Extending the concepts of complex interval valued neutrosophic subbisemiring of bisemiring

M.Palanikumar[1], **Nasreen Kausar**[2], **Emre Özbilge**[3], **Ebru Ozbilge**[4,*]

[1]Department of Mathematics, SRM Valliammai Engineering College, Kattankulathur, 603203,Tamilnadu, India

[2]Department of Mathematics, Faculty of Arts and Science, Yildiz Technical University, Esenler, 34220, Istanbul, Turkey

[3]Department of Computer Engineering, Cyprus International University, Nicosia, 99258, North Cyprus, Turkey

[4]American University of the Middle East, College of Business Administration, 54200, Egaila, Kuwait.

E-mails:[1]palanimaths86@gmail.com,[2]kausar.nasreen57@gmail.com, [3]eozbilge@ciu.edu.tr, [4]ebru.kahveci@aum.edu.kw

[*]Corresponding author: Ebru Ozbilge.

Abstract

The objective of this paper is to investigate the innovative concept of complex neutrosophic subbisemiring. The novelty of the complex neutrosophic subbisemiring lies in its wide range of truth, indeterminacy, and false function values. It goes beyond the range of $[0, 1]$ in the complex plane in contrast to the traditional range $[0, 1]$. Therefore, these three functions can be described mathematically using a complex number in the complex neutrosophic subbisemiring. We develop and analyze the concept of complex interval-valued neutrosophic subbisemiring (CIVNSBS). Moreover, we study homomorphic characteristics and important properties of CIVNSBS. We propose the level sets of CIVNSBS and complex interval valued neutrosophic normal subbisemiring (CIVNNSBS) of bisemirings. Moreover, we introduce (ϱ, σ) CIVNNSBS of bisemiring. Let $\widehat{\Upsilon}$ be a complex neutrosophic subset of bisemiring $\mathcal{S}$. Then $\widehat{R} = (\widehat{R^{\top}_{\Upsilon}}, \widehat{R^{\mathcal{I}}_{\Upsilon}}, \widehat{R^{F}_{\Upsilon}})$ is a CIVNSBS of $\mathcal{S}$ if and only if all non empty level set $\widehat{R^{(\alpha,\beta)}}$ is a subbisemiring, where $\alpha, \beta \in \mathbb{D}[0, 1]$. Let $\widehat{\Upsilon}$ be a CIVNSBS of bisemiring $\mathcal{S}$ and $\widehat{\mathcal{V}}$ be the strongest complex neutrosophic relation of bisemiring $\mathcal{S}$. Then $\widehat{\Upsilon}$ is a CIVNSBS of bisemiring $\mathcal{S}$ if and only if $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S} \times \mathcal{S}$. We illustrate that homomorphic images of every CIVNSBS is a CIVNSBS and homomorphic pre-images of every CIVNSBS is a CIVNSBS. Examples are provided to illustrate our results.

Keywords: CIVNSBS, CIVNNSBS, SBS, homomorphism.

1 Introduction

The algebraic structures play an important role in mathematics, with wide ranging applications in theoretical physics, computer science, control engineering, information science, coding theory, topological spaces, etc. The study of abstract algebra within the framework of fuzzy system provides sufficient motivation for researchers to examine various concepts and results from the field of abstract algebra. Classical mathematics may not always be the solution for practical situations in economics, medical sciences, engineering, social

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sciences, and environmental sciences, which involves various uncertainties, imprecise and incomplete information. The rough set theory and fuzzy set theory each approach the issue of vague, imprecise, inconsistent, and uncertain knowledge from different perspectives. In fuzzy set theory, the object can have a given property to some extent if it has one of the basic principles. Semiring was investigated by Vandever. Fundamentally, it generalized distributive lattices and rings. There have been a number of generalizations and modifications of fuzzy sets theory (FS)[1] is one of the most important works. FSs are described by membership grades (MG), which define them as functions. Real unit intervals are used to measure degrees.

The limitation of classical mathematics that is unable to deal with uncertainties and fuzziness motivated the introduction of mathematical theory such as probability theory, fuzzy set theory,[1] rough set theory,[2] vague set theory,[3] interval mathematics,[4] and soft set theory.[5] However, these theories were insufficient and have limitations in dealing with uncertainties. Probability theory can only deal with stochastically stable problems, which may not apply to many problems in the field of economic, environmental, and social sciences. Interval mathematics takes calculation errors into account by constructing an interval estimate for the solution that is useful in many areas, but it is not appropriately adaptable for problems that arise from unreliable, inadequate, and change of information. On the other hand, the fuzzy set theory introduced by Zadeh[1] is most appropriate for dealing with uncertainties and vagueness. Membership of an element in a fuzzy set is a single value between the interval, but in real-life problems, the degree of non-membership may not always be equal to 1 minus the degree of membership as there may be some degree of hesitation. Works on fuzzy set theory are progressing rapidly and have resulted in the conception of many hybrid fuzzy models. The uncertainties have led to a variety of uncertain theories, FS,[1] intuitionistic FS (IFS),[6] Pythagorean FS (PFS)[7] and spherical FS (SFS).[8] An FS consists of sets with grades between 0 and 1; these grades are called MG. Atanassov[6] discussed that IFS is classified as MG and the value of non-membership grades (NMG) is not greater than 1. When using a decision-making approach, the sum of MGs and NMGs can sometimes exceed 1. Using PFS logic, Yager[7] developed a generalized MG and NMG logic, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs. These hypotheses cannot demonstrate the neutral state (neither favor nor disfavor). Cuong et al.[9] discussed the picture FS used three grade points: positive, neutral, and negative, with the sum of these three grades not exceeding 1. Furthermore, some applications benefit from it more than IFS or PFS. It is a generalization of FS and IFS involving three independent models: truth, indeterminacy, and falsity.

Because this set of information contains many application-related challenges, Smarandache[10] developed neutrosophy to handle unclear and inconsistent information. Recently, there has been a new theory related to sets and logic called neutrosophic logic. The major difference between FS and IFS comes down to neutral cognition, which is the subject matter of neurosophy. Smarandache introduced neutrosophic logic.[10] This logic determines the extent to which an assertion is true, indeterminate, or false. The NSS set of properties is divided into truth, indeterminacy, and falsity components that range from $[0,1]$. According to philosophers, a neutrosophic set is a generalization of a classical set, FS, interval-valued FS, etc. Complex fuzzy set is introduced by.[11] The membership functions of complex fuzzy sets can take on a broad range of values. As opposed to a fuzzy membership function with a fixed unit circle, the complex plane's unit circle extends to $[0,1]$. The complex fuzzy set X is characterized by a membership function $\mu_X(x)$ whose range is not limited to $[0,1]$ but extended to the unit circle in the complex plane. Hence, $\mu_X(x)$ is a complex-valued function that assigns a grade of membership of the form $\xi_X(x)\cdot e^{i\tau_X(x)}$, where $i=\sqrt{-1}$ to any element x in the universe of discourse. The value of $\mu_X(x)$ is defined by the two variables such as $\xi_X(x)$ and $\tau_X(x)$ and both real-valued with $\xi_X(x)\in[0,1]$. The concept of fuzzy membership is modified by complex fuzzy set theory by asserting that, at least in some cases, adding a second dimension to membership expressions is necessary. The basic concept of fuzziness is not altered by the addition of this dimension. Membership in a complex fuzzy set remains "as fuzzy" as membership in a traditional fuzzy set. The fuzziness of membership, i.e., the representation of membership as a value in the range $[0,1]$ is retained in complex fuzzy sets through the amplitude of the grade of membership $\xi_X(x)$. Complex fuzzy sets are distinguished by the additional dimension of membership: $\tau_X(x)$ which comes with the grade of membership. Golan[12] introduced the logic of semirings and its applications. Hussian et al.[13] discussed the concept of bisemirings and its extension. Lee[14] deals with the bipolar-valued fuzzy sets and their operations. Fuzzy semirings were discussed by Ahsan et al.[?,15] Sen et al.[17] introduced the concept of bisemirings. Recently, Palanikumar et al.[18] discussed the notion of intuitionistic fuzzy normal subbisemiring of bisemiring. Palanikumar et al.[19] introduced the concept of bisemiring concept via the bipolar-valued neutrosophic normal sets. In this paper, the following contributions are made:

1. We define complex interval valued neutrosophic subbisemiring.
2. We define of level set based on CIVNSBS.

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3. The intersection of a every CIVNSBSs is again a CIVNSBS of bisemiring $\mathcal{S}$.

4. Let $\widehat{\Upsilon}$ be a CIVNSBS of $\mathcal{S}$ and $\widehat{\mathcal{V}}$ be a strongest complex interval valued neutrosophic relation of $\mathcal{S}$. Then $\widehat{\Upsilon}$ is a CIVNSBS of bisemiring $\mathcal{S}$ if and only if $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S} \times \mathcal{S}$.

5. $\widehat{R} = (\widehat{R^{\top}_{\Upsilon}} \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}, \widehat{R^{\mathcal{I}}_{\Upsilon}} \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}, \widehat{R^{F}_{\Upsilon}} \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}})$ is a CIVNSBS of $\mathcal{S}$ if and only if $\widehat{R^{(\alpha,\beta)}}$ is a subbisemiring of $\mathcal{S}$ for all $\alpha, \beta \in \mathbb{D}[0,1]$.

6. The homomorphic image of every CIVNSBS is a CIVNSBS and homomorphic preimage of every CIVNSBS is a CIVNSBS.

Various aspects of the SBS and CIVNSBS idea will be looked at and findings will be offered. The article consists of the following five components. In Section **??**, you will find an introduction to semirings and SBS. In Section 2, you will find information on the preparation of semirings and SBS. CIVNSBS presents its attributes in Section 3. Section 4 deals with the (ϱ, σ)-CIVNNSBS homomorphism is discussed. CIVNSBS and CIVNNSBS should be evaluated using some numerical examples. Section 5 indicates conclusion and future direction.

2 Preliminaries

The purpose of this section is to contain as much information about semirings and bisemirings.

Definition 2.1. Let $\mathcal{S}$ be the non-empty set with two binary operations " $+$ " and " $\cdot$ " is said to be a semiring, if it satisfies the following conditions:

1. $(\mathcal{S}, +)$ and $(\mathcal{S}, \cdot)$ are semigroups

2. $v_v \cdot (v_\eta + v_\xi) = (v_v \cdot v_\eta) + (v_v \cdot v_\xi)$ and $(v_v + v_\eta) \cdot v_\xi = (v_v \cdot v_\xi) + (v_\eta \cdot v_\xi)$, for all $v_v, v_\eta, v_\xi \in \mathcal{S}$.

Definition 2.2. [17] An algebraic structure $(\mathcal{S}, \oplus, \ominus, \otimes)$ is a bisemiring, if $(\mathcal{S}, \oplus, \ominus)$ and $(\mathcal{S}, \ominus, \otimes)$ are semirings, ie., $(\mathcal{S}, \oplus), (\mathcal{S}, \ominus)$ and $(\mathcal{S}, \otimes)$ are semigroups and

1. $z_v \ominus (z_\eta \oplus z_\xi) = (z_v \ominus z_\eta) \oplus (z_v \ominus z_\xi)$,

2. $(z_\eta \oplus z_\xi) \ominus z_v = (z_\eta \ominus z_v) \oplus (z_\xi \ominus z_v)$,

3. $z_v \otimes (z_\eta \ominus z_\xi) = (z_v \otimes z_\eta) \ominus (z_v \otimes z_\xi)$,

4. $(z_\eta \ominus z_\xi) \otimes z_v = (z_\eta \otimes z_v) \ominus (z_\xi \otimes z_v), \ \forall \ z_v, z_\eta, z_\xi \in \mathcal{S}$.

Definition 2.3. [13,17] A non-empty subset Υ of a bisemiring $(\mathcal{S}, \oplus, \ominus, \otimes)$ is a subbisemiring if $z_v \oplus z_\eta \in \Upsilon$, $z_v \ominus z_\eta \in \Upsilon$ and $z_v \otimes z_\eta \in \Upsilon$ for all $z_v, z_\eta \in \Upsilon$.

Definition 2.4. [10] A neutrosophic set v in the universe $\mathcal{U}$ is $v = \{u, u^{\top}_v(u), u^{\mathcal{I}}_v(u), u^{F}_v(u) | u \in \mathcal{U}\}$, where $u^{\top}_v(u)$, $u^{\mathcal{I}}_v(u)$ $u^{F}_v(u)$ represents the TD, ID and FD of v respectively. Consider the mapping $u^{\top}_v : \mathcal{U} \to [0,1], u^{\mathcal{I}}_v : \mathcal{U} \to [0,1], u^{F}_v : \mathcal{U} \to [0,1]$ and $0 \leqslant u^{\top}_v(u) + u^{\mathcal{I}}_v(u) + u^{F}_v(u) \leqslant 3$.

Definition 2.5. [10] Let $v_1 = \langle u^{\top}_{v_1}, u^{\mathcal{I}}_{v_1}, u^{F}_{v_1} \rangle$, $v_2 = \langle u^{\top}_{v_2}, u^{\mathcal{I}}_{v_2}, u^{F}_{v_2} \rangle$ and $v_3 = \langle u^{\top}_{v_3}, u^{\mathcal{I}}_{v_3}, u^{F}_{v_3} \rangle$ be the three neutrosophic numbers over $\mathcal{U}$. Then

1. ${v_1}^c = \langle u^{F}_{v_1}, u^{\mathcal{I}}_{v_1}, u^{\top}_{v_1} \rangle$,

2. $v_2 \Cup v_3 = \Big\langle \max(u^{\top}_{v_2}, u^{\top}_{v_3}), \min(u^{\mathcal{I}}_{v_2}, u^{\mathcal{I}}_{v_3}), \min(u^{F}_{v_2}, u^{F}_{v_3}) \Big\rangle$,

3. $v_2 \Cap v_3 = \Big\langle \min(u^{\top}_{v_2}, u^{\top}_{v_3}), \max(u^{\mathcal{I}}_{v_2}, u^{\mathcal{I}}_{v_3}), \max(u^{F}_{v_2}, u^{F}_{v_3}) \Big\rangle$,

4. $v_2 \geqslant v_3$ iff $u^{\top}_{v_2} \geqslant u^{\top}_{v_3}$ and $u^{\mathcal{I}}_{v_2} \leqslant u^{\mathcal{I}}_{v_3}$ and $u^{F}_{v_2} \leqslant u^{F}_{v_3}$,

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5. $v_2 = v_3$ iff $u_{v_2}^{\top} = u_{v_3}^{\top}$ and $u_{v_2}^{\mathcal{I}} = u_{v_3}^{\mathcal{I}}$ and $u_{v_2}^{F} = u_{v_3}^{F}$.

Definition 2.6. [10] For any neutrosophic set $v = \{u, u_v^{\top}(u), u_v^{\mathcal{I}}(u), u_v^{F}(u)\}$ of $\mathcal{U}$. Then (ϱ, σ)-cut is defined as $\{u \in U | u_v^{\top}(u) \geqslant \varrho, u_v^{\mathcal{I}}(u) \geqslant \varrho, u_v^{F}(u) \leqslant \sigma\}$.

Definition 2.7. [10] Let V and Y be two neutrosophic sets of $\mathcal{S}$. Then Cartesian product of V and Y is defined as $V \times Y = \{u_{V\times Y}^{\mathcal{T}}(u,v), u_{V\times Y}^{\mathcal{I}}(u,v), u_{V\times Y}^{F}(u,v) |\ for\ all\ u, v \in \mathcal{S}\}$, where $u_{V\times Y}^{\mathcal{T}}(u,v) = \min\{u_V^{\mathcal{T}}(u), u_Y^{\mathcal{T}}(v)\}, u_{V\times Y}^{\mathcal{I}}(u,v) = \frac{u_V^{\mathcal{I}}(u)+u_Y^{\mathcal{I}}(v)}{2}, u_{V\times Y}^{F}(u,v) = \max\{u_V^{F}(u), u_Y^{F}(v)\}$.

Definition 2.8. A fuzzy subset v of a bisemiring $(\mathcal{S}, \dagger_1, \dagger_2, \dagger_3)$ is represents a fuzzy subbisemiring of $\mathcal{S}$ if $u_v(u \dagger_1 \varepsilon) \geqslant \min\{u_v(u), u_v(\varepsilon)\}, u_v(u \dagger_2 \varepsilon) \geqslant \min\{u_v(u), u_v(\varepsilon)\}, u_v(u \dagger_3 \varepsilon) \geqslant \min\{u_v(u), u_v(\varepsilon)\}$, for all $u, \varepsilon \in \mathcal{S}$.

Definition 2.9. A fuzzy subset v of $(\mathcal{S}, \dagger_1, \dagger_2, \dagger_3)$ is represents a normal subbisemiring if $u_v(u \dagger_1 \varepsilon) = u_v(\varepsilon \dagger_1 u), u_v(u \dagger_2 \varepsilon) = u_v(\varepsilon \dagger_2 u), u_v(u \dagger_3 \varepsilon) = u_v(\varepsilon \dagger_3 u), \forall\, u, \varepsilon \in \mathcal{S}$.

Definition 2.10. [13] Let $(T, +, \cdot, \times)$ and $(S, \varnothing, \circ, \otimes)$ be the bisemirings. Then $\varrho : \mathcal{S} \to \mathcal{S}$ is said to be homomorphism if it satisfies the following conditions:

1. $\varrho(u + \varepsilon) = \varrho(u) \varnothing \varrho(\varepsilon)$,
2. $\varrho(u \cdot \varepsilon) = \varrho(u) \circ \varrho(\varepsilon)$,
3. $\varrho(u \times \varepsilon) = \varrho(u) \otimes \varrho(\varepsilon), \forall\, u, \varepsilon \in S$.

3 Complex interval valued neutrosophic subbisemiring

Here $\mathcal{S}$ denotes bisemiring unless other stated, R stands for real part and Θ stands for imaginary part.

Definition 3.1. The complex interval valued neutrosophic set (CIVNS) $\widehat{\Upsilon}$ in universal set O,
$\widehat{\Upsilon} = \left\{ \left\langle u, \widehat{R_{\Upsilon}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\top}(u)}}, \widehat{R_{\Upsilon}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\mathcal{I}(u)}}}, \widehat{R_{\Upsilon}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{F(u)}}} \right\rangle : u \in O \right\}$,
where $\widehat{R_{\Upsilon}^{\top}}(u) = [R_{\Upsilon}^{\top L}, R_{\Upsilon}^{\top U}], \widehat{R_{\Upsilon}^{\mathcal{I}}}(u) = [R_{\Upsilon}^{\mathcal{I}L}, R_{\Upsilon}^{\mathcal{I}U}], \widehat{R_{\Upsilon}^{F}}(u) = [R_{\Upsilon}^{FL}, R_{\Upsilon}^{FU}]$ and $\widehat{R_{\Upsilon}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\top(u)}}}, \widehat{R_{\Upsilon}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\mathcal{I}(u)}}}, \widehat{R_{\Upsilon}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{F(u)}}} : O \to D[0,1]$ represents the truth degree, indeterminacy degree and false degree respectively. For simplicity the symbol $\left\langle \widehat{R_{\Upsilon}^{\top}}, \widehat{R_{\Upsilon}^{\mathcal{I}}}, \widehat{R_{\Upsilon}^{F}} \right\rangle$ is CIVNS
$\widehat{\Upsilon} = \left\{ \left\langle u, \widehat{R_{\Upsilon}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\top(u)}}}, \widehat{R_{\Upsilon}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\mathcal{I}(u)}}}, \widehat{R_{\Upsilon}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{F(u)}}} \right\rangle : u \in O \right\}$.

Definition 3.2. Let $\widehat{\Upsilon} = \left\{ u, \widehat{R_{\Upsilon}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\top(u)}}}, \widehat{R_{\Upsilon}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\mathcal{I}(u)}}}, \widehat{R_{\Upsilon}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{F(u)}}} \right\}$ and $\widehat{\Omega} = \left\{ u, \widehat{R_{\Omega}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{\top(u)}}}, \widehat{R_{\Omega}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{\mathcal{I}(u)}}}, \widehat{R_{\Omega}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{F(u)}}} \right\}$ be two CIVNSs of O. Then we define the intersection and union operation is defined as
(i) $\widehat{\Upsilon} \cap \widehat{\Omega} = \left\{ \left(u, \min\{\widehat{R_{\Upsilon}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\top(u)}}}, \widehat{R_{\Omega}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{\top(u)}}}\}, \min\{\widehat{R_{\Upsilon}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\mathcal{I}(u)}}}, \widehat{R_{\Omega}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{\mathcal{I}(u)}}}\}, \max\{\widehat{R_{\Upsilon}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{F(u)}}}, \widehat{R_{\Omega}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{F(u)}}}\} \right) \Big| u \in O \right\}$.
(ii) $\widehat{\Upsilon} \cup \widehat{\Omega} = \left\{ \left(u, \max\{\widehat{R_{\Upsilon}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\top(u)}}}, \widehat{R_{\Omega}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{\top(u)}}}\}, \max\{\widehat{R_{\Upsilon}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\mathcal{I}(u)}}}, \widehat{R_{\Omega}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{\mathcal{I}(u)}}}\}, \min\{\widehat{R_{\Upsilon}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{F(u)}}}, \widehat{R_{\Omega}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Omega}^{F(u)}}}\} \right) \Big| u \in O \right\}$.

Definition 3.3. For any CIVNS $\widehat{\Upsilon} = \left\{ u, \widehat{R_{\Upsilon}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\top(u)}}}, \widehat{R_{\Upsilon}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\mathcal{I}(u)}}}, \widehat{R_{\Upsilon}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{F(u)}}} \right\}$ of a universal set O. Then (α, β)-cut is defined as $\left\{ u \in O | \widehat{R_{\Upsilon}^{\top}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\top(u)}}} \geqslant \alpha, \widehat{R_{\Upsilon}^{\mathcal{I}}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{\mathcal{I}(u)}}} \geqslant \alpha, \widehat{R_{\Upsilon}^{F}}(u) \cdot e^{i2\pi \widehat{\Theta_{\Upsilon}^{F(u)}}} \leqslant \beta \right\}$.

Definition 3.4. The Cartesian product of $\widehat{\Upsilon}$ and $\widehat{\Omega}$ is defined as $\widehat{\Upsilon \times \Omega} = \Big\{ \widehat{R^{\top}_{\Upsilon\times\Omega}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\top(u,v)}_{\Upsilon\times\Omega}}}, \widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}(u,v)}_{\Upsilon\times\Omega}}}, \widehat{R^{F}_{\Upsilon\times\Omega}}(u\cdot e^{i2\pi\widehat{\Theta^{F(u,v)}_{\Upsilon\times\Omega}}}, v) | \; for\ all\ u,v \in S \Big\}$, where $\widehat{\Upsilon}$ and $\widehat{\Omega}$ be the CIVNS of O, where

$$\left\{\begin{array}{l} \widehat{R^{\top}_{\Upsilon\times\Omega}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\top(u,v)}_{\Upsilon\times\Omega}}} = \min\Big\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top(u)}_{\Upsilon}}}, \widehat{R^{\top}_{\Omega}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top(v)}_{\Omega}}}\Big\} \\ \widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}(u,v)}_{\Upsilon\times\Omega}}} = \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}(u)}_{\Upsilon}}} + \widehat{R^{\mathcal{I}}_{\Omega}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}(v)}_{\Omega}}}}{2} \\ \widehat{R^{F}_{\Upsilon\times\Omega}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{F(u,v)}_{\Upsilon\times\Omega}}} = \max\Big\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F(u)}_{\Upsilon}}}, \widehat{R^{F}_{\Omega}}(v)\cdot e^{i2\pi\widehat{\Theta^{F(v)}_{\Omega}}}\Big\} \end{array}\right\}$$

Definition 3.5. For any complex fuzzy set Υ of $(\mathcal{S}, \triangledown_1, \triangledown_2, \triangledown_3)$ is said to be a CNSBS of $\mathcal{S}$ if it satisfies the following conditions:

$$\left\{\begin{array}{l} R_{\Upsilon}(u\triangledown_1 v)\cdot e^{i2\pi\Theta_{\Upsilon}(u\triangledown_1 v)} \geqslant \min\{R_{\Upsilon}(u)\cdot e^{i2\pi\Theta_{\Upsilon}(u)}, R_{\Upsilon}(v)\cdot e^{i2\pi\Theta_{\Upsilon}(v)}\} \\ R_{\Upsilon}(u\triangledown_2 v)\cdot e^{i2\pi\Theta_{\Upsilon}(u\triangledown_2 v)} \geqslant \min\{R_{\Upsilon}(u)\cdot e^{i2\pi\Theta_{\Upsilon}(u)}, R_{\Upsilon}(v)\cdot e^{i2\pi\Theta_{\Upsilon}(v)}\} \\ R_{\Upsilon}(u\triangledown_3 v)\cdot e^{i2\pi\Theta_{\Upsilon}(u\triangledown_3 v)} \geqslant \min\{R_{\Upsilon}(u)\cdot e^{i2\pi\Theta_{\Upsilon}(u)}, R_{\Upsilon}(v)\cdot e^{i2\pi\Theta_{\Upsilon}(v)}\} \end{array}\right\}$$

$\forall\, u,v \in \mathcal{S}$.

Definition 3.6. For any complex fuzzy set Υ of $(\mathcal{S}, \triangledown_1, \triangledown_2, \triangledown_3)$ is represents a CNNSBS of $\mathcal{S}$ if it satisfies the following conditions:

$$\left\{\begin{array}{l} R_{\Upsilon}(u\triangledown_1 v)\cdot e^{i2\pi\Theta_{\Upsilon}(u\triangledown_1 v)} = R_{\Upsilon}(v\triangledown_1 u)\cdot e^{i2\pi\Theta_{\Upsilon}(v\triangledown_1 u)} \\ R_{\Upsilon}(u\triangledown_2 v)\cdot e^{i2\pi\Theta_{\Upsilon}(u\triangledown_2 v)} = R_{\Upsilon}(v\triangledown_2 u)\cdot e^{i2\pi\Theta_{\Upsilon}(v\triangledown_2 u)} \\ R_{\Upsilon}(u\triangledown_3 v)\cdot e^{i2\pi\Theta_{\Upsilon}(u\triangledown_3 v)} = R_{\Upsilon}(v\triangledown_3 u)\cdot e^{i2\pi\Theta_{\Upsilon}(v\triangledown_3 u)} \end{array}\right\}$$

for all $u, v \in \mathcal{S}$.

Definition 3.7. For any CIVNS $\widehat{\Upsilon}$ of $\mathcal{S}$ is said to be a CIVNSBS of $\mathcal{S}$ if

$$\left\{\begin{array}{l} \widehat{R^{\top}_{\Upsilon}}(u\triangledown_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)}\} \\ \widehat{R^{\top}_{\Upsilon}}(u\triangledown_2 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)}\} \\ \widehat{R^{\top}_{\Upsilon}}(u\triangledown_3 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)}\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \widehat{R^{\mathcal{I}}_{\Upsilon}}(u\triangledown_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\triangledown_1 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\triangledown_1 v)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\triangledown_1 v)}}{2} \\ \qquad \text{OR} \\ \widehat{R^{\mathcal{I}}_{\Upsilon}}(u\triangledown_2 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\triangledown_1 v)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\triangledown_1 v)}}{2} \\ \qquad \text{OR} \\ \widehat{R^{\mathcal{I}}_{\Upsilon}}(u\triangledown_3 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\triangledown_1 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\triangledown_1 v)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\triangledown_1 v)}}{2} \end{array}\right\}$$

$$\left\{\begin{array}{l} \widehat{R^{F}_{\Upsilon}}(u\triangledown_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)}\} \\ \widehat{R^{F}_{\Upsilon}}(u\triangledown_2 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)}\} \\ \widehat{R^{F}_{\Upsilon}}(u\triangledown_3 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\triangledown_1 v)}\} \end{array}\right\}$$

for all $u, v \in \mathcal{S}$.

Example 3.8. Consider the bisemiring $\mathcal{S} = \{v, \nu, \eta, \xi\}$ with the Cayley table:

$\triangledown_1$	v	ν	η	ξ
v	v	v	v	v
ν	v	ν	v	ν
η	v	v	η	η
ξ	v	ν	η	ξ

$\triangledown_2$	v	ν	η	ξ
v	v	ν	η	ξ
ν	ν	ν	ξ	ξ
η	η	ξ	η	ξ
ξ	ξ	ξ	ξ	ξ

$\triangledown_3$	v	ν	η	ξ
v	v	v	v	v
ν	v	ν	η	ξ
η	ξ	ξ	ξ	ξ
ξ	ξ	ξ	ξ	ξ

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	$w=v$	$w=\nu$	$w=\eta$	$w=\xi$
$\widehat{R^{\top}_{\Upsilon}}(w)$	$[0.8e^{i2\pi(0.65)}, 0.9e^{i2\pi(0.75)}]$	$[0.7e^{i2\pi(0.55)}, 0.8e^{i2\pi(0.65)}]$	$[0.4e^{i2\pi(0.25)}, 0.5e^{i2\pi(0.35)}]$	$[0.6e^{i2\pi(0.45)}, 0.7e^{i2\pi(0.55)}]$
$\widehat{R^{\mathcal{I}}_{\Upsilon}}(w)$	$[0.5e^{i2\pi(0.35)}, 0.6e^{i2\pi(0.45)}]$	$[0.4e^{i2\pi(0.25)}, 0.5e^{i2\pi(0.35)}]$	$[0.2e^{i2\pi(0.15)}, 0.3e^{i2\pi(0.25)}]$	$[0.3e^{i2\pi(0.15)}, 0.4e^{i2\pi(0.25)}]$
$\widehat{R^{F}_{\Upsilon}}(w)$	$[0.6e^{i2\pi(0.45)}, 0.7e^{i2\pi(0.55)}]$	$[0.7e^{i2\pi(0.55)}, 0.8e^{i2\pi(0.6)}]$	$[0.90e^{i2\pi(0.75)}, 0.8e^{i2\pi(0.65)}]$	$[0.85e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.75)}]$

Hence, $\widehat{\Upsilon}$ is a CIVNSBS of $\mathcal{S}$.

Theorem 3.9. *The intersection of a every CIVNSBSs is again a CIVNSBS of $\mathcal{S}$.*

Proof. Let $\{\widehat{\mathcal{V}}_i : i \in I\}$ be the family of CIVNSBSs of $\mathcal{S}$ and $\widehat{\Upsilon} = \bigcap_{i\in I}\widehat{\mathcal{V}}_i$. Let $u, v \in \mathcal{S}$.

Now,

$$\begin{aligned}
\widehat{R^{\top}_{\Upsilon}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)} &= \inf_{i\in I}\ \widehat{R^{\top}_{\mathcal{V}_i}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}\\
&\geqslant \inf_{i\in I}\ \min\{\widehat{R^{\top}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}, \widehat{R^{\top}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}\}\\
&= \min\left\{\inf_{i\in I}\ \widehat{R^{\top}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}, \inf_{i\in I}\ \widehat{R^{\top}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}\right\}\\
&= \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}\}
\end{aligned}$$

Similarly,
$\widehat{R^{\top}_{\Upsilon}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}\}$,
$\widehat{R^{\top}_{\Upsilon}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\nabla_1 v)}\}$.
Now,

$$\begin{aligned}
\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)} &= \inf_{i\in I}\ \widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)}\\
&\geqslant \inf_{i\in I}\ \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)} + \widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)}}{2}\\
&= \frac{\inf_{i\in I}\ \widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)} + \inf_{i\in I}\ \widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)}}{2}\\
&= \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)}}{2}
\end{aligned}$$

Similarly,
$\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)}}{2}$ and
$\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)}}{2}$.
Now,

$$\begin{aligned}
\widehat{R^{F}_{\Upsilon}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)} &= \sup_{i\in I}\ \widehat{R^{F}_{\mathcal{V}_i}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}\\
&\leqslant \sup_{i\in I}\ \max\{\widehat{R^{F}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}, \widehat{R^{F}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}\}\\
&= \max\left\{\sup_{i\in I}\ \widehat{R^{F}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}, \sup_{i\in I}\ \widehat{R^{F}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}\right\}\\
&= \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}\}
\end{aligned}$$

Similarly,
$\widehat{R^{F}_{\Upsilon}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}\}$,and
$\widehat{R^{F}_{\Upsilon}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\nabla_1 v)}\}$.
Thus, $\widehat{\Upsilon}$ is a CIVNSBS of $\mathcal{S}$.

Since, the union of two CIVNSBSs is not a CIVNSBS of $\mathcal{S}$.

Theorem 3.10. *If $\widehat{\Upsilon}$ and $\widehat{\Omega}$ be the CIVNSBSs of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, then $\widehat{\Upsilon \times \Omega}$is a CIVNSBS of $\mathcal{S}_1 \times \mathcal{S}_2$.*

Proof. Let $u_1, u_2 \in \mathcal{S}_1$ and $v_1, v_2 \in \mathcal{S}_2$. Then (u_1, v_1) and (u_2, v_2) are in $\mathcal{S}_1 \times \mathcal{S}_2$. Now

$$\begin{aligned}
& \widehat{R^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)] \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)]} \\
= \; & \widehat{R^{\top}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2, v_1\nabla_1v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2, v_1\nabla_1v_2)} \\
= \; & \min\{\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1u_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1\nabla_1u_2)}, \widehat{R^{\top}_{\Omega}}(v_1\nabla_1v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Omega}}(v_1\nabla_1v_2)}\} \\
\geq \; & \min\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1}), \widehat{R^{\top}_{\Upsilon}}(u_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2)}\}, \min\{\widehat{R^{\top}_{\Omega}}(v_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_1)}, \widehat{R^{\top}_{\Omega}}(v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_2)}\}\} \\
= \; & \min\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1}), \widehat{R^{\top}_{\Omega}}(v_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_1)})\}, \min\{\widehat{R^{\top}_{\Upsilon}}(u_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2)}, \widehat{R^{\top}_{\Omega}}(v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_2)}\}\} \\
= \; & \min\{\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_1,v_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_1,v_1)}, \widehat{R^{\top}_{\Upsilon\times\Omega}}(u_2,v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_2,v_2)}\}
\end{aligned}$$

Also $\widehat{R^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)] \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)]}$
$\geqslant \min\{\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_1,v_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)]}, \widehat{R^{\top}_{\Upsilon\times\Omega}}(u_2,v_2)e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_2,v_2)]}\}$

and $\widehat{R^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)] \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)]}$
$\geqslant \min\{\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_1,v_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)]}, \widehat{R^{\top}_{\Upsilon\times\Omega}}(u_2,v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_2,v_2)]}\}$.
Now,

$$\begin{aligned}
& \widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)] \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)]} \\
= \; & \widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2, v_1\nabla_1v_2) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2, v_1\nabla_1v_2)} \\
= \; & \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1u_2) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2)} + \widehat{R^{\mathcal{I}}_{\Omega}}(v_1\nabla_1v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(v_1\nabla_1v_2)}}{2} \\
\geq \; & \frac{1}{2}\left[\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_2)}}{2} + \frac{\widehat{R^{\mathcal{I}}_{\Omega}}(v_1) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Omega}}(v_1)} + \widehat{R^{\mathcal{I}}_{\Omega}}(v_2) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Omega}}(v_2)}}{2}\right] \\
= \; & \frac{1}{2}\left[\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)} + \widehat{R^{\mathcal{I}}_{\Omega}}(v_1) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Omega}}(v_1)}}{2} + \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_2)} + \widehat{R^{\mathcal{I}}_{\Omega}}(v_2) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Omega}}(v_2)}}{2}\right] \\
= \; & \frac{1}{2}\left[\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1,v_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_1,v_1)} + \widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_2,v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_2,v_2)}\right]
\end{aligned}$$

Also
$\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)] \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)]} \geqslant \frac{1}{2}\Big[\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1,v_1) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1,v_1)} + \widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_2,v_2) \cdot$
$e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_2,v_2)}\Big]$
and
$\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)] \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)]} \geqslant \frac{1}{2}\Big[\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1,v_1) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1,v_1)} + \widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_2,v_2) \cdot$
$e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_2,v_2)}\Big]$.
Now,

$$\begin{aligned}
& \widehat{R^{F}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)] \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)]} \\
= \; & \widehat{R^{F}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2, v_1\nabla_1v_2) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2, v_1\nabla_1v_2)} \\
= \; & \max\{\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1u_2) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2)}, \widehat{R^{F}_{\Omega}}(v_1\nabla_1v_2) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon\times\Omega}}(v_1\nabla_1v_2)}\} \\
\leq \; & \max\{\max\{\widehat{R^{F}_{\Upsilon}}(u_1) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)}, \widehat{R^{F}_{\Upsilon}}(u_2) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_2)}\}, \max\{\widehat{R^{F}_{\Omega}}(v_1) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Omega}}(v_1)}, \widehat{R^{F}_{\Omega}}(v_2) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Omega}}(v_2)}\}\} \\
= \; & \max\{\max\{\widehat{R^{F}_{\Upsilon}}(u_1) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)}, \widehat{R^{F}_{\Omega}}(v_1) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Omega}}(v_1)}\}, \max\{\widehat{R^{F}_{\Upsilon}}(u_2) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_2)}, \widehat{R^{F}_{\Omega}}(v_2) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Omega}}(v_2)}\}\} \\
= \; & \max\{\widehat{R^{F}_{\Upsilon\times\Omega}}(u_1,v_1) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_1,v_1)}, \widehat{R^{F}_{\Upsilon\times\Omega}}(u_2,v_2) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_2,v_2)}\}
\end{aligned}$$

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Also $\widehat{R^{F}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)]\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)]}$
$\leqslant \max\{\widehat{R^{F}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_1,v_1)},\widehat{R^{F}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}}(u_2,v_2)\}$,
$\widehat{R^{F}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)]\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)]}$
$\leqslant \max\{\widehat{R^{F}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_1,v_1)},\widehat{R^{F}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon\times\Omega}}(u_2,v_2)}\}$.
Thus, $\widehat{\Upsilon\times\Omega}$ is a CIVNSBS of $\mathcal{S}$.

Corollary 3.11. *If $\widehat{\Upsilon_1},\widehat{\Upsilon_2},...,\widehat{\Upsilon_n}$ be the finite collection of CIVNSBSs of $\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n$ respectively. Then $\Upsilon_1\times\widehat{\Upsilon_2\times...}\times\Upsilon_n$ is a CIVNSBS of $\mathcal{S}_1\times\mathcal{S}_2\times...\times\mathcal{S}_n$.*

Definition 3.12. Let $\widehat{\Upsilon}\subseteq\mathcal{S}$, the strongest CIVN relation on $\mathcal{S}$ is

$$\left\{\begin{array}{l}\widehat{R^{\top}_{\mathcal{V}}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u,v)}=min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u)},\widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v)}\}\\ \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u,v)}=\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v)}}{2}\\ \widehat{R^{F}_{\mathcal{V}}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(u,v)}=max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)},\widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(v)}\}\end{array}\right\}$$

Theorem 3.13. *Let $\widehat{\Upsilon}$ be a CIVNSBS of $\mathcal{S}$ and $\widehat{\mathcal{V}}$ be a strongest complex interval valued neutrosophic relation of $\mathcal{S}$. Then $\widehat{\Upsilon}$ is a CIVNSBS of $\mathcal{S}\times\mathcal{S}$ if and only if $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S}\times\mathcal{S}$.*

Proof. Suppose $\widehat{\Upsilon}$ is a CIVNSBS of $\mathcal{S}\times\mathcal{S}$ and $\widehat{\mathcal{V}}$ be the strongest complex interval valued neutrosophic relation of $\mathcal{S}$.
For any $u=(u_1,u_2),v=(v_1,v_2)\in\mathcal{S}\times\mathcal{S}$. Now,

$$\begin{aligned}\widehat{R^{\top}_{\mathcal{V}}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u\nabla_1 v)} &= \widehat{R^{\top}_{\mathcal{V}}}[((u_1,u_2)\nabla_1(v_1,v_2)]\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}[((u_1,u_2)\nabla_1(v_1,v_2)]}\\ &= \widehat{R^{\top}_{\mathcal{V}}}(u_1\nabla_1 v_1,u_2\nabla_1 v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u_1\nabla_1 v_1,u_2\nabla_1 v_2)}\\ &= \min\{\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1\nabla_1 v_1)},\widehat{R^{\top}_{\Upsilon}}(u_2\nabla_1 v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2\nabla_1 v_2)}\}\\ &\geq \min\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)},\widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_1)}\},\\ &\quad \min\{\widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2)},\widehat{R^{\top}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_2)}\}\}\\ &= \min\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)},\widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2)}\},\\ &\quad \min\{\widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_1)},\widehat{R^{\top}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_2)}\}\}\\ &= \min\{\widehat{R^{\top}_{\mathcal{V}}}(u_1,u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u_1,u_2)},\widehat{R^{\top}_{\mathcal{V}}}(v_1,v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(v_1,v_2)}\}\\ &= \min\{\widehat{R^{\top}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u)},\widehat{R^{\top}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(v)}\}\end{aligned}$$

Also $\widehat{R^{\top}_{\mathcal{V}}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u\nabla_2 v)}\geqslant\min\{\widehat{R^{\top}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u)},\widehat{R^{\top}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(v)}\}$,
$\widehat{R^{\top}_{\mathcal{V}}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u\nabla_3 v)}\geqslant\min\{\widehat{R^{\top}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u)},\widehat{R^{\top}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(v)}\}$.
Now, $\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u\nabla_1 v)}$

$$\begin{aligned}&=\widehat{R^{\mathcal{I}}_{\mathcal{V}}}[((u_1,u_2)\nabla_1(v_1,v_2)]\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}[((u_1,u_2)\nabla_1(v_1,v_2)]}\\ &=\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u_1\nabla_1 v_1,u_2\nabla_1 v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u_1\nabla_1 v_1,u_2\nabla_1 v_2)}\\ &=\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 v_1)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2\nabla_1 v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_2\nabla_1 v_2)}{2}\\ &\geqslant\frac{1}{2}\left[\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v_1)}}{2}+\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_2)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v_2)}}{2}\right]\\ &=\frac{1}{2}\left[\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_2)}}{2}+\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v_1)}}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v_2)}}{2}\right]\\ &=\frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u_1,u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u_1,u_2)}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(v_1,v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(v_1,v_2)}}{2}\\ &=\frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}(u)}}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u)}}{2}\end{aligned}$$

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Also $\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u\nabla_2 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}(u)}}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(v)}}{2}$
and $\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u\nabla_3 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}(u)}}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(v)}}{2}$.
Similarly, $\widehat{R^{F}_{\mathcal{V}}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(u\nabla_1 v)} \leqslant \max\{\widehat{R^{F}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}(u)}}, \widehat{R^{F}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}(v)}}\}$,
$\widehat{R^{F}_{\mathcal{V}}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(u\nabla_2 v)} \leqslant \max\{\widehat{R^{F}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}(u)}}, \widehat{R^{F}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}(v)}}\}$ and
$\widehat{R^{F}_{\mathcal{V}}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(u\nabla_3 v)} \leqslant \max\{\widehat{R^{F}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}(u)}}, \widehat{R^{F}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}(v)}}\}$.
Therefore, $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S}\times\mathcal{S}$.

Conversely, suppose that $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S}\times\mathcal{S}$. Let $u=(u_1,u_2), v=(v_1,v_2)\in\mathcal{S}\times\mathcal{S}$. Now,

$$
\begin{aligned}
&\min\{\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_1\nabla_1 v_1)}}, \widehat{R^{\top}_{\Upsilon}}(u_2\nabla_1 v_2)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_2\nabla_1 v_2)}}\}\\
&= \widehat{R^{\top}_{\mathcal{V}}}(u_1\nabla_1 v_1, u_2\nabla_1 v_2)\cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}(u_1\widehat{\nabla_1 v_1, u_2}\nabla_1 v_2)}\\
&= \widehat{R^{\top}_{\mathcal{V}}}[(u_1,u_2)\nabla_1(v_1,v_2)]\cdot e^{i2\pi\Theta^{\top}_{v}[(u_1,\widehat{u_2)}\nabla_1(v_1,v_2)]}\\
&= \widehat{R^{\top}_{\mathcal{V}}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}(u\nabla_1 v)}}\\
&\geqslant \min\{\widehat{R^{\top}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}(u)}}, \widehat{R^{\top}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}(v)}}\}\\
&= \min\{\widehat{R^{\top}_{\mathcal{V}}}(u_1,u_2)\}\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}(u_1,u_2)}\}}, \widehat{R^{\top}_{\mathcal{V}}}(v_1,v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(v_1,v_2)}\}\\
&= \min\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)}, \widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2)}\}, \min\{\widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_1)}}, \widehat{R^{\top}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_2)}}\}\}
\end{aligned}
$$

If $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_1\nabla_1 v_1)}} \leqslant \widehat{R^{\top}_{\Upsilon}}(u_2\nabla_1 v_2)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_2\nabla_1 v_2)}}$, then $\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)} \leqslant \widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2)}$ and $\widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_1)}} \leqslant \widehat{R^{\top}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_2)}}$. We get $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_1\nabla_1 v_1)}} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)}, \widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_1)}}\}$ for all $u_1, v_1\in\mathcal{S}$, and
$\min\{\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_2 v_1)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_1\nabla_2 v_1)}}, \widehat{R^{\top}_{\Upsilon}}(u_2\nabla_2 v_2)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_2\nabla_2 v_2)}}\} \geqslant \min\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)}, \widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2)}\}, \min\{\widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_1)}}, \widehat{R^{\top}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_2)}}\}\}$
If $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_2 v_1)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_1\nabla_2 v_1)}} \leqslant \widehat{R^{\top}_{\Upsilon}}(u_2\nabla_2 v_2)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_2\nabla_2 v_2)}}$, then $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_2 v_1)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_1\nabla_2 v_1)}} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)}, \widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_1)}}\}$.
$\min\{\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_3 v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1\nabla_3 v_1)}, \widehat{R^{\top}_{\Upsilon}}(u_2\nabla_3 v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2\nabla_3 v_2)}\} \geqslant \min\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)}, \widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2)}\}, \min\{\widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_1)}, \widehat{R^{\top}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v_2)}\}\}$
If $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_3 v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1\nabla_3 v_1)} \leqslant \widehat{R^{\top}_{\Upsilon}}(u_2\nabla_3 v_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_2\nabla_3 v_2)}$, then $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_3 v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1\nabla_3 v_1)} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u_1)}, \widehat{R^{\top}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_1)}}\}$.
Now,

$$
\begin{aligned}
&\frac{1}{2}\Big[\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon}\widehat{(u_1\nabla_1 v_1)}} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2\nabla_1 v_2)\cdot e^{i2\pi\Theta_{\Upsilon}^{\mathcal{I}\widehat{(u_2\nabla_1 v_2)}}}\Big]\\
&= \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u_1\nabla_1 v_1, u_2\nabla_1 v_2)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}(u_1\widehat{\nabla_1 v_1, u_2}\nabla_1 v_2)}\\
&= \widehat{R^{\mathcal{I}}_{\mathcal{V}}}[(u_1,u_2)\nabla_1(v_1,v_2)]\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}[(u_1,\widehat{u_2)}\nabla_1(v_1,v_2)]}\\
&= \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}(u\nabla_1 v)}}\\
&\geqslant \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u)\cdot e^{i2\pi\widehat{\Theta_{\mathcal{V}}^{\mathcal{I}(u)}}} + \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}(v)}}}{2}\\
&= \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u_1,u_2)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}\widehat{(u_1,u_2)}} + \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(v_1,v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}}}{2}\\
&= \frac{1}{2}\left[\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_2)}}{2} + \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v_1)}} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v_2)}}}{2}\right]
\end{aligned}
$$

If $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 v_1)} \leqslant \widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2\nabla_1 v_2)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon}\widehat{(u_2\nabla_1 v_2)}}$, then $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)} \leqslant \widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_2)}$ and $\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v_1)}} \leqslant \widehat{R^{\mathcal{I}}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v_2)}}$.

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We get $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u_1\nabla_1 v_1)}} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_1)}}}{2}$.
Similarly, $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_2 v_1)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon}\widehat{(u_1\nabla_2 v_1)}} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v_1)}}{2}$
and $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_3 v_1)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon}\widehat{(u_1\nabla_3 v_1)}} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u_1)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v_1)}}}{2}$.
Similarly to prove that
$\max\{\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_1\nabla_1 v_1)}}, \widehat{R^{F}_{\Upsilon}}(u_2\nabla_1 v_2)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_2\nabla_1 v_2)}}\} \leqslant \max\{\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)}, \widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_2)}\}, \max\{\widehat{R^{F}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_1)}}, \widehat{R^{F}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_2)}}\}\}$
If $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_1\nabla_1 v_1)}} \geqslant \widehat{R^{F}_{\Upsilon}}(u_2\nabla_1 v_2)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_2\nabla_1 v_2)}}$, then $\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)} \geqslant \widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_2)}$ and $\widehat{R^{F}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_1)}} \geqslant \widehat{R^{F}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_2)}}$.
We get $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1 v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_1\nabla_1 v_1)}} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)}, \widehat{R^{F}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_1)}}\}$.
$\max\{\widehat{R^{F}_{\Upsilon}}(u_1\nabla_2 v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_1\nabla_2 v_1)}}, \widehat{R^{F}_{\Upsilon}}(u_2\nabla_2 v_2)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_2\nabla_2 v_2)}}\}$
$\leqslant \max\{\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)}, \widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_2)}\}, \max\{\widehat{R^{F}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_1)}}, \widehat{R^{F}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}\}\}$
If $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_2 v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_1\nabla_2 v_1)}} \geqslant \widehat{R^{F}_{\Upsilon}}(u_2\nabla_2 v_2)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_2\nabla_2 v_2)}}$, then $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_2 v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_1\nabla_2 v_1)}} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)}, \widehat{R^{F}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_1)}}\}$.
$\max\{\widehat{R^{F}_{\Upsilon}}(u_1\nabla_3 v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_1\nabla_3 v_1)}}, \widehat{R^{F}_{\Upsilon}}(u_2\nabla_3 v_2)\cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u_2\nabla_3 v_2)}}\} \leqslant \max\{\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)}, \widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_2)}\}, \max\{\widehat{R^{F}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_1)}}, \widehat{R^{F}_{\Upsilon}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v_2)}}\}\}$
If $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_3 v_1)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_1\nabla_3 v_1)}} \geqslant \widehat{R^{F}_{\Upsilon}}(u_2\nabla_3 v_2)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_2\nabla_3 v_2)}}$, then $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_3 v_1)\cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u_1\nabla_3 v_1)}} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u_1)}, \widehat{R^{F}_{\Upsilon}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v_1)}}\}$.
Therefore, $\widehat{\Upsilon}$ is a CIVNSBS of $\mathcal{S}$.

Theorem 3.14. *Suppose that $\widehat{\Upsilon}$ is a subset of $\mathcal{S}$. Then $\widehat{R} = (\widehat{R^{\top}_{\Upsilon}}\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}, \widehat{R^{\mathcal{I}}_{\Upsilon}}\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}, \widehat{R^{F}_{\Upsilon}}\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}})$ is a CIVNSBS of $\mathcal{S}$ if and only if $\widehat{R^{(\alpha,\beta)}}$ is a subbisemiring of $\mathcal{S}$ for all $\alpha,\beta\in\mathbb{D}[0,1]$.*

Proof. Assume that $\widehat{R}$ is a CIVNSBS of $\mathcal{S}$. For each $\alpha,\beta\in\mathbb{D}[0,1]$ and $u_1,u_2\in\widehat{R^{(\alpha,\beta)}}$. Now, $\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1)\geqslant\alpha$, $\widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_2)\geqslant\alpha$ and $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1)\geqslant\alpha$, $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_2)\geqslant\alpha$ and $\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1)\leqslant\beta$, $\widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_2)\leqslant\beta$. Now, $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1\nabla_1 u_2)\geqslant\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1), \widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_2)\}\geqslant\alpha$ and $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1\nabla_1 u_2)\geqslant\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1)+\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_2)}{2}$ $\geqslant\frac{\alpha+\alpha}{2}=\alpha$ and $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1\nabla_1 u_2)\leqslant\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1), \widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_2)\}\leqslant\beta$. This implies that $u_1\nabla_1 u_2\in\widehat{R^{(\alpha,\beta)}}$. Similarly, $u_1\nabla_2 u_2\in\widehat{R^{(\alpha,\beta)}}$ and $u_1\nabla_3 u_2\in\widehat{R^{(\alpha,\beta)}}$. Hence, $\widehat{R^{(\alpha,\beta)}}$ is a subbisemiring of $\mathcal{S}$, for all $\alpha,\beta\in\mathbb{D}[0,1]$.

Conversely, assume that $\widehat{R^{(\alpha,\beta)}}$ is a subbisemiring of $\mathcal{S}$ and $\alpha,\beta\in\mathbb{D}[0,1]$. Suppose if there exist $u_1,u_2\in\mathcal{S}$ such that $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1\nabla_1 u_2)<\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1), \widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_2)\}$, $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1\nabla_1 u_2)<\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1)+\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_2)}{2}$ and $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1\nabla_1 u_2)>\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1), \widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_2)\}$. For $\alpha,\beta\in\mathbb{D}[0,1]$ such that $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1\nabla_1 u_2)<\alpha\leqslant\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1), \widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_2)\}$ and $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1\nabla_1 u_2)<\alpha$ $\leqslant\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1)+\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_2)}{2}$ and $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1\nabla_1 u_2)>\beta$ $\geqslant\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1), \widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_2)\}$. Thus, $u_1,u_2\in\widehat{R^{(\alpha,\beta)}}$, but $u_1\nabla_1 u_2\notin\widehat{R^{(\alpha,\beta)}}$. This contradicts, $\widehat{R^{(\alpha,\beta)}}$ is a SBS of $\mathcal{S}$. Therefore $\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1\nabla_1 u_2)\geqslant\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_1), \widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u_2)\}$, $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1\nabla_1 u_2)\geqslant\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_1)+\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u_2)}{2}$ and $\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1 u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1\nabla_1 u_2)\leqslant\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_1), \widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u_2)\}$. Similarly, ∇_2 and ∇_3 cases. Hence $\widehat{R}=(\widehat{R^{\top}_{\Upsilon}}\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}, \widehat{R^{\mathcal{I}}_{\Upsilon}}\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}, \widehat{R^{F}_{\Upsilon}}\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}})$ is a CIVNSBS of $\mathcal{S}$.

Definition 3.15. Let $(\mathcal{S}_1,\varnothing_1,\varnothing_2,\varnothing_3)$ and $(\mathcal{S}_2,\square_1,\square_2,\square_3)$ be any two bisemirings. The mapping $H:\mathcal{S}_1\to\mathcal{S}_2$ and $\widehat{\Upsilon}$ be any CIVNSBS in $\mathcal{S}_1$, $\widehat{\mathcal{V}}$ be any CIVNSBS in $H(\mathcal{S}_1)=\mathcal{S}_2$. If $\widehat{R_{\Upsilon}}\cdot e^{i2\pi\widehat{\Theta_{\Upsilon}}}=[\widehat{R^{\top}_{\Upsilon}}\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}, \widehat{R^{\mathcal{I}}_{\Upsilon}}\cdot$

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$e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}, \widehat{R^{F}_{\Upsilon}} \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}]$ is a CIVNS in $\mathcal{S}_1$, then $\widehat{R_{\mathcal{V}}}$ is a CIVNS in $\mathcal{S}_2$, defined by

$$\widehat{R^{\top}_{\mathcal{V}}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}} = \begin{cases} \sup \widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}(u) & if \;\; u \in H^{-1}v \\ 0 & otherwise \end{cases}$$

$$\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u) = \begin{cases} \sup \widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}}(u) & if \;\; u \in H^{-1}v \\ 0 & otherwise \end{cases}$$

$$\widehat{R^{F}_{\mathcal{V}}}(v) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u) = \begin{cases} \inf \widehat{R^{F}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}}(u) & if \;\; u \in H^{-1}v \\ 1 & otherwise \end{cases}$$

for all $u \in \mathcal{S}_1$ and $v \in \mathcal{S}_2$ is represents the image of R_{Υ} under H.

Similarly, If $\widehat{R_{\mathcal{V}}} \cdot e^{i2\pi\widehat{\Theta_{\mathcal{V}}}} = [\widehat{R^{\top}_{\mathcal{V}}} \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}}, \widehat{R^{\mathcal{I}}_{\mathcal{V}}} \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}}, \widehat{R^{F}_{\mathcal{V}}} \cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}}]$ is a CIVNS in $\mathcal{S}_2$, then CIVNS $\widehat{R_{\Upsilon}} = H \circ \widehat{R_{\mathcal{V}}}$ in $\mathcal{S}_1$ [ie, the CIVNS defined by $\widehat{R_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta_{\Upsilon}(u)}} = \widehat{R_{\mathcal{V}}}(H(u)) \cdot e^{i2\pi\widehat{\Theta_{\Upsilon}(H(u))}}$] is represents the preimage of $\widehat{R_{\mathcal{V}}}$ under H.

Theorem 3.16. *The homomorphic image of every CIVNSBS is a CIVNSBS.*

Proof. The mapping $H : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Now, $H(u\varnothing_1 v) = H(u)\Box_1 H(v), H(u\varnothing_2 v) = H(u)\Box_2 H(v)$ and $H(u\varnothing_3 v) = H(u)\Box_3 H(v)$ for all $u, v \in \mathcal{S}_1$. Let $\widehat{\mathcal{V}} = H(\widehat{\Upsilon})$, $\widehat{\Upsilon}$ is any CIVNSBS of $\mathcal{S}_1$. Let $H(u), H(v) \in \mathcal{S}_2$. Let $u \in uH^{-1}(H(u))$ and $v \in H^{-1}(H(v))$ be such that $\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}} = \sup_{u \in H^{-1}(H(u))} \widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}$ and $\widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}} = \sup_{u \in H^{-1}(H(v))} \widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}$. Now,

$$\begin{aligned} \widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_1 H(v)) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(H(u)\Box_1 H(v))}} &= \sup_{u' \in H^{-1}(H(u)\Box_1 H(v))} \widehat{R^{\top}_{\Upsilon}}(u') \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u')}} \\ &= \sup_{u' \in H^{-1}(H(u\varnothing_1 v))} \widehat{R^{\top}_{\Upsilon}}(u') \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u')}} \\ &= \widehat{R^{\top}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u\varnothing_1 v)}} \\ &\geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}\} \\ &= \min\{\widehat{R^{\top}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(u)}, \widehat{R^{\top}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(v)}\}. \end{aligned}$$

Thus, $\widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_1 H(v)) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(u)\Box_1 H(v))} \geqslant \min\{\widehat{R^{\top}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(u)}, \widehat{R^{\top}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(v)}\}$.

Similarly, $\widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_2 H(v)) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(u)\Box_2 H(v))} \geqslant \min\{\widehat{R^{\top}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(u)}, \widehat{R^{\top}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(v)}\}$ and

$\widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_3 H(v)) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(u)\Box_3 H(v))} \geqslant \min\{\widehat{R^{\top}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(u)}, \widehat{R^{\top}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(v)}\}$.

Let $u \in H^{-1}(H(u))$ and $v \in H^{-1}(H(v))$ be such that $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u)} = \sup_{u \in H^{-1}(H(u))} \widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(u)}$ and $\widehat{R^{\mathcal{I}}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(v)} = \sup_{u \in H^{-1}(H(v))} \widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(u)}$. Now,

$$\begin{aligned} \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_1 H(v)) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_1 H(v))} &= \sup_{u' \in H^{-1}(H(u)\Box_1 H(v))} \widehat{R^{\mathcal{I}}_{\Upsilon}}(u') \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u')} \\ &= \sup_{u' \in H^{-1}(H(u\varnothing_1 v))} \widehat{R^{\mathcal{I}}_{\Upsilon}}(u') \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u')} \\ &= \widehat{R^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)} \\ &\geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v)}}{2} \\ &= \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(u)} + \widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(v)}}{2}. \end{aligned}$$

Thus, $\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_1 H(v)) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_1 H(v))} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(u)} + \widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(v)}}{2}$.

Similarly, $\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_2 H(v)) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_2 H(v))} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(u)} + \widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(v)}}{2}$ and

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$\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\square_3 H(v))\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(u)\square_3 H(v))} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(u)}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(v)}}{2}$.

Let $H(u), H(v)\in\mathcal{S}_2$. Let $u\in H^{-1}(H(u))$ and $v\in H^{-1}(H(v))$ be such that $\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)} = \inf_{u\in H^{-1}(H(u))}\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)}$ and $\widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(v)} = \inf_{u\in H^{-1}(H(v))}\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)}$. Now,

$$\begin{aligned}\widehat{R^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))} &= \inf_{u'\in H^{-1}(H(u)\square_1 H(v))}\widehat{R^{F}_{\Upsilon}}(u')\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u')}\\ &= \inf_{u'\in H^{-1}(H(u\varnothing_1 v)}\widehat{R^{F}_{\Upsilon}}(u')\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u')}\\ &= \widehat{R^{F}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\varnothing_1 v)}\\ &\leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(v)}\}\\ &= \max\{\widehat{R^{F}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(u)}, \widehat{R^{F}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(v)}\}.\end{aligned}$$

Thus, $\widehat{R^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))} \leqslant \max\{\widehat{R^{F}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(u)}, \widehat{R^{F}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(v)}\}$. Similarly, $\widehat{R^{F}_{\mathcal{V}}}(H(u)\square_2 H(v))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u)\square_2 H(v))} \leqslant \max\{\widehat{R^{F}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(u)}, \widehat{R^{F}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(v)}\}$ and $\widehat{R^{F}_{\mathcal{V}}}(H(u)\square_3 H(v))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))} \leqslant \max\{\widehat{R^{F}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(v)}, \widehat{R^{F}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(v)}\}$. Thus, $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S}_2$.

Theorem 3.17. *The homomorphic preimage of every CIVNSBS is a CIVNSBS.*

Proof. The mapping $H:\mathcal{S}_1\to\mathcal{S}_2$ be a homomorphism. Now, $H(u\varnothing_1 v) = H(u)\square_1 H(v), H(u\varnothing_2 v) = H(u)\square_2 H(v)$ and $H(u\varnothing_3 v) = H(u)\square_3 H(v)$ for all $u, v\in\mathcal{S}_1$. Let $\widehat{\mathcal{V}} = H(\widehat{\Upsilon})$, $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S}_2$. Let $u, v\in\mathcal{S}_1$. Now, $\widehat{R^{\top}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\varnothing_1 v)} = \widehat{R^{\top}_{\mathcal{V}}}(H(u\varnothing_1 v))\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(u\varnothing_1 v))} = \widehat{R^{\top}_{\mathcal{V}}}(H(u)\square_1 H(v))\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(u)\square_1 H(v))}$ $\geqslant \min\{\widehat{R^{\top}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(u)}, \widehat{R^{\top}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}H(v)}\} = \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u)}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v)}\}$. Thus, $\widehat{R^{\top}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\varnothing_1 v)} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u)}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v)}\}$. Now, $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)} = \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u\varnothing_1 v))\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u\varnothing_1 v)} = \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\square_1 H(v))\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(u\varnothing_1 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(u)}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}H(v)}}{2} = \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v)}}{2}$. Thus, $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u)}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v)}}{2}$. Now, $\widehat{R^{F}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\varnothing_1 v)} = \widehat{R^{F}_{\mathcal{V}}}(H(u\varnothing_1 v))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u\varnothing_1 v))} = \widehat{R^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))} \leqslant \max\{\widehat{R^{F}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(u)}, \widehat{R^{F}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}H(v)}\} = \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(v)}\}$. Thus, $\widehat{R^{F}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\varnothing_1 v)} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(v)}\}$.

Theorem 3.18. *If $H:\mathcal{S}_1\to\mathcal{S}_2$ is a homomorphism, then $H(\widehat{\Upsilon_{(\alpha,\beta)}})$ is a subbisemiring of CIVNSBS $\widehat{\mathcal{V}}$ of $\mathcal{S}_2$.*

Proof. The mapping $H:\mathcal{S}_1\to\mathcal{S}_2$ be a homomorphism. Now, $H(u\varnothing_1 v) = H(u)\square_1 H(v), H(u\varnothing_2 v) = H(u)\square_2 H(v)$ and $H(u\varnothing_3 v) = H(u)\square_3 H(v)$ for all $u, v\in\mathcal{S}_1$. Let $\widehat{\mathcal{V}} = H(\widehat{\Upsilon})$, $\widehat{\Upsilon}$ is a CIVNSBS of $\mathcal{S}_1$. By Theorem 3.16, $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S}_2$. Let $\widehat{\Upsilon_{(\alpha,\beta)}}$ be any subbisemiring of $\widehat{\Upsilon}$. Suppose that $u, v\in\widehat{\Upsilon_{(\alpha,\beta)}}$. Then $u\varnothing_1 v, u\varnothing_2 v$ and $u\varnothing_3 v\in\widehat{\Upsilon_{(\alpha,\beta)}}$. Now, $\widehat{R^{\top}_{\mathcal{V}}}(H(u))\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(u))} = \widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u)} \geqslant \alpha, \widehat{R^{\top}_{\mathcal{V}}}(H(v))\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(v))} = \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v)} \geqslant \alpha$. Thus, $\widehat{R^{\top}_{\mathcal{V}}}(H(u)\square_1 H(v))\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(u)\square_1 H(v))} \geqslant \widehat{R^{\top}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\varnothing_1 v)} \geqslant \alpha$. Now, $\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u))\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(u))} = \widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u)} \geqslant \alpha, \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(v))\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(v))} = \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v)} \geqslant \alpha$. Thus, $\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\square_1 H(v))\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(u)\square_1 H(v))} \geqslant \widehat{R^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)} \geqslant \alpha$. Now, $\widehat{R^{F}_{\mathcal{V}}}(H(u))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u))} = \widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)} \leqslant \beta, \widehat{R^{F}_{\mathcal{V}}}(H(v))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(v))} = \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(v)} \leqslant \beta$. Thus, $\widehat{R^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u)\square_1 H(v))} \leqslant \widehat{R^{F}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\varnothing_1 v)} \leqslant \beta$, for all $H(u), H(v)\in\mathcal{S}_2$. Similarly other operations, $H(\widehat{\Upsilon_{(\alpha,\beta)}})$ is a subbisemiring of CIVNSBS $\widehat{\mathcal{V}}$ of $\mathcal{S}_2$.

Theorem 3.19. *If $H:\mathcal{S}_1\to\mathcal{S}_2$ is any homomorphism, then $\widehat{\Upsilon_{(\alpha,\beta)}}$ is a subbisemiring of CIVNSBS $\widehat{\Upsilon}$ of $\mathcal{S}_1$.*

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Proof. The mapping $H : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. We have $H(u\varnothing_1 v) = H(u)\Box_1 H(v), H(u\varnothing_2 v) = H(u)\Box_2 H(v)$ and $H(u\varnothing_3 v) = H(u)\Box_3 H(v)$ for all $u, v \in \mathcal{S}_1$. Let $\widehat{\mathcal{V}} = H(\widehat{\Upsilon})$, $\widehat{\mathcal{V}}$ is a CIVNSBS of $\mathcal{S}_2$. By Theorem 3.17, $\widehat{\Upsilon}$ is a CIVNSBS of $\mathcal{S}_1$. Let $H(\widehat{\Upsilon_{(\alpha,\beta)}})$ be a subbisemiring of $\widehat{\mathcal{V}}$. Suppose that $H(u), H(v) \in H(\widehat{\Upsilon_{(\alpha,\beta)}})$. Now, $H(u\varnothing_1 v), H(u\varnothing_2 v)$ and $H(u\varnothing_3 v) \in H(\widehat{\Upsilon_{(\alpha,\beta)}})$. Now, $\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u)} = \widehat{R^{\top}_{\mathcal{V}}}(H(u)) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(u))} \geqslant \alpha, \widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v)} = \widehat{R^{\top}_{\mathcal{V}}}(H(v)) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}}(H(v))} \geqslant \alpha$. Thus, $\widehat{R^{\top}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u\varnothing_1 v)} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(u)}, \widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}}(v)}\} \geqslant \alpha$. Now, $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u)} = \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(u))} \geqslant \alpha, \widehat{R^{\mathcal{I}}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v)} = \widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(v)) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}}(H(v))} \geqslant \alpha$. Thus, $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)} \geqslant \frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(u)} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}}(v)}}{2} \geqslant \alpha$. Now, $\widehat{R^{F}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)} = \widehat{R^{F}_{\mathcal{V}}}(H(u)) \cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u))} \leqslant \beta, \widehat{R^{F}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(v)} = \widehat{R^{F}_{\mathcal{V}}}(H(v)) \cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(v))} \leqslant \beta$. Thus, $\widehat{R^{F}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u\varnothing_1 v)} = \widehat{R^{F}_{\mathcal{V}}}(H(u)\Box_1 H(v)) \cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}}}(H(u)\Box_1 H(v))} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(u)}, \widehat{R^{F}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}}(v)}\} \leqslant \beta$, for all $u, v \in \mathcal{S}_1$. Similarly other operations, $\widehat{\Upsilon_{(\alpha,\beta)}}$ is a subbisemiring of CIVNSBS $\widehat{\Upsilon}$ of $\mathcal{S}_1$.

4 (ϱ, σ) CIVNSBS

In this section $(\varrho, \sigma) \in \mathbb{D}[0, 1]$ and $0 \leqslant \varrho < \sigma \leqslant 1$.

Definition 4.1. Let $\widehat{\Upsilon}$ be a CIVN subset of $\mathcal{S}$ is called a (ϱ, σ) CIVNSBS of $\mathcal{S}$ if

$$\begin{cases} \max\{\widehat{R^{\top}_{\Upsilon}}(u\nabla_1 v) \cdot e^{i2\pi\Theta^{\widehat{\top(u\nabla_1 v)}}_{\Upsilon}}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}, \widehat{\sigma}\} \\ \max\{\widehat{R^{\top}_{\Upsilon}}(u\nabla_2 v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u\nabla_2 v)}}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}, \widehat{\sigma}\} \\ \max\{\widehat{R^{\top}_{\Upsilon}}(u\nabla_3 v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u\nabla_3 v)}}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}, \widehat{\sigma}\} \end{cases}$$

$$\begin{cases} \max\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v), \widehat{\varrho}\} \geqslant \min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)}{2}, \widehat{\sigma}\right\} \\ \text{OR} \\ \max\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_2 v), \widehat{\varrho}\} \geqslant \min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)}{2}, \widehat{\sigma}\right\} \\ \text{OR} \\ \max\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_3 v), \widehat{\varrho}\} \geqslant \min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)}{2}, \widehat{\sigma}\right\} \end{cases}$$

$$\begin{cases} \min\{\widehat{R^{F}_{\Upsilon}}(u\nabla_1 v), \widehat{\varrho}\} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u), \widehat{R^{F}_{\Upsilon}}(v), \widehat{\sigma}\} \\ \min\{\widehat{R^{F}_{\Upsilon}}(u\nabla_2 v), \widehat{\varrho}\} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u), \widehat{R^{F}_{\Upsilon}}(v), \widehat{\sigma}\} \\ \min\{\widehat{R^{F}_{\Upsilon}}(u\nabla_3 v), \widehat{\varrho}\} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u), \widehat{R^{F}_{\Upsilon}}(v), \widehat{\sigma}\} \end{cases}$$

for all $u, v \in \mathcal{S}$.

Example 4.2. By the Example 3.8,

	$w = v$	$w = \nu$	$w = \eta$	$w = \xi$
$\widehat{R^{\top}_{\Upsilon}}(w)$	$[0.65e^{i2\pi(0.5)}, 0.75e^{i2\pi(0.6)}]$	$[0.55e^{i2\pi(0.4)}, 0.65e^{i2\pi(0.5)}]$	$[0.25e^{i2\pi(0.1)}, 0.35e^{i2\pi(0.2)}]$	$[0.45e^{i2\pi(0.3)}, 0.55e^{i2\pi(0.4)}]$
$\widehat{R^{\mathcal{I}}_{\Upsilon}}(w)$	$[0.35e^{i2\pi(0.2)}, 0.45e^{i2\pi(0.3)}]$	$[0.25e^{i2\pi(0.1)}, 0.35e^{i2\pi(0.2)}]$	$[0.1e^{i2\pi(0.1)}, 0.15e^{i2\pi(0.1)}]$	$[0.15e^{i2\pi(0.1)}, 0.25e^{i2\pi(0.1)}]$
$\widehat{R^{F}_{\Upsilon}}(w)$	$[0.45e^{i2\pi(0.3)}, 0.55e^{i2\pi(0.4)}]$	$[0.55e^{i2\pi(0.4)}, 0.65e^{i2\pi(0.45)}]$	$[0.75e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.8)}]$	$[0.7e^{i2\pi(0.55)}, 0.75e^{i2\pi(0.75)}]$

Clearly $\widehat{\Upsilon}$ is a $([0.2e^{0.05\pi i}, 0.3e^{0.15\pi i}], [0.5e^{0.35\pi i}, 0.6e^{0.45\pi i}])$ CIVNSBS of $\mathcal{S}$.

Theorem 4.3. *The intersection of every (ϱ, σ) CIVNSBSs is a (ϱ, σ) CIVNSBS.*

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Proof. Let $\{\widehat{\mathcal{V}}_i : i \in I\}$ be a family of (ϱ, σ) CIVNSBSs of $\mathcal{S}$ and $\widehat{\Upsilon} = \bigcap_{i\in I}\widehat{\mathcal{V}}_i$. Let $u, v \in \mathcal{S}$. Now,

$$\begin{aligned}
\max\{\widehat{R^{\top}_{\Upsilon}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u\nabla_1 v)}}, \widehat{\varrho}\} &= \inf_{i\in I}\ \max\{\widehat{R^{\top}_{\mathcal{V}_i}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}_i}(u\nabla_1 v)}}, \widehat{\varrho}\}\\
&\geqslant \inf_{i\in I}\ \min\{\widehat{R^{\top}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}_i}(u)}}, \widehat{R^{\top}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}_i}(v)}}, \widehat{\sigma}\}\\
&= \min\left\{\inf_{i\in I}\ \widehat{R^{\top}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}_i}(u)}}, \inf_{i\in I}\ \widehat{R^{\top}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}_i}(v)}}, \widehat{\sigma}\right\}\\
&= \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}, \widehat{\sigma}\}
\end{aligned}$$

Similarly, $\max\{\widehat{R^{\top}_{\Upsilon}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u\nabla_2 v)}}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}, \widehat{\sigma}\}$ and $\max\{\widehat{R^{\top}_{\Upsilon}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u\nabla_3 v)}}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}, \widehat{\sigma}\}$. Now,

$$\begin{aligned}
\max\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u\nabla_1 v)}}, \widehat{\varrho}\} &= \inf_{i\in I}\ \max\{\widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}_i}(u\nabla_1 v)}}, \widehat{\varrho}\}\\
&\geqslant \inf_{i\in I}\ \min\left\{\frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}_i}}} + \widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}_i}(v)}}}{2}, \widehat{\sigma}\right\}\\
&= \min\left\{\frac{\inf_{i\in I}\ \widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}_i}(u)}} + \inf_{i\in I}\ \widehat{R^{\mathcal{I}}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}_i}(v)}}}{2}, \widehat{\sigma}\right\}\\
&= \min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v)}}}{2}, \widehat{\sigma}\right\}
\end{aligned}$$

Similarly, $\max\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u\nabla_2 v)}}, \widehat{\varrho}\} \geqslant \min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v)}}}{2}, \widehat{\sigma}\right\}$ and $\max\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u\nabla_3 v)}}, \widehat{\varrho}\} \geqslant \min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}} + \widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v)}}}{2}, \widehat{\sigma}\right\}$. Now,

$$\begin{aligned}
\min\{\widehat{R^{F}_{\Upsilon}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u\nabla_1 v)}}, \widehat{\varrho}\} &= \sup_{i\in I}\ \min\{\widehat{R^{F}_{\mathcal{V}_i}}(u\nabla_1 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}_i}(u\nabla_1 v)}}, \widehat{\varrho}\}\\
&\leqslant \sup_{i\in I}\ \max\{\widehat{R^{F}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}_i}(u)}}, \widehat{R^{F}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}_i}(v)}}, \widehat{\sigma}\}\\
&= \max\left\{\sup_{i\in I}\ \widehat{R^{F}_{\mathcal{V}_i}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\mathcal{V}_i}(u)}}, \sup_{i\in I}\ \widehat{R^{F}_{\mathcal{V}_i}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v)}}, \widehat{\sigma}\right\}\\
&= \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u)}}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v)}}, \widehat{\sigma}\}
\end{aligned}$$

Similarly, $\min\{\widehat{R^{F}_{\Upsilon}}(u\nabla_2 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u\nabla_2 v)}}, \widehat{\varrho}\} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u)}}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v)}}, \widehat{\sigma}\}$ and $\min\{\widehat{R^{F}_{\Upsilon}}(u\nabla_3 v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u\nabla_3 v)}}, \widehat{\varrho}\} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u)}}, \widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v)}}, \widehat{\sigma}\}$. Hence $\widehat{\Upsilon}$ is a (ϱ, σ) CIVNSBS of $\mathcal{S}$.

Theorem 4.4. *If $\widehat{\Upsilon}$ and $\widehat{\Omega}$ are any two (ϱ, σ) CIVNSBSs of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, then $\widehat{\Upsilon \times \Omega}$ is a (ϱ, σ) CIVNSBS of $\mathcal{S}_1 \times \mathcal{S}_2$.*

Proof. Let $\widehat{\Upsilon}$ and $\widehat{\Omega}$ be two (ϱ, σ) CIVNSBSs of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Let $u_1, u_2 \in \mathcal{S}_1$ and $v_1, v_2 \in \mathcal{S}_2$.

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Then (u_1, v_1) and (u_2, v_2) are in $\mathcal{S}_1 \times \mathcal{S}_2$. Now

$$\begin{aligned}
&\max\left\{\widehat{R^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)]\cdot e^{i2\pi\Theta^{\top[(u_1,\widehat{v_1)\nabla_1}(u_2,v_2)]}_{\Upsilon\times\Omega}},\widehat{\varrho}\right\}\\
=\;&\max\left\{\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2,v_1\nabla_1v_2)\cdot e^{i2\pi\Theta^{F(u_1\widehat{\nabla_1u_2,v_1}\nabla_1v_2)}_{\Upsilon\times\Omega}},\widehat{\varrho}\right\}\\
=\;&\min\left\{\max\{\widehat{R^{\top}_{\Upsilon}}(u_1\nabla_1u_2)\cdot e^{i2\pi\Theta^{\top\widehat{(u_1\nabla_1u_2)}}_{\Upsilon}},\widehat{\varrho}\},\max\{\widehat{R^{\top}_{\Omega}}(v_1\nabla_1v_2)\cdot e^{i2\pi\Theta^{F\widehat{(v_1\nabla_1v_2)}}_{\Omega}},\widehat{\varrho}\}\right\}\\
\geq\;&\min\left\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u_1)}},\widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u_2)}},\widehat{\sigma}\},\min\{\widehat{R^{\top}_{\Omega}}(v_1)\cdot e^{i2\pi\Theta^{\top\widehat{(u(v_1),v)}}_{\Omega}},\widehat{R^{\top}_{\Omega}}(v_2)\cdot e^{i2\pi\Theta^{\widehat{F(u,v)}}_{\Omega}},\widehat{\sigma}\}\right\}\\
=\;&\min\left\{\{\min\{\widehat{R^{\top}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u_1)}},\widehat{R^{\top}_{\Omega}}(v_1)\cdot e^{i2\pi\Theta^{\widehat{F(u,v)}}_{\Upsilon}}\},\min\{\widehat{R^{\top}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u_2)}},\widehat{R^{\top}_{\Omega}}(v_2)\cdot e^{i2\pi\Theta^{\widehat{F(u,v)}}_{\Omega}}\}\},\widehat{\sigma}\right\}\\
=\;&\min\left\{\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\Theta^{\widehat{F(u,v)}}_{\Upsilon\times\Omega}},\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\Theta^{F\widehat{(u,v)}(u_2,v_2)}_{\Upsilon\times\Omega}},\widehat{\sigma}\right\}
\end{aligned}$$

Also $\max\left\{\widehat{R^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)]\cdot e^{i2\pi\Theta^{\top[(u_1,\widehat{v_1)\nabla_2}(u_2,v_2)]}_{\Upsilon\times\Omega}},\widehat{\varrho}\right\}\geq$
$\min\left\{\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\Theta^{\top\widehat{(u_1,v_1)}}_{\Upsilon\times\Omega}},\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\Theta^{\top\widehat{(u_2,v_2)}}_{\Upsilon\times\Omega}},\widehat{\sigma}\right\}$ and
$\max\left\{\widehat{R^{\top}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)]\cdot e^{i2\pi\Theta^{\top[(u_1,\widehat{v_1)\nabla_3}(u_2,v_2)]}_{\Upsilon\times\Omega}},\widehat{\varrho}\right\}\geq$
$\min\left\{\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\Theta^{\top\widehat{(u_1,v_1)}}_{\Upsilon\times\Omega}},\widehat{R^{\top}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\Theta^{\widehat{F(u,v)}}_{\Upsilon\times\Omega}},\widehat{\sigma}\right\}$.
Now, $\max\left\{\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)]\cdot e^{i2\pi\Theta^{\mathcal{I}[(u_1,\widehat{v_1)\nabla_1}(u_2,v_2)]}_{\Upsilon\times\Omega}},\widehat{\varrho}\right\}$

$$\begin{aligned}
&=\max\left\{\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1\nabla_1u_2,v_1\nabla_1v_2)\cdot e^{i2\pi\Theta^{\mathcal{I}(u_1\widehat{\nabla_1u_2,v_1}\nabla_1v_2)]}_{\Upsilon\times\Omega}},\widehat{\varrho}\right\}\\
&=\min\left\{\frac{1}{2}\left[\max\left\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1\nabla_1u_2)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}\widehat{(u_1\nabla_1u_2)}},\widehat{\varrho}\right\}+\max\left\{\widehat{R^{\mathcal{I}}_{\Omega}}(v_1\nabla_1v_2)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}\widehat{(v_1\nabla_1v_2)}},\widehat{\varrho}\right\}\right]\right\}\\
&\geqslant\min\left\{\frac{1}{2}\left[\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u_1)}}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u_2)}}}{2},\widehat{\sigma}\right\}\right.\right.\\
&\left.\left.+\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Omega}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}(v_1)}}+\widehat{R^{\mathcal{I}}_{\Omega}}(v_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}(v_2)}}}{2},\widehat{\sigma}\right\}\right]\right\}\\
&=\min\left\{\frac{1}{2}\left[\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u_1)}}+\widehat{R^{\mathcal{I}}_{\Omega}}(v_1)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Omega}(v_1)}}}{2}+\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u_2)}}+\widehat{R^{\mathcal{I}}_{\Omega}}(v_2)}{2}\right],\widehat{\sigma}\right\}\\
&=\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}\widehat{(u_1,v_1)}}+\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}\widehat{(u_2,v_2)}}}{2},\widehat{\sigma}\right\}
\end{aligned}$$

Also $\max\left\{\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)]\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}[(u_1,\widehat{v_1)\nabla_2}(u_2,v_2)]},\widehat{\varrho}\right\}$
$\geqslant\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}\widehat{(u_1,v_1)}}+\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}\widehat{(u_2,v_2)}}}{2},\widehat{\sigma}\right\}$ and
$\max\left\{\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)]\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}[(u_1,\widehat{v_1)\nabla_3}(u_2,v_2)]},\widehat{\varrho}\right\}\geq$
$\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}\widehat{(u_1,v_1)}}+\widehat{R^{\mathcal{I}}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon\times\Omega}\widehat{(u_2,v_2)}}}{2},\widehat{\sigma}\right\}$.

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Similarly, $\min\left\{\widehat{R^{F}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_1(u_2,v_2)],\widehat{\varrho}\right\}$

$$
\begin{aligned}
&=\min\left\{\widehat{R^{F}_{\Upsilon\times\Omega}}(u_1\nabla_1 u_2, v_1\nabla_1 v_2),\widehat{\varrho}\right\}\\
&=\max\left\{\min\{\widehat{R^{F}_{\Upsilon}}(u_1\nabla_1 u_2),\widehat{\varrho}\},\min\{\widehat{R^{F}_{\Omega}}(v_1\nabla_1 v_2),\widehat{\varrho}\}\right\}\\
&\leqslant\max\left\{\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u_1)}},\widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u_2)}},\widehat{\sigma}\},\max\{\widehat{R^{F}_{\Omega}}(v_1),\widehat{R^{F}_{\Omega}}(v_2),\widehat{\sigma}\}\right\}\\
&=\max\left\{\{\max\{\widehat{R^{F}_{\Upsilon}}(u_1)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u_1)}},\widehat{R^{F}_{\Omega}}(v_1)\},\max\{\widehat{R^{F}_{\Upsilon}}(u_2)\cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u_2)}},\widehat{R^{F}_{\Omega}}(v_2)\}\},\widehat{\sigma}\right\}\\
&=\max\left\{\widehat{R^{F}_{\Upsilon\times\Omega}}(u_1,v_1),\widehat{R^{F}_{\Upsilon\times\Omega}}(u_2,v_2),\widehat{\sigma}\right\}
\end{aligned}
$$

Also $\min\left\{\widehat{R^{F}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_2(u_2,v_2)]\cdot e^{i2\pi\Theta^{F}_{\Upsilon\times\Omega}\widehat{[(u_1,v_1)\nabla_2(u_2,v_2)]}},\widehat{\varrho}\right\}$
$\leqslant\max\left\{\widehat{R^{F}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon\times\Omega}\widehat{(u_1,v_1)}},\widehat{R^{F}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\Theta^{F}_{\Upsilon\times\Omega}\widehat{(u_2,v_2)}},\widehat{\sigma}\right\}$,
$\min\left\{\widehat{R^{F}_{\Upsilon\times\Omega}}[(u_1,v_1)\nabla_3(u_2,v_2)]\cdot e^{i2\pi\Theta^{F}_{\Upsilon\times\Omega}\widehat{[(u_1,v_1)\nabla_3(u_2,v_2)]}},\widehat{\varrho}\right\}$
$\leqslant\max\left\{\widehat{R^{F}_{\Upsilon\times\Omega}}(u_1,v_1)\cdot e^{i2\pi\Theta^{F}_{\Upsilon\times\Omega}\widehat{(u_1,v_1)}},\widehat{R^{F}_{\Upsilon\times\Omega}}(u_2,v_2)\cdot e^{i2\pi\Theta^{F}_{\Upsilon\times\Omega}\widehat{(u_2,v_2)}},\widehat{\sigma}\right\}$.
Hence $\widehat{\Upsilon\times\Omega}$ is a (ϱ,σ) CIVNSBS s of $\mathcal{S}$.

Corollary 4.5. *If $\widehat{\Upsilon_1},\widehat{\Upsilon_2},...,\widehat{\Upsilon_n}$ are the family of (ϱ,σ) CIVNSBSs of $\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n$ respectively. Then $\Upsilon_1\times\widehat{\Upsilon_2\times...}\times\Upsilon_n$ is a (ϱ,σ) CIVNSBS of $\mathcal{S}_1\times\mathcal{S}_2\times...\times\mathcal{S}_n$.*

Definition 4.6. Let $\widehat{\Upsilon}$ be a CIVNS subset of $\mathcal{S}$, the strongest (ϱ,σ)CIVN relation on $\mathcal{S}$. We define the (ϱ,σ) complex interval valued neutrosophic relation $\widehat{\Upsilon}$ on $\widehat{\mathcal{V}}$ is given by

$$
\left\{\begin{array}{l}
\max\{\widehat{R^{\top}_{\mathcal{V}}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u,v)}},\widehat{\varrho}\}=\min\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}},\widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}},\widehat{\sigma}\}\\
\max\{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u,v)}},\widehat{\varrho}\}=\min\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}},\widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}},\widehat{\sigma}\}\\
\min\{\widehat{R^{F}_{\mathcal{V}}}(u,v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u,v)}},\widehat{\varrho}\}=\max\{\widehat{R^{F}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}},\widehat{R^{F}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}},\widehat{\sigma}\}
\end{array}\right\}
$$

Theorem 4.7. *Let $\widehat{\Upsilon}$ be a (ϱ,σ) CIVNSBS of $\mathcal{S}$ and $\widehat{\mathcal{V}}$ be the strongest (ϱ,σ)CIVN relation of $\mathcal{S}$. Then $\widehat{\Upsilon}$ is a (ϱ,σ) CIVNSBS of $\mathcal{S}$ if and only if $\widehat{\mathcal{V}}$ is a (ϱ,σ) CIVNSBS of $\mathcal{S}\times\mathcal{S}$.*

The proof follows from Theorem 3.13

Theorem 4.8. *The homomorphic image of every (ϱ,σ) CIVNSBS is a (ϱ,σ) CIVNSBS.*

Proof. Let $H:\mathcal{S}_1\to\mathcal{S}_2$ be any homomorphism. Now, $H(u\varnothing_1 v)=H(u)\Box_1 H(v), H(u\varnothing_2 v)=H(u)\Box_2 H(v)$ and $H(u\varnothing_3 v)=H(u)\Box_3 H(v)$ for all $u,v\in\mathcal{S}_1$. Let $\widehat{\mathcal{V}}=H(\widehat{\Upsilon})$, $\widehat{\Upsilon}$ is any (ϱ,σ) CIVNSBS of $\mathcal{S}_1$. Let $H(u),H(v)\in\mathcal{S}_2$. Let $u\in H^{-1}(H(u))$ and $v\in H^{-1}(H(v))$ be such that $\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}=\sup\limits_{u\in H^{-1}(H(u))}\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}$ and $\widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}=\sup\limits_{u\in H^{-1}(H(v))}\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}$. Now,

$$
\begin{aligned}
\max\left[\widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_1 H(v))\cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}\widehat{(H(u)\Box_1 H(v))}},\widehat{\varrho}\right]&=\max\left[\sup_{u'\in H^{-1}(H(u)\Box_1 H(v))}\widehat{R^{\top}_{\Upsilon}}(u')\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u')}},\widehat{\varrho}\right]\\
&=\max\left[\sup_{u'\in H^{-1}(H(u\varnothing_1 v)}\widehat{R^{\top}_{\Upsilon}}(u')\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u')}},\widehat{\varrho}\right]\\
&=\max\left[\widehat{R^{\top}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u\varnothing_1 v)}},\widehat{\varrho}\right]\\
&\geqslant\min\left\{\widehat{R^{\top}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}},\widehat{R^{\top}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}},\widehat{\sigma}\right\}\\
&=\min\left\{\widehat{R^{\top}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}H(u)}},\widehat{R^{\top}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}H(u)}},\widehat{\sigma}\right\}.
\end{aligned}
$$

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Thus, $\max\left[\widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_1 H(v))\cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}\widehat{(H(u)\Box_1 H(v))}},\widehat{\varrho}\right]\geqslant\min\left\{\widehat{R^{\top}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}H(u)}},\widehat{R^{\top}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}H(v)}},\widehat{\sigma}\right\}$.
Similarly, $\max\left[\widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_2 H(v))\cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}\widehat{(H(u)\Box_2 H(v))}},\widehat{\varrho}\right]\geqslant\min\left\{\widehat{R^{\top}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}H(u)}},\widehat{R^{\top}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}H(v)}},\widehat{\sigma}\right\}$ and
$\max\left[\widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_3 H(v))\cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}\widehat{(H(u)\Box_3 H(v))}},\widehat{\varrho}\right]\geqslant\min\left\{\widehat{R^{\top}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}H(u)}},\widehat{R^{\top}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\top}_{\mathcal{V}}H(v)}},\widehat{\sigma}\right\}$.
Let $H(u),H(v)\in\mathcal{S}_2$. Let $u\in H^{-1}(H(u))$ and $v\in H^{-1}(H(v))$ be such that $\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}}=\sup_{u\in H^{-1}(H(u))}\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}}$ and $\widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v)}}=\sup_{u\in H^{-1}(H(v))}\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}}$. Now,

$$\begin{aligned}\max\left[\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_1 H(v))\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}\widehat{(H(u)\Box_1 H(v))}},\widehat{\varrho}\right]&=\max\left[\sup_{u'\in H^{-1}(H(u)\Box_1 H(v))}\widehat{R^{\mathcal{I}}_{\Upsilon}}(u')\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u')}},\widehat{\varrho}\right]\\&=\max\left[\sup_{u'\in H^{-1}(H(u\varnothing_1 v)}\widehat{R^{\mathcal{I}}_{\Upsilon}}(u')\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u')}},\widehat{\varrho}\right]\\&=\max\left[\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u\varnothing_1 v)}},\widehat{\varrho}\right]\\&\geqslant\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}}+\widehat{R^{\mathcal{I}}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v)}}}{2},\widehat{\sigma}\right\}\\&=\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}H(u)}}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}H(v)}}}{2},\widehat{\sigma}\right\}\end{aligned}$$

Thus, $\max\left[\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_1 H(v))\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}\widehat{(H(u)\Box_1 H(v))}},\widehat{\varrho}\right]\geqslant\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}H(u)}}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}H(v)}}}{2},\widehat{\sigma}\right\}$.
Similarly, $\max\left[\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_2 H(v))\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}\widehat{(H(u)\Box_2 H(v))}},\widehat{\varrho}\right]\geqslant\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}H(u)}}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}H(v)}}}{2},\widehat{\sigma}\right\}$
and
$\max\left[\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_3 H(v))\cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}\widehat{(H(u)\Box_3 H(v))}},\widehat{\varrho}\right]\geqslant\min\left\{\frac{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}H(u)}}+\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\mathcal{V}}H(v)}}}{2},\widehat{\sigma}\right\}$.
Let $u\in H^{-1}(H(u))$ and $v\in H^{-1}(H(v))$ be such that $\widehat{R^{\digamma}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(u)}}=\inf_{u\in H^{-1}(H(u))}\widehat{R^{\digamma}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(u)}}$ and $\widehat{R^{\digamma}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(v)}}=\inf_{u\in H^{-1}(H(v))}\widehat{R^{\digamma}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(u)}}$. Now,

$$\begin{aligned}\min\left[\widehat{R^{\digamma}_{\mathcal{V}}}(H(u)\Box_1 H(v))\cdot e^{i2\pi\Theta^{\digamma}_{\mathcal{V}}\widehat{(H(u)\Box_1 H(v))}},\widehat{\varrho}\right]&=\min\left[\inf_{u'\in H^{-1}(H(u)\Box_1 H(v))}\widehat{R^{\digamma}_{\Upsilon}}(u')\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(u')}},\widehat{\varrho}\right]\\&=\min\left[\inf_{u'\in H^{-1}(H(u\varnothing_1 v)}\widehat{R^{\digamma}_{\Upsilon}}(u')\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(u')}},\widehat{\varrho}\right]\\&=\min\left[\widehat{R^{\digamma}_{\Upsilon}}(u\varnothing_1 v)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(u\varnothing_1 v)}},\widehat{\varrho}\right]\\&\leqslant\max\left\{\widehat{R^{\digamma}_{\Upsilon}}(u)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(u)}},\widehat{R^{\digamma}_{\Upsilon}}(v)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\Upsilon}(v)}},\widehat{\sigma}\right\}\\&=\max\left\{\widehat{R^{\digamma}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\mathcal{V}}H(u)}},\widehat{R^{\digamma}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\mathcal{V}}H(v)}},\widehat{\sigma}\right\}.\end{aligned}$$

Thus, $\min\left[\widehat{R^{\digamma}_{\mathcal{V}}}(H(u)\Box_1 H(v))\cdot e^{i2\pi\Theta^{\digamma}_{\mathcal{V}}\widehat{(H(u)\Box_1 H(v))}},\widehat{\varrho}\right]\leqslant\max\left\{\widehat{R^{\digamma}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\mathcal{V}}H(u)}},\widehat{R^{\digamma}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\mathcal{V}}H(v)}},\widehat{\sigma}\right\}$.
Similarly, $\min\left[\widehat{R^{\digamma}_{\mathcal{V}}}(H(u)\Box_2 H(v))\cdot e^{i2\pi\Theta^{\digamma}_{\mathcal{V}}\widehat{(H(u)\Box_2 H(v))}},\widehat{\varrho}\right]\leqslant\max\left\{\widehat{R^{\digamma}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\mathcal{V}}H(u)}},\widehat{R^{\digamma}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\mathcal{V}}H(v)}},\widehat{\sigma}\right\}$ and $\min\left[\widehat{R^{\digamma}_{\mathcal{V}}}(H(u)\Box_3 H(v))\cdot e^{i2\pi\Theta^{\digamma}_{\mathcal{V}}\widehat{(H(u)\Box_3 H(v))}},\widehat{\varrho}\right]\leqslant\max\left\{\widehat{R^{\digamma}_{\mathcal{V}}}H(u)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\mathcal{V}}H(u)}},\widehat{R^{\digamma}_{\mathcal{V}}}H(v)\cdot e^{i2\pi\widehat{\Theta^{\digamma}_{\mathcal{V}}H(v)}},\widehat{\sigma}\right\}$. Hence $\widehat{\mathcal{V}}$ is a (ϱ,σ) CIVNSBS of $\mathcal{S}_2$.

Theorem 4.9. *The homomorphic pre-image of every (ϱ,σ) CIVNSBS is a (ϱ,σ) CIVNSBS.*

Proof. Let $H:\mathcal{S}_1\to\mathcal{S}_2$ be a homomrphism. Now, $H(u\varnothing_1 v)=H(u)\Box_1 H(v), H(u\varnothing_2 v)=H(u)\Box_2 H(v)$ and $H(u\varnothing_3 v)=H(u)\Box_3 H(v)$ for all $u,v\in\mathcal{S}_1$. Let $\widehat{\mathcal{V}}=H(\widehat{\Upsilon})$, where $\widehat{\mathcal{V}}$ is any (ϱ,σ) CIVNSBS of $\mathcal{S}_2$.

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Let $u, v \in \mathcal{S}_1$. Then $\max\{\widehat{R^{\top}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u\varnothing_1 v)}}, \widehat{\varrho}\} = \max\{\widehat{R^{\top}_{\mathcal{V}}}(H(u\varnothing_1 v)) \cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}(\widehat{H(u\varnothing_1 v)})}, \widehat{\varrho}\} = \max\{\widehat{R^{\top}_{\mathcal{V}}}(H(u)\Box_1 H(v)) \cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}(\widehat{H(u)\Box_1 H(v)})}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\top}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}\widehat{H(u)}}, \widehat{R^{\top}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\Theta^{\top}_{\mathcal{V}}\widehat{H(v)}}, \widehat{\sigma}\} = \min\{\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}, \widehat{\sigma}\}$. Thus, $\max\{\widehat{R^{\top}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\Theta^{\top}_{\Upsilon}\widehat{(u\varnothing_1 v)}}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\top}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(u)}}, \widehat{R^{\top}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\top}_{\Upsilon}(v)}}, \widehat{\sigma}\}$. Now, $\max\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon}\widehat{(u\varnothing_1 v)}}, \widehat{\varrho}\}$ $= \max\{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u\varnothing_1 v)) \cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}(\widehat{H(u\varnothing_1 v)})}, \widehat{\varrho}\} = \max\{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}(H(u)\Box_1 H(v)) \cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}(\widehat{H(u\varnothing_1 v)})}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}\widehat{H(u)}}, \widehat{R^{\mathcal{I}}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\Theta^{\mathcal{I}}_{\mathcal{V}}\widehat{H(v)}}, \widehat{\sigma}\} = \min\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}}, \widehat{R^{\mathcal{I}}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v)}}, \widehat{\sigma}\}$. Thus, $\max\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\Theta^{\mathcal{I}}_{\Upsilon}\widehat{(u\varnothing_1 v)}}, \widehat{\varrho}\} \geqslant \min\{\widehat{R^{\mathcal{I}}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(u)}}, \widehat{R^{\mathcal{I}}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{\mathcal{I}}_{\Upsilon}(v)}}, \widehat{\sigma}\}$. Now, $\min\{\widehat{R^{F}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u\varnothing_1 v)}}, \widehat{\varrho}\}$ $= \min\{\widehat{R^{F}_{\mathcal{V}}}(H(u\varnothing_1 v)) \cdot e^{i2\pi\Theta^{F}_{\mathcal{V}}(\widehat{H(u\varnothing_1 v)})}, \widehat{\varrho}\} = \min\{\widehat{R^{F}_{\mathcal{V}}}(H(u)\Box_1 H(v)) \cdot e^{i2\pi\Theta^{F}_{\mathcal{V}}(\widehat{H(u)\Box_1 H(v)})}, \widehat{\varrho}\} \leqslant \max\{\widehat{R^{F}_{\mathcal{V}}}H(u) \cdot e^{i2\pi\Theta^{F}_{\mathcal{V}}\widehat{H(u)}}, \widehat{R^{F}_{\mathcal{V}}}H(v) \cdot e^{i2\pi\Theta^{F}_{\mathcal{V}}\widehat{H(v)}}, \widehat{\sigma}\} = \max\{\widehat{R^{F}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u)}}, \widehat{R^{F}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v)}}, \widehat{\sigma}\}$. Thus, $\min\{\widehat{R^{F}_{\Upsilon}}(u\varnothing_1 v) \cdot e^{i2\pi\Theta^{F}_{\Upsilon}\widehat{(u\varnothing_1 v)}}, \widehat{\varrho}\} \leqslant \max\{\widehat{R^{F}_{\Upsilon}}(u) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(u)}}, \widehat{R^{F}_{\Upsilon}}(v) \cdot e^{i2\pi\widehat{\Theta^{F}_{\Upsilon}(v)}}, \widehat{\sigma}\}$. Hence $\widehat{\Upsilon}$ is a (ϱ, σ) CIVNSBS of $\mathcal{S}_1$.

5 Conclusion and future direction

This paper presents a new type of neutrosophic subbisemiring. By describing three grades in terms of a complex number, the complex neutrosophic subbisemiring presents a novel approach to the concept of three grades. By transforming three grades into a two-dimensional parameter, a complex form of three grades represents a paradigm shift. In order to understand complex neutrosophic subbisemiring, we first considered the basic set theoretic operations of complement, union, and intersection. As a result, set theory operations were introduced. We defined complex interval valued neutrosophic subbisemiring. We developed the concepts of level sets of CIVNSBS, CIVNNSBS and CIVNSBS. We introduced an approach for (ϱ, σ) CIVNSBS and CIVNNSBS over bisemiring. Our goal is to apply the Q-fuzzy set and anti Q-fuzzy set to bisemiring. It is also attempted to investigate the properties of various transformations. We are attempting to process images and signals as well as analyze data, so it makes sense to use F-transforms as a way to increase the application of new fuzzy structures. Therefore, in the future, we should think about the applications of cubic subbisemiring, soft set CIVNSBS and soft set CIVNNSBS.

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