamma(x) &= \pi_i \circ \gamma(0 * x) \\
&\ge \inf\{\pi_i \circ \gamma((0 * (y * x)), \pi_i \circ \gamma(y)\} \\
&= \inf\{\pi_i \circ \gamma(y * x), \pi_i \circ \gamma(y)\} \\
&= \inf\{\pi_i \circ \gamma(0), \pi_i \circ \gamma(y)\} \\
&= \pi_i \circ \gamma(y),
\end{aligned}$$

$$\begin{aligned}
\pi_i \circ \psi(x) &= \pi_i \circ \psi(0 * x) \\
&\ge \sup\{\pi_i \circ \psi(0 * (y * x)), \pi_i \circ \psi(y)\} \\
&= \sup\{\pi_i \circ \psi(y * x), \pi_i \circ \varphi(y)\} \\
&= \sup\{\pi_i \circ \psi(0), \pi_i \circ \psi(y)\} \\
&= \pi_i \circ \psi(y).
\end{aligned}$$

□

Lemma 3.8. *If $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-ideal of X, then*

$$(\forall\, w, x, y, z \in X) \left(x \le w * (y * z) \;\Rightarrow \begin{cases} \Phi(x * z) \ge \inf\{\Phi(w), \Phi(y)\} \\ \Upsilon(x * z) \ge \inf\{\Upsilon(w), \Upsilon(y)\} \\ \Psi(x * z) \le \sup\{\Psi(w), \Psi(y)\} \end{cases} \right). \tag{24}$$

Proof. Let $w, x, y, z \in X$ be such that $x \le w * (y * z)$. Then $x * (w * (y * z)) = 0$ and so

$$\begin{aligned}
\pi_i \circ \varphi(x * z) &\ge \inf\{\pi_i \circ \varphi(x * (y * z)), \pi_i \circ \varphi(y)\} \\
&\ge \inf\{\inf\{\pi_i \circ \varphi(x * (w * (y * z))), \pi_i \circ \varphi(w)\}, \pi_i \circ \varphi(y)\} \\
&= \inf\{\inf\{\pi_i \circ \varphi(0), \pi_i \circ \varphi(w)\}, \pi_i \circ \varphi(y)\} \\
&= \inf\{\pi_i \circ \varphi(w), \pi_i \circ \varphi(y)\},
\end{aligned}$$

$$\begin{aligned}
\pi_i \circ \gamma(x * z) &\ge \inf\{\pi_i \circ \gamma(x * (y * z)), \pi_i \circ \gamma(y)\} \\
&\ge \inf\{\inf\{\pi_i \circ \gamma(x * (w * (y * z))), \pi_i \circ \gamma(w)\}, \pi_i \circ \gamma(y)\} \\
&= \inf\{\inf\{\pi_i \circ \gamma(0), \pi_i \circ \gamma(w)\}, \pi_i \circ \gamma(y)\} \\
&= \inf\{\pi_i \circ \gamma(w), \pi_i \circ \gamma(y)\},
\end{aligned}$$

$$\begin{aligned}
\pi_i \circ \psi(x * z) &\le \sup\{\pi_i \circ \psi(x * (y * z)), \pi_i \circ \psi(y)\} \\
&\le \sup\{\sup\{\pi_i \circ \psi(x * (w * (y * z))), \pi_i \circ \psi(w)\}, \pi_i \circ \psi(y)\} \\
&= \sup\{\pi_i \circ \psi(0), \pi_i \circ \psi(w), \pi_i \circ \psi(y)\} \\
&= \sup\{\pi_i \circ \psi(w), \pi_i \circ \psi(y)\}.
\end{aligned}$$

□

Lemma 3.9. *If $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-ideal of X, then*

$$(\forall\, x, y, z \in X) \left(x \le y * z \;\Rightarrow \begin{cases} \Phi(x * z) \ge \Phi(y) \\ \Upsilon(x * z) \ge \Upsilon(y) \\ \Psi(x * z) \le \Psi(y) \end{cases} \right). \tag{25}$$

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Proof. Let $x, y, z \in X$ be such that $x \leq y * z$. Then $x * (y * z) = 0$ and so

$$\begin{array}{rcl} \pi_i \circ \varphi(x * z) & \geq & \inf\{\pi_i \circ \varphi(x * (y * z)), \pi_i \circ \varphi(y)\} \\ & = & \inf\{\pi_i \circ \varphi(0), \pi_i \circ \varphi(y)\} \\ & = & \pi_i \circ \varphi(y), \\ \pi_i \circ \gamma(x * z) & \geq & \inf\{\pi_i \circ \gamma(x * (y * z)), \pi_i \circ \gamma(y)\} \\ & = & \inf\{\pi_i \circ \gamma(0), \pi_i \circ \gamma(y)\} \\ & = & \pi_i \circ \gamma(y), \\ \pi_i \circ \psi(x * z) & \leq & \sup\{\pi_i \circ \psi(x * (y * z)), \pi_i \circ \psi(y)\} \\ & = & \sup\{\pi_i \circ \psi(0), \pi_i \circ \psi(y)\} \\ & = & \pi_i \circ \psi(y). \end{array}$$

□

Theorem 3.10. *Every m-polar neutrosophic UP-ideal of X is a m-polar neutrosophic UP-subalgebra of X.*

Proof. Let $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ be a m-polar neutrosophic UP-ideal of X, and let $x, y \in X$. Since $x \leq y * x$, we have

$$\begin{array}{c} \pi_i \circ \varphi(y * x) \geq \pi_i \circ \varphi(x) \geq \inf\{\pi_i \circ \varphi(y), \pi_i \circ \varphi(x)\} \\ \pi_i \circ \gamma(y * x) \geq \pi_i \circ \gamma(x) \geq \inf\{\pi_i \circ \gamma(y), \pi_i \circ \gamma(x)\} \\ \pi_i \circ \psi(y * x) \leq \pi_i \circ \psi(x) \leq \sup\{\pi_i \circ \psi(y), \pi_i \circ \psi(x)\}. \end{array}$$

Hence $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-subalgebra of X. □

Theorem 3.11. *If a m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ of X satisfies*

$$(\forall\, x, y, z \in X) \left(z \leq x * y \Rightarrow \begin{cases} \Phi(z) \geq \inf\{\Phi(x), \Phi(y)\} \\ \Upsilon(z) \geq \inf\{\Upsilon(x), \Upsilon(y)\} \\ \Psi(z) \leq \sup\{\Psi(x), \Psi(y)\} \end{cases} \right), \tag{26}$$

then $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-subalgebra of X.

Proof. Let $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ be a m-polar neutrosophic set of X satisfying the condition (26). Next, let $x, y \in X$. By (6), we have $(x * y) * (x * y) = 0$, that is, $x * y \leq x * y$. It follows from (26) that

$$\pi_i \circ \varphi(x * y) \geq \inf\{\pi_i \circ \varphi(x), \pi_i \circ \varphi(y)\},$$

$$\pi_i \circ \gamma(x * y) \geq \inf\{\pi_i \circ \gamma(x), \pi_i \circ \gamma(y)\},$$

$$\pi_i \circ \psi(x * y) \leq \sup\{\pi_i \circ \psi(x), \pi_i \circ \psi(y)\}.$$

Hence $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-subalgebra of X. □

Theorem 3.12. *If a m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ of X satisfies*

$$(\forall a, x, y, z \in X) \left(a \leq x * (y * z) \Rightarrow \begin{cases} \Phi(x * z) \geq \inf\{\Phi(a), \Phi(y)\} \\ \Upsilon(x * z) \geq \inf\{\Upsilon(a), \Upsilon(y)\} \\ \Psi(x * z) \leq \sup\{\Psi(a), \Psi(y)\} \end{cases} \right), \tag{27}$$

then $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-ideal of X.

Proof. Let $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ be a m-polar neutrosophic set of X satisfying the condition (27). Next, let $x, y \in X$. By (3), we have $x * (0 * (x * 0)) = 0$, that is, $x \leq 0 * (x * 0)$. It follows from (27) that

$$\pi_i \circ \varphi(0) = \pi_i \circ \varphi(0 * 0) \geq \inf\{\pi_i \circ \varphi(x), \pi_i \circ \varphi(x)\} = \pi_i \circ \varphi(x),$$

$$\pi_i \circ \gamma(0) = \pi_i \circ \gamma(0 * 0) \geq \inf\{\pi_i \circ \gamma(x), \pi_i \circ \gamma(x)\} = \pi_i \circ \gamma(x),$$

$$\pi_i \circ \psi(0) = \pi_i \circ \psi(0 * 0) \leq \sup\{\pi_i \circ \psi(x), \pi_i \circ \psi(x)\} = \pi_i \circ \psi(x).$$

Next, let $x, y, z \in X$. By (6), we have $(x * (y * z)) * (x * (y * z)) = 0$, that is, $(x * (y * z)) \leq (x * (y * z))$. It follows from (27) that

$$\pi_i \circ \varphi(x * z) \geq \inf\{\pi_i \circ \varphi(x * (y * z)), \pi_i \circ \varphi(y)\},$$

$$\pi_i \circ \gamma(x * z) \geq \inf\{\pi_i \circ \gamma(x * (y * z)), \pi_i \circ \gamma(y)\},$$

$$\pi_i \circ \psi(x * z) \leq \sup\{\pi_i \circ \psi(x * (y * z)), \pi_i \circ \psi(y)\}.$$

Hence $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic UP-ideal of X. □

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Theorem 3.13. *If a m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ of X satisfies*

$$(\forall x, y, z \in X) \left(z \leq x * y \Rightarrow \begin{cases} \Phi(z) \geq \Phi(y) \\ \Upsilon(z) \geq \Upsilon(y) \\ \Psi(z) \leq \Psi(y) \end{cases} \right), \tag{28}$$

then $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic strong UP-ideal of X.

Proof. Let $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ be a m-polar neutrosophic set of X satisfying the condition (28). Next, let $x, y \in X$. By (3), we have $x * (y * y) = 0$, that is, $x \leq y * y$. It follows from (28) that $\pi_i \circ \varphi(x) \geq \pi_i \circ \varphi(y)$, $\pi_i \circ \gamma(x) \geq \pi_i \circ \gamma(y)$ and $\pi_i \circ \psi(x) \leq \pi_i \circ \psi(y)$. Similarly, $\pi_i \circ \varphi(y) \geq \pi_i \circ \varphi(x)$, $\pi_i \circ \gamma(y) \geq \pi_i \circ \gamma(x)$ and $\pi_i \circ \psi(y) \leq \pi_i \circ \psi(x)$. Then $\pi_i \circ \varphi(x) = \pi_i \circ \varphi(y)$, $\pi_i \circ \gamma(x) = \pi_i \circ \gamma(y)$ and $\pi_i \circ \psi(x) = \pi_i \circ \psi(y)$ for all $x, y \in X$. Hence $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a constant. By Theorem 3.4, $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a m-polar neutrosophic strong UP-ideal of X. ☐

Definition 3.14. Let $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ be an m-polar neutrosophic set defined on X. The operators $\oplus\mathcal{P}$ and $\otimes\mathcal{P}$ are defined as $\oplus\mathcal{P} = (\Phi, \Upsilon, \overline{\Phi})$ and $\otimes\mathcal{P} = (\overline{\Psi}, \overline{\Psi}, \Psi)$, where $\overline{\Psi} = 1 - \Psi$, that is, $\overline{(\pi_i \circ \psi)}(x) = 1 - (\pi_i \circ \psi)(x)$ for all $i = 1, 2, 3, \cdots, m$.

Theorem 3.15. *An m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is an m-polar neutrosophic UP-subalgebra of X if and only if $\oplus\mathcal{P}$ and $\otimes\mathcal{H}$ are neutrosophic UP-subalgebras.*

Proof. Let $\mathcal{P}$ be an m-polar neutrosophic UP-subalgebra of X. Let $x, y \in X$. Then

$$\begin{aligned} \overline{\Phi}(x * y) &= 1 - \Phi(x * y) \\ &\leq 1 - \inf\{\Phi(x), \Phi(y)\} \\ &= \sup\{1 - \Phi(x), 1 - \Phi(y)\} \\ &= \sup\{\overline{\Phi}(x), \overline{\Phi}(y)\}. \end{aligned}$$

Hence $\oplus\mathcal{P}$ is an m-polar neutrosophic UP-subalgebra of X.
Let $x, y \in X$. Then

$$\begin{aligned} \overline{\Psi}(x * y) &= 1 - \Psi(x * y) \\ &\geq 1 - \sup\{\Psi(x), \Psi(y)\} \\ &= \inf\{1 - \Psi(x), 1 - \Psi(y)\} \\ &= \inf\{\overline{\Psi}(x), \overline{\Psi}(y)\}. \end{aligned}$$

Hence $\otimes\mathcal{P}$ is an m-polar neutrosophic UP-subalgebra of X. Conversely, assume that $\oplus\mathcal{P}$ and $\otimes\mathcal{P}$ are m-polar neutrosophic UP-subalgebras of X. Then for any $x, y \in X$, we have $\Phi(x * y) \geq \inf\{\Phi(x), \Phi(y)\}$, $\Upsilon(x * y) \geq \inf\{\Upsilon(x), \Upsilon(y)\}$ and $\Psi(x * y) \leq \sup\{\Psi(x), \Psi(y)\}$. Hence $\mathcal{P}$ is an m-polar neutrosophic UP-subalgebra of X. ☐

Theorem 3.16. *An m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is an m-polar neutrosophic UP-ideal of X if and only if $\oplus\mathcal{P}$ and $\otimes\mathcal{H}$ are m-polar neutrosophic UP-ideals.*

Proof. Let $\mathcal{P}$ be an m-polar neutrosophic UP-ideal of X. Let $x, y, z \in X$. Then

$$\begin{aligned} \overline{\Phi}(x * y) &= 1 - \Phi(x * y) \\ &\leq 1 - \inf\{\Phi(x * (y * z)), \Phi(y)\} \\ &= \sup\{1 - \Phi(x * (y * z)), 1 - \Phi(y)\} \\ &= \sup\{\overline{\Phi}(x * (y * z)), \overline{\Phi}(y)\}. \end{aligned}$$

Hence $\oplus\mathcal{P}$ is an m-polar neutrosophic UP-ideal of X.
Let $x, y, z \in X$. Then

$$\begin{aligned} \overline{\Psi}(x * y) &= 1 - \Psi(x * y) \\ &\geq 1 - \sup\{\Psi(x * (y * z)), \Psi(y)\} \\ &= \inf\{1 - \Psi(x * (y * z)), 1 - \Psi(y)\} \\ &= \inf\{\overline{\Psi}(x * (y * z)), \overline{\Psi}(y)\}. \end{aligned}$$

Hence $\otimes\mathcal{P}$ is an m-polar neutrosophic UP-ideal of X. Conversely, assume that $\oplus\mathcal{P}$ and $\otimes\mathcal{P}$ are m-polar neutrosophic UP-ideal of X. Then for any $x, y, z \in X$, we have $\Phi(x * y) \geq \inf\{\Phi(x * (y * z)), \Phi(y)\}$, $\Upsilon(x * y) \geq \inf\{\Upsilon(x * (y * z)), \Upsilon(y)\}$ and $\Psi(x * y) \leq \sup\{\Psi(x * (y * z)), \Psi(y)\}$. Hence $\mathcal{P}$ is an m-polar neutrosophic UP-ideal of X. ☐

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Theorem 3.17. *An m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is an m-polar neutrosophic strong UP-ideal of X if and only if $\oplus\mathcal{P}$ and $\otimes\mathcal{H}$ are m-polar neutrosophic strong UP-ideals.*

Proof. Let $\mathcal{P}$ be an m-polar neutrosophic strong UP-ideal of X. Let $x, y, z \in X$. Then

$$\begin{array}{rcl}\overline{\Phi}(x) & = & 1 - \Phi(x) \\ & \leq & 1 - \inf\{\Phi((z * y) * (z * x)), \Phi(y)\} \\ & = & \sup\{1 - \Phi((z * y) * (z * x)), 1 - \Phi(y)\} \\ & = & \sup\{\overline{\Phi}((z * y) * (z * x)), \overline{\Phi}(y)\}.\end{array}$$

Hence $\oplus\mathcal{P}$ is an m-polar neutrosophic strong UP-ideal of X.
Let $x, y, z \in X$. Then

$$\begin{array}{rcl}\overline{\Psi}(x) & = & 1 - \Psi(x) \\ & \geq & 1 - \sup\{\Psi((z * y) * (z * x)), \Psi(y)\} \\ & = & \inf\{1 - \Psi((z * y) * (z * x)), 1 - \Psi(y)\} \\ & = & \inf\{\overline{\Psi}((z * y) * (z * x)), \overline{\Psi}(y)\}.\end{array}$$

Hence $\otimes\mathcal{P}$ is an m-polar neutrosophic strong UP-ideal of X. Conversely, assume that $\oplus\mathcal{P}$ and $\otimes\mathcal{P}$ are m-polar neutrosophic strong UP-ideal of X. Then for any $x, y, z \in X$, we have $\Phi(x) \geq \inf\{\Phi((z*y)*(z*x)), \Phi(y)\}$, $\Upsilon(x) \geq \inf\{\Upsilon((z * y) * (z * x)), \Upsilon(y)\}$ and $\Psi(x) \leq \sup\{\Psi((z * y) * (z * x)), \Psi(y)\}$. Hence $\mathcal{P}$ is an m-polar neutrosophic strong UP-ideal of X. $\square$

Definition 3.18. Given an m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ over X and $(\zeta, \omega, \xi) \in (0, 1]^m \times (0, 1]^m \times [0, 1)^m$, we consider the sets, $U(\Phi, \zeta) = \{x \in X : \varphi(x) + \zeta > 1\}$, $U(\Upsilon, \omega) = \{x \in X : \gamma(x) + \omega > 1\}$ and $L(\Psi, \xi) = \{x \in X : \psi(x) + \xi < 1\}$. Then $U(\Phi, \zeta) = \bigcap_{i=1}^{m} U(\Phi, \zeta)^i$, $U(\Upsilon, \omega) = \bigcap_{i=1}^{m} U(\Upsilon, \omega)^i$ and $L(\Psi, \xi) = \bigcap_{i=1}^{m} L(\Psi, \xi)^i$, where $U(\Phi, \zeta)^i = \{x \in X : (\pi_i \circ \varphi)(x) + \zeta_i > 1\}$, $U(\Upsilon, \omega)^i = \{x \in X : (\pi_i \circ \gamma)(x) + \omega_i > 1\}$ and $L(\Psi, \xi)^i = \{x \in X : (\pi_i \circ \psi)(x) + \xi_i < 1\}$ for all $i = 1, 2, 3, \cdots, m$.

Theorem 3.19. *An m-polar neutrosophic set $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ of X is an m-polar neutrosophic UP-subalgebra of X if and only if for any $(\zeta, \omega, \xi) \in (0, 1]^m \times (0, 1]^m \times [0, 1)^m$, the nonempty sets $U(\Phi, \zeta)$, $U(\Upsilon, \omega)$ and $L(\Psi, \xi)$ are UP-subalgebras of X.*

Proof. Let $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ be a m-polar neutrosophic UP-subalgebra of X. Let $(\zeta, \omega, \xi) \in (0, 1]^m \times (0, 1]^m \times [0, 1)^m$ such that $U(\Phi, \zeta), U(\Upsilon, \omega), L(\Psi, \xi) \neq \emptyset$. Let $x, y \in U(\Phi, \zeta) = \bigcap_{i=1}^{m} U(\Phi, \zeta)^i$, $x, y \in U(\Upsilon, \omega) = \bigcap_{i=1}^{m} U(\Upsilon, \omega)^i$ and let $x, y \in L(\Psi, \xi) = \bigcap_{i=1}^{m} L(\Psi, \xi)^i$. Then $(\pi_i \circ \varphi)(x) + \zeta_i > 1$, $(\pi_i \circ \varphi)(y) + \zeta_i > 1$, $(\pi_i \circ \gamma)(x) + \omega_i > 1$, $(\pi_i \circ \gamma)(y) + \omega_i > 1$, $(\pi_i \circ \psi)(x) + \xi_i < 1$, $(\pi_i \circ \psi)(y) + \xi_i < 1, \forall i = 1, 2, 3 \cdots, m$. It follows from (17) that, for all $i = 1, 2, 3 \cdots, m$,

$$\begin{array}{rcl}(\pi_i \circ \varphi)(x * y) + \zeta_i & \geq & \inf\{\pi_i \circ \varphi(x), \pi_i \circ \varphi(y)\} + \zeta_i \\ & \geq & \inf\{\pi_i \circ \varphi(x) + \zeta_i, \pi_i \circ \varphi(y) + \zeta_i\} \\ & > & 1,\end{array}$$

$$\begin{array}{rcl}(\pi_i \circ \gamma)(x * y) + \omega_i & \geq & \inf\{\pi_i \circ \gamma(x), \pi_i \circ \gamma(y)\} + \omega_i \\ & \geq & \inf\{\pi_i \circ \gamma(x) + \omega_i, \pi_i \circ \gamma(y) + \omega_i\} \\ & > & 1,\end{array}$$

$$\begin{array}{rcl}(\pi_i \circ \psi)(x * y) + \xi_i & \leq & \sup\{(\pi_i \circ \psi)(x), (\pi_i \circ \psi)(y)\} + \xi_i \\ & \leq & \sup\{(\pi_i \circ \psi)(x) + \xi_i, (\pi_i \circ \psi)(y) + \xi_i\} \\ & < & 1.\end{array}$$

Hence $x * y \in U(\Phi, \zeta) = \bigcap_{i=1}^{m} U(\Phi, \zeta)^i$, $x * y \in U(\Upsilon, \omega) = \bigcap_{i=1}^{m} U(\Upsilon, \omega)^i$ and $x * y \in L(\Psi, \xi) = \bigcap_{i=1}^{m} L(\Psi, \xi)^i$. Hence $U(\Phi, \zeta)$, $U(\Upsilon, \omega)$ and $L(\Psi, \xi)$ are UP-subalgebras of X for all $(\zeta, \omega, \xi) \in (0, 1]^m \times (0, 1]^m \times [0, 1)^m$ with $U(\Phi, \zeta), U(\Upsilon, \omega), L(\Psi, \xi) \neq \emptyset$. Conversely, assume that $U(\Phi, \zeta)$, $U(\Upsilon, \omega)$ and $L(\Psi, \xi)$ are UP-subalgebras of X for all $(\zeta, \omega, \xi) \in (0, 1]^m \times (0, 1]^m \times [0, 1)^m$ with $U(\Phi, \zeta), U(\Upsilon, \omega), L(\Psi, \xi) \neq \emptyset$. On contrary, there exist $a, b \in X$ such that $\Phi(a*b) < \inf\{\Phi(a), \Phi(b)\}$. Then $\Phi(a*b)+\zeta \leq 1 < \inf\{\Phi(a), \Phi(b)\}+\zeta$

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for some $\zeta \in (0,1]^m$. It follows that $a, b \in U(\Phi,\zeta)$, which implies $a*b \in U(\Phi,\zeta)$, since $U(\Phi,\zeta)$ is a UP-subalgebra of X. Hence $\Phi(a*b)+\zeta > 1$, which is a contradiction. If $\Upsilon(a*b) < \inf\{\Upsilon(a),\Upsilon(b)\}$. Then $\Upsilon(a*b)+\omega \leq 1 < \inf\{\Upsilon(a),\Upsilon(b)\}+\omega$ for some $\omega \in (0,1]^m$. It follows that $a,b \in U(\Upsilon,\omega)$, which implies $a*b \in U(\Upsilon,\omega)$, since $U(\Upsilon,\omega)$ is a UP-subalgebra of X. Hence $\Upsilon(a*b)+\omega > 1$, which is a contradiction. If $\Psi(a*b) > \sup\{\Psi(a),\Psi(b)\}$, then $\Psi(a*b)+\xi \geq 1 > \sup\{\Psi(a),\Psi(b)\}+\xi$ for some $\xi \in [0,1)^m$. It follows that $a,b \in L(\Psi,\xi)$, which implies $a*b \in L(\Psi,\xi)$, since $L(\Psi,\xi)$ is a UP-subalgebra of X. Hence $\Psi(a*b)+\xi < 1$, which is a contradiction. Hence $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ is a m-polar neutrosophic UP-subalgebra of X.

Theorem 3.20. *An m-polar neutrosophic set $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ of X is an m-polar neutrosophic UP-ideal of X if and only if for any $(\zeta,\omega,\xi) \in (0,1]^m \times (0,1]^m \times [0,1)^m$, the nonempty sets $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ are UP-ideals of X.*

Proof. Let $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ be a m-polar neutrosophic UP-ideal of X. Let $(\zeta,\omega,\xi) \in (0,1]^m \times (0,1]^m \times [0,1)^m$ such that $U(\Phi,\zeta), U(\Upsilon,\omega), L(\Psi,\xi) \neq \emptyset$. Let $x*(y*z), y \in U(\Phi,\zeta) = \bigcap_{i=1}^{m} U(\Phi,\zeta)^i$, $x*(y*z), y \in U(\Upsilon,\omega) = \bigcap_{i=1}^{m} U(\Upsilon,\omega)^i$ and let $x*(y*z), y \in L(\Psi,\xi) = \bigcap_{i=1}^{m} L(\Psi,\xi)^i$. Then $(\pi_i \circ \varphi)(x*(y*z))+\zeta_i > 1$, $(\pi_i \circ \varphi)(y)+\zeta_i > 1$, $(\pi_i \circ \gamma)(x*(y*z))+\omega_i > 1$, $(\pi_i \circ \gamma)(y)+\omega_i > 1$, $(\pi_i \circ \psi)(x*(y*z))+\xi_i < 1$, $(\pi_i \circ \psi)(y)+\xi_i < 1$, $\forall i = 1,2,3\cdots,m$. It follows from (17) that, for all $i = 1,2,3\cdots,m$,

$$\begin{aligned}
(\pi_i \circ \varphi)(x*y)+\zeta_i &\geq \inf\{\pi_i \circ \varphi(x*(y*z)), \pi_i \circ \varphi(y)\}+\zeta_i \\
&\geq \inf\{\pi_i \circ \varphi(x*(y*z))+\zeta_i, \pi_i \circ \varphi(y)+\zeta_i\} \\
&> 1,
\end{aligned}$$

$$\begin{aligned}
(\pi_i \circ \gamma)(x*y)+\omega_i &\geq \inf\{\pi_i \circ \gamma(x*(y*z)), \pi_i \circ \gamma(y)\}+\omega_i \\
&\geq \inf\{\pi_i \circ \gamma(x*(y*z))+\omega_i, \pi_i \circ \gamma(y)+\omega_i\} \\
&> 1,
\end{aligned}$$

$$\begin{aligned}
(\pi_i \circ \psi)(x*y)+\xi_i &\leq \sup\{(\pi_i \circ \psi)(x*(y*z)), (\pi_i \circ \psi)(y)\}+\xi_i \\
&\leq \sup\{(\pi_i \circ \psi)(x*(y*z))+\xi_i, (\pi_i \circ \psi)(y)+\xi_i\} \\
&< 1.
\end{aligned}$$

Hence $x*y \in U(\Phi,\zeta) = \bigcap_{i=1}^{m} U(\Phi,\zeta)^i$, $x*y \in U(\Upsilon,\omega) = \bigcap_{i=1}^{m} U(\Upsilon,\omega)^i$ and $x*y \in L(\Psi,\xi) = \bigcap_{i=1}^{m} L(\Psi,\xi)^i$. Hence $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ are UP-ideals of X for all $(\zeta,\omega,\xi) \in (0,1]^m \times (0,1]^m \times [0,1)^m$ with $U(\Phi,\zeta), U(\Upsilon,\omega), L(\Psi,\xi) \neq \emptyset$. Conversely, assume that $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ are UP-ideals of X for all $(\zeta,\omega,\xi) \in (0,1]^m \times (0,1]^m \times [0,1)^m$ with $U(\Phi,\zeta), U(\Upsilon,\omega), L(\Psi,\xi) \neq \emptyset$. On contrary, there exist $a,b,c \in X$ such that $\Phi(a*b) < \inf\{\Phi(a*(b*c)),\Phi(b)\}$. Then $\Phi(a*b)+\zeta \leq 1 < \inf\{\Phi(a*(b*c)),\Phi(b)\}+\zeta$ for some $\zeta \in (0,1]^m$. It follows that $a*(b*c), b \in U(\Phi,\zeta)$, which implies $a*b \in U(\Phi,\zeta)$, since $U(\Phi,\zeta)$ is a UP-ideal of X. Hence $\Phi(a*b)+\zeta > 1$, which is a contradiction. If $\Upsilon(a*b) < \inf\{\Upsilon(a*(b*c)),\Upsilon(b)\}$. Then $\Upsilon(a*b)+\omega \leq 1 < \inf\{\Upsilon(a*(b*c)),\Upsilon(b)\}+\omega$ for some $\omega \in (0,1]^m$. It follows that $a*(b*c), b \in U(\Upsilon,\omega)$, which implies $a*b \in U(\Upsilon,\omega)$, since $U(\Upsilon,\omega)$ is a UP-ideal of X. Hence $\Upsilon(a*b)+\omega > 1$, which is a contradiction. If $\Psi(a*b) > \sup\{\Psi(a*(b*c)),\Psi(b)\}$, then $\Psi(a*b)+\xi \geq 1 > \sup\{\Psi(a*(b*c)),\Psi(b)\}+\xi$ for some $\xi \in [0,1)^m$. It follows that $a*(b*c), b \in L(\Psi,\xi)$, which implies $a*b \in L(\Psi,\xi)$, since $L(\Psi,\xi)$ is a UP-ideal of X. Hence $\Psi(a*b)+\xi < 1$, which is a contradiction. Hence $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ is a m-polar neutrosophic UP-ideal of X. ☐

Theorem 3.21. *An m-polar neutrosophic set $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ of X is an m-polar neutrosophic strong UP-ideal of X if and only if for any $(\zeta,\omega,\xi) \in (0,1]^m \times (0,1]^m \times [0,1)^m$, the nonempty sets $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ are strong UP-ideals of X.*

Proof. Let $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ be a m-polar neutrosophic strong UP-ideal of X. Let $(\zeta,\omega,\xi) \in (0,1]^m \times (0,1]^m \times [0,1)^m$ such that $U(\Phi,\zeta), U(\Upsilon,\omega), L(\Psi,\xi) \neq \emptyset$. Let $(z*y)*(z*x), y \in U(\Phi,\zeta) = \bigcap_{i=1}^{m} U(\Phi,\zeta)^i$, $(z*y)*(z*x), y \in U(\Upsilon,\omega) = \bigcap_{i=1}^{m} U(\Upsilon,\omega)^i$ and let $(z*y)*(z*x), y \in L(\Psi,\xi) = \bigcap_{i=1}^{m} L(\Psi,\xi)^i$. Then $(\pi_i \circ \varphi)((z*y)*(z*x))+\zeta_i > 1$, $(\pi_i \circ \varphi)(y)+\zeta_i > 1$, $(\pi_i \circ \gamma)((z*y)*(z*x))+\omega_i > 1$, $(\pi_i \circ \gamma)(y)+\omega_i > 1$,

$(\pi_i \circ \psi)((z*y)*(z*x)) + \xi_i < 1$, $(\pi_i \circ \psi)(y) + \xi_i < 1, \forall i = 1,2,3\cdots,m$. It follows from (17) that, for all $i = 1,2,3\cdots,m$,

$$\begin{aligned}(\pi_i \circ \varphi)(x) + \zeta_i &\geq \inf\{\pi_i \circ \varphi((z*y)*(z*x)), \pi_i \circ \varphi(y)\} + \zeta_i \\ &\geq \inf\{\pi_i \circ \varphi((z*y)*(z*x)) + \zeta_i, \pi_i \circ \varphi(y) + \zeta_i\} \\ &> 1,\end{aligned}$$

$$\begin{aligned}(\pi_i \circ \gamma)(x) + \omega_i &\geq \inf\{\pi_i \circ \gamma((z*y)*(z*x)), \pi_i \circ \gamma(y)\} + \omega_i \\ &\geq \inf\{\pi_i \circ \gamma((z*y)*(z*x)) + \omega_i, \pi_i \circ \gamma(y) + \omega_i\} \\ &> 1,\end{aligned}$$

$$\begin{aligned}(\pi_i \circ \psi)(x) + \xi_i &\leq \sup\{(\pi_i \circ \psi)((z*y)*(z*x)), (\pi_i \circ \psi)(y)\} + \xi_i \\ &\leq \sup\{(\pi_i \circ \psi)((z*y)*(z*x)) + \xi_i, (\pi_i \circ \psi)(y) + \xi_i\} \\ &< 1.\end{aligned}$$

Hence $x \in U(\Phi,\zeta) = \bigcap_{i=1}^{m} U(\Phi,\zeta)^i$, $x \in U(\Upsilon,\omega) = \bigcap_{i=1}^{m} U(\Upsilon,\omega)^i$ and $x \in L(\Psi,\xi) = \bigcap_{i=1}^{m} L(\Psi,\xi)^i$. Hence $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ are strong UP-ideals of X for all $(\zeta,\omega,\xi) \in (0,1]^m \times (0,1]^m \times [0,1)^m$ with $U(\Phi,\zeta), U(\Upsilon,\omega), L(\Psi,\xi) \neq \emptyset$. Conversely, assume that $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ are strong UP-ideals of X for all $(\zeta,\omega,\xi) \in (0,1]^m \times (0,1]^m \times [0,1)^m$ with $U(\Phi,\zeta), U(\Upsilon,\omega), L(\Psi,\xi) \neq \emptyset$. On contrary, there exist $a,b,c \in X$ such that $\Phi(a) < \inf\{\Phi((c*b)*(c*a)), \Phi(b)\}$. Then $\Phi(a) + \zeta \leq 1 < \inf\{\Phi((c*b)*(c*a)), \Phi(b)\} + \zeta$ for some $\zeta \in (0,1]^m$. It follows that $(c*b)*(c*a), b \in U(\Phi,\zeta)$, which implies $a \in U(\Phi,\zeta)$, since $U(\Phi,\zeta)$ is a strong UP-ideal of X. Hence $\Phi(a) + \zeta > 1$, which is a contradiction. If $\Upsilon(a) < \inf\{\Upsilon((c*b)*(c*a)), \Upsilon(b)\}$. Then $\Upsilon(a) + \omega \leq 1 < \inf\{\Upsilon((c*b)*(c*a)), \Upsilon(b)\} + \omega$ for some $\omega \in (0,1]^m$. It follows that $(c*b)*(c*a), b \in U(\Upsilon,\omega)$, which implies $a \in U(\Upsilon,\omega)$, since $U(\Upsilon,\omega)$ is a strong UP-ideal of X. Hence $\Upsilon(a) + \omega > 1$, which is a contradiction. If $\Psi(a) > \sup\{\Psi((c*b)*(c*a)), \Psi(b)\}$, then $\Psi(a) + \xi \geq 1 > \sup\{\Psi((c*b)*(c*a)), \Psi(b)\} + \xi$ for some $\xi \in [0,1)^m$. It follows that $(c*b)*(c*a), b \in L(\Psi,\xi)$, which implies $a \in L(\Psi,\xi)$, since $L(\Psi,\xi)$ is a UP-ideal of X. Hence $\Psi(a) + \xi < 1$, which is a contradiction. Hence $\mathcal{P} = (\Phi,\Upsilon,\Psi)$ is a m-polar neutrosophic strong UP-ideal of X. □

Theorem 3.22. *Let A be a subset of X and let $\mathcal{P}_A = (\Phi_A,\Upsilon_A,\Psi_A)$ be an m-polar neutrosophic set on X defined by* $\Phi_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{1} & \text{if } x \in A \\ \overline{0} & \text{otherwise,} \end{cases}$ $\Upsilon_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{1} & \text{if } x \in A \\ \overline{0} & \text{otherwise,} \end{cases}$ $\Psi_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{0} & \text{if } x \in A \\ \overline{1} & \text{otherwise,} \end{cases}$ *where* $\overline{1} = (1,1,1,\cdots,1) \in [0,1]^m$ *and* $\overline{0} = (0,0,0,\cdots,0) = [0,1]^m$. *Then $\mathcal{P}_A = (\Phi_A,\Upsilon_A,\Psi_A)$ is an m-polar neutrosophic UP-subalgebra of X if and only if A is a UP-subalgebra of X.*

Proof. Assume that A is a UP-subalgebra of X. Let $x,y \in X$.
Case 1 : If $x,y \in A$, then $\Phi_A(x) = \overline{1}$ and $\Phi_A(y) = \overline{1}$. Thus $\inf\{\Phi_A(x), \Phi_A(y)\} = \overline{1}$. Since A is a UP-subalgebra of X, $x*y \in A$ and so $\Phi_A(x*y) = \overline{1}$. Then $\Phi_A(x*y) = \overline{1} \geq \overline{1} = \inf\{\Phi_A(x), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x*y) \geq \inf\{\Upsilon_A(x), \Upsilon_A(y)\}$. Also $\Psi_A(x) = \overline{0}$ and $\Psi_A(y) = \overline{0}$. Thus $\sup\{\Psi_A(x), \Psi_A(y)\} = \overline{0}$. Since A is a UP-subalgebra of X, $x*y \in A$ and so $\Psi_A(x*y) = \overline{0}$. Then $\Psi_A(x*y) = \overline{0} \leq \overline{0} = \sup\{\Psi_A(x), \Psi_A(y)\}$.
Case 2 : If $x \in A$ and $y \notin A$, then $\Phi_A(x) = \overline{1}$ and $\Phi_A(y) = \overline{0}$. Thus $\inf\{\Phi_A(x), \Phi_A(y)\} = \overline{0}$. Then $\Phi_A(x*y) \geq \overline{0} = \inf\{\Phi_A(x), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x*y) \geq \inf\{\Upsilon_A(x), \Upsilon_A(y)\}$. Also $\Psi_A(x) = \overline{0}$ and $\Psi_A(y) = \overline{1}$. Thus $\sup\{\Psi_A(x), \Psi_A(y)\} = \overline{1}$. Then $\Psi_A(x*y) \leq \overline{1} = \sup\{\Psi_A(x), \Psi_A(y)\}$.
Case 3 : If $x \notin A$ and $y \in A$, then $\Phi_A(x) = \overline{0}$ and $\Phi_A(y) = \overline{1}$. Thus $\inf\{\Phi_A(x), \Phi_A(y)\} = \overline{0}$. Then $\Phi_A(x*y) \geq \overline{0} = \inf\{\Phi_A(x), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x*y) \geq \inf\{\Upsilon_A(x), \Upsilon_A(y)\}$. Also $\Psi_A(x) = \overline{1}$ and $\Psi_A(y) = \overline{0}$. Thus $\sup\{\Psi_A(x), \Psi_A(y)\} = \overline{1}$. Then $\Psi_A(x*y) \leq \overline{1} = \sup\{\Psi_A(x), \Psi_A(y)\}$.
Case 4 : If $x \notin A$ and $y \notin A$, then $\Phi_A(x) = \overline{0}$ and $\Phi_A(y) = \overline{0}$. Thus $\inf\{\Phi_A(x), \Phi_A(y)\} = \overline{0}$. Hence $\Phi_A(x*y) \geq \overline{0} = \inf\{\Phi_A(x), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x*y) \geq \inf\{\Upsilon_A(x), \Upsilon_A(y)\}$. Also $\Psi_A(x) = \overline{1}$ and $\Psi_A(y) = \overline{1}$. Thus $\sup\{\Psi_A(x), \Psi_A(y)\} = \overline{1}$. Then $\Psi_A(x*y) \leq \overline{1} = \sup\{\Psi_A(x), \Psi_A(y)\}$.
Hence $\mathcal{P}_A = (\Phi_A,\Upsilon_A,\Psi_A)$ is an m-polar neutrosophic UP-subalgebra of X. Conversely, assume that $\mathcal{P}_A = (\Phi_A,\Upsilon_A,\Psi_A)$ is an m-polar neutrosophic UP-subalgebra of X. Let $x,y \in A$. Then $\Phi_A(x) = \overline{1}$ and $\Phi_A(y) = \overline{1}$. Thus $\Phi_A(x*y) \leq \sup\{\Phi_A(x), \Phi_A(y)\} = \overline{1}$, so $\Phi_A(x*y) = \overline{1}$. Hence $x*y \in A$ and so A is a UP-subalgebra of X. □

Lemma 3.23. *The constant 0 of X is in a nonempty subset B of X if and only if $\Phi_B(0) \geq \Phi_B(x)$, $\Upsilon_B(0) \geq \Upsilon_B(x)$ and $\Psi_B(0) \leq \Psi_B(x)$ for all $x \in X$.*

Proof. If $0 \in B$, then $\Phi_B(0) = \overline{1}$ and $\Upsilon_B(0) = \overline{1}$. Thus $\Phi_B(0) = \overline{1} \geq \Phi_B(x)$ and $\Upsilon_B(0) \geq \Upsilon_B(x)$ for all $x \in X$. Also $\Psi_B(0) = \overline{0}$. Then $\Psi_B(0) = \overline{0} \leq \Psi_B(x)$ for all $x \in X$. Conversely, assume that $\Phi_B(0) \geq \Phi_B(x)$, $\Upsilon_B(0) \geq \Upsilon_B(x)$ and $\Psi_B(0) \leq \Psi_B(x)$ for all $x \in X$. Since B is a nonempty subset of X, $a \in B$ for some $a \in X$. Then $\Phi_B(0) \geq \Phi_B(a) = \overline{1}$, so $\Phi_B(0) = \overline{1}$. Hence $0 \in B$. ☐

Theorem 3.24. *Let A be a subset of X and let $\mathcal{P}_A = (\Phi_A, \Upsilon_A, \Psi_A)$ be an m-polar neutrosophic set on X defined by $\Phi_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{1} & \text{if } x \in A \\ \overline{0} & \text{otherwise,} \end{cases}$ $\Upsilon_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{1} & \text{if } x \in A \\ \overline{0} & \text{otherwise,} \end{cases}$ $\Psi_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{0} & \text{if } x \in A \\ \overline{1} & \text{otherwise,} \end{cases}$ where $\overline{1} = (1,1,1,\cdots,1) \in [0,1]^m$ and $\overline{0} = (0,0,0,\cdots,0) = [0,1]^m$. Then $\mathcal{P}_A = (\Phi_A, \Upsilon_A, \Psi_A)$ is an m-polar neutrosophic UP-ideal of X if and only if A is a UP-ideal of X.*

Proof. Assume that A is a UP-ideal of X. Since $0 \in A$, it follows from Lemma 3.23 that $\Phi_B(0) \geq \Phi_B(x)$, $\Upsilon_B(0) \geq \Upsilon_B(x)$ and $\Psi_B(0) \leq \Psi_B(x)$ for all $x \in X$. Next, let $x, y, z \in X$.
Case 1 : If $x*(y*z), y \in A$, then $\Phi_A(x*(y*z)) = \overline{1}$ and $\Phi_A(y) = \overline{1}$. Thus $\inf\{\Phi_A(x*(y*z)), \Phi_A(y)\} = \overline{1}$. Since A is a UP-ideal of X, $x * y \in A$ and so $\Phi_A(x * y) = \overline{1}$. Then $\Phi_A(x * y) = \overline{1} \geq \overline{1} = \inf\{\Phi_A(x * (y * z)), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x * y) \geq \inf\{\Upsilon_A(x * (y * z)), \Upsilon_A(y)\}$. Also $\Psi_A(x * (y * z)) = \overline{0}$ and $\Psi_A(y) = \overline{0}$. Thus $\sup\{\Psi_A(x * (y * z)), \Psi_A(y)\} = \overline{0}$. Since A is a UP-ideal of X, $x * y \in A$ and so $\Psi_A(x * y) = \overline{0}$. Then $\Psi_A(x * y) = \overline{0} \leq \overline{0} = \sup\{\Psi_A(x * (y * z)), \Psi_A(y)\}$.
Case 2 : If $x * (y * z) \in A$ and $y \notin A$, then $\Phi_A(x * (y * z)) = \overline{1}$ and $\Phi_A(y) = \overline{0}$. Thus $\inf\{\Phi_A(x * (y * z)), \Phi_A(y)\} = \overline{0}$. Then $\Phi_A(x*y) \geq \overline{0} = \inf\{\Phi_A(x*(y*z)), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x*y) \geq \inf\{\Upsilon_A(x*(y*z)), \Upsilon_A(y)\}$. Also $\Psi_A(x*(y*z)) = \overline{0}$ and $\Psi_A(y) = \overline{1}$. Thus $\sup\{\Psi_A(x*(y*z)), \Psi_A(y)\} = \overline{1}$. Then $\Psi_A(x * y) \leq \overline{1} = \sup\{\Psi_A(x * (y * z)), \Psi_A(y)\}$.
Case 3 : If $x * (y * z) \notin A$ and $y \in A$, then $\Phi_A(x * (y * z)) = \overline{0}$ and $\Phi_A(y) = \overline{1}$. Thus $\inf\{\Phi_A(x * (y * z)), \Phi_A(y)\} = \overline{0}$. Then $\Phi_A(x*y) \geq \overline{0} = \inf\{\Phi_A(x*(y*z)), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x*y) \geq \inf\{\Upsilon_A(x*(y*z)), \Upsilon_A(y)\}$. Also $\Psi_A(x*(y*z)) = \overline{1}$ and $\Psi_A(y) = \overline{0}$. Thus $\sup\{\Psi_A(x*(y*z)), \Psi_A(y)\} = \overline{1}$. Then $\Psi_A(x * y) \leq \overline{1} = \sup\{\Psi_A(x * (y * z)), \Psi_A(y)\}$.
Case 4 : If $x * (y * z) \notin A$ and $y \notin A$, then $\Phi_A(x * (y * z)) = \overline{0}$ and $\Phi_A(y) = \overline{0}$. Thus $\inf\{\Phi_A(x * (y * z)), \Phi_A(y)\} = \overline{0}$. Hence $\Phi_A(x*y) \geq \overline{0} = \inf\{\Phi_A(x*(y*z)), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x*y) \geq \inf\{\Upsilon_A(x*(y*z)), \Upsilon_A(y)\}$. Also $\Psi_A(x*(y*z)) = \overline{1}$ and $\Psi_A(y) = \overline{1}$. Thus $\sup\{\Psi_A(x*(y*z)), \Psi_A(y)\} = \overline{1}$. Then $\Psi_A(x * y) \leq \overline{1} = \sup\{\Psi_A(x * (y * z)), \Psi_A(y)\}$.
Hence $\mathcal{P}_A = (\Phi_A, \Upsilon_A, \Psi_A)$ is an m-polar neutrosophic UP-ideal of X. Conversely, assume that $\mathcal{P}_A = (\Phi_A, \Upsilon_A, \Psi_A)$ is an m-polar neutrosophic UP-ideal of X. Let $x * (y * z), y \in A$. Then $\Phi_A(x * (y * z)) = \overline{1}$ and $\Phi_A(y) = \overline{1}$. Thus $\Phi_A(x * y) \geq \inf\{\Phi_A(x * (y * z)), \Phi_A(y)\} = \overline{1}$, so $\Phi_A(x * y) = \overline{1}$. Hence $x * y \in A$ and so A is a UP-ideal of X. ☐

Theorem 3.25. *Let A be a subset of X and let $\mathcal{P}_A = (\Phi_A, \Upsilon_A, \Psi_A)$ be an m-polar neutrosophic set on X defined by $\Phi_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{1} & \text{if } x \in A \\ \overline{0} & \text{otherwise,} \end{cases}$ $\Upsilon_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{1} & \text{if } x \in A \\ \overline{0} & \text{otherwise,} \end{cases}$ $\Psi_A : X \to [0,1]^m, x \mapsto \begin{cases} \overline{0} & \text{if } x \in A \\ \overline{1} & \text{otherwise,} \end{cases}$ where $\overline{1} = (1,1,1,\cdots,1) \in [0,1]^m$ and $\overline{0} = (0,0,0,\cdots,0) = [0,1]^m$. Then $\mathcal{P}_A = (\Phi_A, \Upsilon_A, \Psi_A)$ is an m-polar neutrosophic strong UP-ideal of X if and only if A is a strong UP-ideal of X.*

Proof. Assume that A is a strong UP-ideal of X. Since $0 \in A$, it follows from Lemma 3.23 that $\Phi_B(0) \geq \Phi_B(x)$, $\Upsilon_B(0) \geq \Upsilon_B(x)$ and $\Psi_B(0) \leq \Psi_B(x)$ for all $x \in X$. Next, let $x, y, z \in X$.
Case 1 : If $(z * y) * (z * x), y \in A$, then $\Phi_A((z * y) * (z * x)) = \overline{1}$ and $\Phi_A(y) = \overline{1}$. Thus $\inf\{\Phi_A((z * y) * (z * x)), \Phi_A(y)\} = \overline{1}$. Since A is a strong UP-ideal of X, $x \in A$ and so $\Phi_A(x) = \overline{1}$. Then $\Phi_A(x) = \overline{1} \geq \overline{1} = \inf\{\Phi_A((z*y)*(z*x)), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x) \geq \inf\{\Upsilon_A((z*y)*(z*x)), \Upsilon_A(y)\}$. Also $\Psi_A((z * y) * (z * x)) = \overline{0}$ and $\Psi_A(y) = \overline{0}$. Thus $\sup\{\Psi_A((z * y) * (z * x)), \Psi_A(y)\} = \overline{0}$. Since A is a strong UP-ideal of X, $x \in A$ and so $\Psi_A(x) = \overline{0}$. Then $\Psi_A(x) = \overline{0} \leq \overline{0} = \sup\{\Psi_A((z * y) * (z * x)), \Psi_A(y)\}$.
Case 2 : If $(z * y) * (z * x) \in A$ and $y \notin A$, then $\Phi_A((z * y) * (z * x)) = \overline{1}$ and $\Phi_A(y) = \overline{0}$. Thus $\inf\{\Phi_A((z * y) * (z * x)), \Phi_A(y)\} = \overline{0}$. Then $\Phi_A(x) \geq \overline{0} = \inf\{\Phi_A((z * y) * (z * x)), \Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x) \geq \inf\{\Upsilon_A((z * y) * (z * x)), \Upsilon_A(y)\}$. Also $\Psi_A((z * y) * (z * x)) = \overline{0}$ and $\Psi_A(y) = \overline{1}$. Thus $\sup\{\Psi_A((z * y) * (z * x)), \Psi_A(y)\} = \overline{1}$. Then $\Psi_A(x) \leq \overline{1} = \sup\{\Psi_A((z * y) * (z * x)), \Psi_A(y)\}$.
Case 3 : If $(z * y) * (z * x) \notin A$ and $y \in A$, then $\Phi_A((z * y) * (z * x)) = \overline{0}$ and $\Phi_A(y) = \overline{1}$. Thus

$\inf\{\Phi_A((z*y)*(z*x)),\Phi_A(y)\}=\overline{0}$. Then $\Phi_A(x)\geq\overline{0}=\inf\{\Phi_A((z*y)*(z*x)),\Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x)\geq\inf\{\Upsilon_A((z*y)*(z*x)),\Upsilon_A(y)\}$. Also $\Psi_A((z*y)*(z*x))=\overline{1}$ and $\Psi_A(y)=\overline{0}$. Thus $\sup\{\Psi_A((z*y)*(z*x)),\Psi_A(y)\}=\overline{1}$. Then $\Psi_A(x)\leq\overline{1}=\sup\{\Psi_A((z*y)*(z*x)),\Psi_A(y)\}$.
Case 4 : If $(z*y)*(z*x)\notin A$ and $y\notin A$, then $\Phi_A((z*y)*(z*x))=\overline{0}$ and $\Phi_A(y)=\overline{0}$. Thus $\inf\{\Phi_A((z*y)*(z*x)),\Phi_A(y)\}=\overline{0}$. Hence $\Phi_A(x)\geq\overline{0}=\inf\{\Phi_A((z*y)*(z*x)),\Phi_A(y)\}$. Similarly, we can prove $\Upsilon_A(x)\geq\inf\{\Upsilon_A((z*y)*(z*x)),\Upsilon_A(y)\}$. Also $\Psi_A((z*y)*(z*x))=\overline{1}$ and $\Psi_A(y)=\overline{1}$. Thus $\sup\{\Psi_A((z*y)*(z*x)),\Psi_A(y)\}=\overline{1}$. Then $\Psi_A(x)\leq\overline{1}=\sup\{\Psi_A((z*y)*(z*x)),\Psi_A(y)\}$.
Hence $\mathcal{P}_A=(\Phi_A,\Upsilon_A,\Psi_A)$ is an m-polar neutrosophic strong UP-ideal of X. Conversely, assume that $\mathcal{P}_A=(\Phi_A,\Upsilon_A,\Psi_A)$ is an m-polar neutrosophic strong UP-ideal of X. Let $(z*y)*(z*x),y\in A$. Then $\Phi_A((z*y)*(z*x))=\overline{1}$ and $\Phi_A(y)=\overline{1}$. Thus $\Phi_A(x)\geq\inf\{\Phi_A((z*y)*(z*x)),\Phi_A(y)\}=\overline{1}$, so $\Phi_A(x)=\overline{1}$. Hence $x\in A$ and so A is a strong UP-ideal of X. □

Given an m-polar neutrosophic set $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ over X, we define an m-polar neutrosophic set $\mathcal{P}^*=(\Phi^*,\Upsilon^*,\Psi^*)$ over U by

$$\Phi^*:X\to[0,1]^m, x\mapsto\begin{cases}\overline{1} & \text{if } x\in U(\Phi,\zeta)\\ \overline{0} & otherwise,\end{cases}\quad \Upsilon^*:X\to[0,1]^m, x\mapsto\begin{cases}\overline{1} & \text{if } x\in U(\Upsilon,\omega)\\ \overline{0} & otherwise.\end{cases}$$

$$\Psi^*:X\to[0,1]^m, x\mapsto\begin{cases}\overline{0} & \text{if } x\in L(\Psi,\xi)\\ \overline{1} & otherwise.\end{cases}$$

□

Theorem 3.26. *If $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ is an m-polar neutrosophic UP-subalgebra of X, so is $\mathcal{P}^*=(\Phi^*,\Upsilon^*,\Psi^*)$.*

Proof. If $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ is an m-polar neutrosophic UP-subalgebra of X, then the nonempty m-polar level sets $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ of $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ are UP-subalgebras of X for all $\zeta,\omega,\xi\in[0,1]^m$. Let $x,y\in X$. If $x,y\in U(\Phi,\zeta)$, then $x*y\in U(\Phi,\zeta)$. Then $\Phi^*(x*y)=\overline{1}\geq\inf\{\Phi(x),\Phi(y)\}=\inf\{\Phi^*(x),\Phi^*(y)\}$. If $x\notin U(\Phi,\zeta)$ or $y\notin U(\Phi,\zeta)$, then $\Phi^*(x)=\overline{0}$ or $\Phi^*(y)=\overline{0}$. Hence $\Phi^*(x*y)\geq\inf\{\Phi^*(x),\Phi^*(y)\}$. Similarly, we can prove $\Upsilon^*(x*y)\geq\inf\{\Upsilon^*(x),\Upsilon^*(y)\}$ for any $x,y\in X$. Now if $x,y\in L(\Psi,\xi)$, then $x*y\in L(\Psi,\xi)$. Then $\Psi^*(x*y)=\overline{0}\leq\sup\{\Psi(x),\Psi(y)\}=\sup\{\Psi^*(x),\Psi^*(y)\}$. If $x\notin L(\Psi,\xi)$ or $y\notin L(\Psi,\xi)$, then $\Psi^*(x)=\overline{1}$ or $\Psi^*(y)=\overline{1}$. Hence $\Psi^*(x*y)\leq\overline{1}=\sup\{\Psi^*(x),\Psi^*(y)\}$. Therefore, $\mathcal{P}^*=(\Phi^*,\Upsilon^*,\Psi^*)$ is an m-polar neutrosophic UP-subalgebra of X. □

Theorem 3.27. *If $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ is an m-polar neutrosophic UP-ideal of X, then so is $\mathcal{P}^*=(\Phi^*,\Upsilon^*,\Psi^*)$.*

Proof. If $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ is an m-polar neutrosophic UP-ideal of X, then the nonempty m-polar level sets $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ of $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ are UP-ideals of X for all $\zeta,\omega,\xi\in[0,1]^m$. Obviously, $0\in U(\Phi,\zeta)\cap U(\Upsilon,\omega)\cap L(\Psi,\xi)$. Let $x,y,z\in X$. If $x*(y*z),y\in U(\Phi,\zeta)$, then $x*y\in U(\Phi,\zeta)$. Then $\Phi^*(x*y)=\overline{1}\geq\inf\{\Phi(x*(y*z)),\Phi(y)\}=\inf\{\Phi^*(x*(y*z)),\Phi^*(y)\}$. If $x*(y*z)\notin U(\Phi,\zeta)$ or $y\notin U(\Phi,\zeta)$, then $\Phi^*(x*(y*z))=\overline{0}$ or $\Phi^*(y)=\overline{0}$. Hence $\Phi^*(x*y)\geq\inf\{\Phi^*(x*(y*z)),\Phi^*(y)\}$. Similarly, we can prove $\Upsilon^*(x*y)\geq\inf\{\Upsilon^*(x*(y*z)),\Upsilon^*(y)\}$ for any $x,y,z\in X$. Now if $x*(y*z),y\in L(\Psi,\xi)$, then $x*y\in L(\Psi,\xi)$. Then $\Psi^*(x*y)=\overline{0}\leq\sup\{\Psi(x*(y*z)),\Psi(y)\}=\sup\{\Psi^*(x*(y*z)),\Psi^*(y)\}$. If $x*(y*z)\notin L(\Psi,\xi)$ or $y\notin L(\Psi,\xi)$, then $\Psi^*(x*(y*z))=\overline{1}$ or $\Psi^*(y)=\overline{1}$. Hence $\Psi^*(x*y)\leq\overline{1}=\sup\{\Psi^*(x*(y*z)),\Psi^*(y)\}$. Therefore, $\mathcal{P}^*=(\Phi^*,\Upsilon^*,\Psi^*)$ is an m-polar neutrosophic UP-ideal of X. □

Theorem 3.28. *If $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ is an m-polar neutrosophic strong UP-ideal of X, then so is $\mathcal{P}^*=(\Phi^*,\Upsilon^*,\Psi^*)$.*

Proof. If $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ is an m-polar neutrosophic strong UP-ideal of X, then the nonempty m-polar level sets $U(\Phi,\zeta)$, $U(\Upsilon,\omega)$ and $L(\Psi,\xi)$ of $\mathcal{P}=(\Phi,\Upsilon,\Psi)$ are strong UP-ideals of X for all $\zeta,\omega,\xi\in[0,1]^m$. Obviously, $0\in U(\Phi,\zeta)\cap U(\Upsilon,\omega)\cap L(\Psi,\xi)$. Let $x,y,z\in X$. If $(z*y)*(z*x),y\in U(\Phi,\zeta)$, then $x\in U(\Phi,\zeta)$. Then $\Phi^*(x)=\overline{1}\geq\inf\{\Phi((z*y)*(z*x)),\Phi(y)\}=\inf\{\Phi^*((z*y)*(z*x)),\Phi^*(y)\}$. If $(z*y)*(z*x)\notin U(\Phi,\zeta)$ or $y\notin U(\Phi,\zeta)$, then $\Phi^*((z*y)*(z*x))=\overline{0}$ or $\Phi^*(y)=\overline{0}$. Hence $\Phi^*(x)\geq\inf\{\Phi^*((z*y)*(z*x)),\Phi^*(y)\}$. Similarly, we can prove $\Upsilon^*(x)\geq\inf\{\Upsilon^*((z*y)*(z*x)),\Upsilon^*(y)\}$ for any $x,y,z\in X$. Now if $(z*y)*(z*x),y\in L(\Psi,\xi)$, then $x\in L(\Psi,\xi)$. Then $\Psi^*(x)=\overline{0}\leq\sup\{\Psi((z*y)*(z*x)),\Psi(y)\}=\sup\{\Psi^*((z*y)*(z*x)),\Psi^*(y)\}$. If $(z*y)*(z*x)\notin L(\Psi,\xi)$ or $y\notin L(\Psi,\xi)$, then $\Psi^*((z*y)*(z*x))=\overline{1}$ or $\Psi^*(y)=\overline{1}$. Hence $\Psi^*(x)\leq\overline{1}=\sup\{\Psi^*((z*y)*(z*x)),\Psi^*(y)\}$. Therefore, $\mathcal{P}^*=(\Phi^*,\Upsilon^*,\Psi^*)$ is an m-polar neutrosophic strong UP-ideal of X. □

Proposition 3.29. *If* $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ *is an* m*-polar neutrosophic UP-subalgebra of* X*, then* $X_{(\Phi,\Upsilon,\Psi)} = \{x \in X : \Phi(x) = \Phi(0), \Upsilon(x) = \Upsilon(0), \Psi(x) = \Psi(0)\}$ *is a UP-subalgebra of* X*.*

Proof. Let $x, y \in X$ be such that $x, y \in X_{(\Phi,\Upsilon,\Psi)}$. Then $(\pi_i \circ \varphi)(x * y) = (\pi_i \circ \varphi)(0)$ and $(\pi_i \circ \psi)(x * y) = (\pi_i \circ \psi)(0)$ for all $i = 1, 2, 3, \cdots, m$. Since $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is an m-polar neutrosophic UP-subalgebra of X, $(\pi_i \circ \varphi)(x * y) \geq \inf\{(\pi_i \circ \varphi)(x), (\pi_i \circ \varphi)(y)\} = (\pi_i \circ \varphi)(0)$ and $(\pi_i \circ \psi)(x * y) \leq \sup\{(\pi_i \circ \psi)(x), (\pi_i \circ \psi)(y)\} = (\pi_i \circ \psi)(0)$ for all $i = 1, 2, 3, \cdots, m$. Hence $x * y \in X_{(\Phi,\Upsilon,\Psi)}$. Therefore, $X_{(\Phi,\Upsilon,\Psi)}$ is a UP-subalgebra of X. ☐

Proposition 3.30. *If* $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ *is an* m*-polar neutrosophic UP-ideal of* X*, then* $X_{(\Phi,\Upsilon,\Psi)} = \{x \in X : \Phi(x) = \Phi(0), \Upsilon(x) = \Upsilon(0), \Psi(x) = \Psi(0)\}$ *is a UP-ideal of* X*.*

Proof. Clearly, $0 \in X_{(\Phi,\Upsilon,\Psi)}$. Let $x, y, z \in X$ such that $x * (y * z) \in X_{(\Phi,\Upsilon,\Psi)}$ and $y \in X_{(\Phi,\Upsilon,\Psi)}$. Then $(\pi_i \circ \varphi)(x * (y * z)) = (\pi_i \circ \varphi)(0)$ and $(\pi_i \circ \varphi)(y) = (\pi_i \circ \varphi)(0)$. Since $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a neutrosophic UP-ideal of X, by (19), $(\pi_i \circ \varphi)(x * y) \geq \inf\{(\pi_i \circ \varphi)(x * (y * z)), (\pi_i \circ \varphi)(y)\} = (\pi_i \circ \varphi)(0)$, whence $(\pi_i \circ \varphi)(x * y) = (\pi_i \circ \varphi)(0)$, by (18). Then $x * y \in X_{(\Phi,\Upsilon,\Psi)}$. Also $(\pi_i \circ \gamma)(x * (y * z)) = (\pi_i \circ \gamma)(0)$ and $(\pi_i \circ \gamma)(y) = (\pi_i \circ \gamma)(0)$. Since $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a neutrosophic UP-ideal of X, by (19), $(\pi_i \circ \gamma)(x * y) \geq \inf\{(\pi_i \circ \gamma)(x * (y * z)), (\pi_i \circ \gamma)(y)\} = (\pi_i \circ \gamma)(0)$, whence $(\pi_i \circ \gamma)(x * y) = (\pi_i \circ \gamma)(0)$, by (18). Then $x * y \in X_{(\Phi,\Upsilon,\Psi)}$. Finally, $(\pi_i \circ \psi)(x * (y * z)) = (\pi_i \circ \psi)(0)$ and $(\pi_i \circ \psi)(y) = (\pi_i \circ \psi)(0)$. Since $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a neutrosophic UP-ideal of X, by (19), $(\pi_i \circ \psi)(x * y) \leq \sup\{(\pi_i \circ \psi)(x * (y * z)), (\pi_i \circ \psi)(y)\} = (\pi_i \circ \psi)(0)$, whence $(\pi_i \circ \psi)(x * y) = (\pi_i \circ \psi)(0)$, by (18). Then $x * y \in X_{(\Phi,\Upsilon,\Psi)}$. Hence $X_{(\Phi,\Upsilon,\Psi)}$ is a UP-ideal of X. ☐

Proposition 3.31. *If* $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ *is an* m*-polar neutrosophic strong UP-ideal of* X*, then* $X_{(\Phi,\Upsilon,\Psi)} = \{x \in X : \Phi(x) = \Phi(0), \Upsilon(x) = \Upsilon(0), \Psi(x) = \Psi(0)\}$ *is a strong UP-ideal of* X*.*

Proof. Clearly, $0 \in X_{(\Phi,\Upsilon,\Psi)}$. Let $x, y, z \in X$ such that $(z * y) * (z * x) \in X_{(\Phi,\Upsilon,\Psi)}$ and $y \in X_{(\Phi,\Upsilon,\Psi)}$. Then $(\pi_i \circ \varphi)((z * y) * (z * x)) = (\pi_i \circ \varphi)(0)$ and $(\pi_i \circ \varphi)(y) = (\pi_i \circ \varphi)(0)$. Since $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a neutrosophic strong UP-ideal of X, by (20), $(\pi_i \circ \varphi)(x) \geq \inf\{(\pi_i \circ \varphi)((z * y) * (z * x)), (\pi_i \circ \varphi)(y)\} = (\pi_i \circ \varphi)(0)$, whence $(\pi_i \circ \varphi)(x) = (\pi_i \circ \varphi)(0)$, by (18). Then $x \in X_{(\Phi,\Upsilon,\Psi)}$. Also $(\pi_i \circ \gamma)((z * y) * (z * x)) = (\pi_i \circ \gamma)(0)$ and $(\pi_i \circ \gamma)(y) = (\pi_i \circ \gamma)(0)$. Since $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a neutrosophic strong UP-ideal of X, by (20), $(\pi_i \circ \gamma)(x) \geq \inf\{(\pi_i \circ \gamma)((z * y) * (z * x)), (\pi_i \circ \gamma)(y)\} = (\pi_i \circ \gamma)(0)$, whence $(\pi_i \circ \gamma)(x) = (\pi_i \circ \gamma)(0)$, by (18). Then $x \in X_{(\Phi,\Upsilon,\Psi)}$. Finally, $(\pi_i \circ \psi)((z * y) * (z * x)) = (\pi_i \circ \psi)(0)$ and $(\pi_i \circ \psi)(y) = (\pi_i \circ \psi)(0)$. Since $\mathcal{P} = (\Phi, \Upsilon, \Psi)$ is a neutrosophic strong UP-ideal of X, by (20), $(\pi_i \circ \psi)(x) \leq \sup\{(\pi_i \circ \psi)((z * y) * (z * x)), (\pi_i \circ \psi)(y)\} = (\pi_i \circ \psi)(0)$, whence $(\pi_i \circ \psi)(x) = (\pi_i \circ \psi)(0)$, by (18). Then $x \in X_{(\Phi,\Upsilon,\Psi)}$. Hence $X_{(\Phi,\Upsilon,\Psi)}$ is a strong UP-ideal of X. ☐

Conclusion: This work contributes to the integration of algebraic structures with fuzzy set theory and extends the application of neutrosophic m-polar fuzzy sets to the realm of UP-algebras. This interdisciplinary approach offers a more comprehensive understanding of uncertainty and vague information in mathematical modelling and analysis.

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