# On the Approximation of 3-Monotone Functions in $L_P$ Spaces

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**Abstract**

The main goal of this study is to obtain a direct theorem for the simultaneous approximation of 3 monotone functions in $L_P$ by splines in the space $L_P$. Also, we have illustrated many examples to clarify the validity of our work.

**Keywords:** approximation theory; functional space; monotone; $L_P$ space

## 1. Introduction and Basic Definitions

A function $x:[a,b] \to R$ is said to be *k-monotone*, $k \geq 1$, on $[a,b]$ if and only if for all hoices of $k$ +1 distinct points $x_0, x_1, \dots, x_k$ in $[a,b]$ the inequality $x[x_0, x_1, \dots, x_k] > 0$, holds, where $x[x_0, x_1, \dots, x_k] = \sum_{j=0}^{k} \frac{f(x_j)}{\omega'(x_j)}$, denotes the kth divided difference of $x$ at $x_0, x_1, \dots, x_k$, **and** $\omega(y) = \prod_{j=0}^{k}(y - x_j)$.
Note that 1-monotone and 2-monotone functions are just non-decreasing and convex functions, respectively. We denote the class of all k-monotone functions on $[a,b]$ **by** $\Delta^k$. **If** $x \in C^k[a,b]$, then $x \in \Delta^k$ **if and only if** $x^{(k)}(y) \geq 0$, $y \in [a,b]$.
In recent years the first author has many papers dealing with the degree of approximation of nonnegative, of monotone and convex functions by algebraic polynomials and splines that are similarly nonnegative, monotone and convex, the so-called positive, monotone and convex approximation( see [2], [1]). Also, She has papers dealing with the degree of approximation of functions that change their positivity, monotonicity, or convexity finitely many times in [-1,1], by polynomials and splines that have the same changes at exactly the same points. This is the copositive comonotone and coconvex approximation( see [2], [3], [4]). On the other hand, very little is known on the degree of 3-monotone functions by 3-monotone polynomials and splines. We are aware of only one paper [5] by Konovalov and Leviatan on the uniform Jackson estimates of 3-monotone functions by 3-monotone splines. In this paper we will obtain the degree of simultaneous approximation of such function in $L_P$.
We need a few notations. Let $\|.\|_P$ $P > 0$ , denote the $L_P$ quasi norm. The *rth symmetric difference* of $x$ is given by

$$\Delta_h^r(x, y, [a,b]) := \Delta_h^r(f, x) := \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} x\left(y - \frac{rh}{2} + ih\right), & y \pm \frac{rh}{2} \in [a,b] \\ 0 & o.w. \end{cases}$$

Then the *rth usual modulus of smoothness* of $x \in L_P[a,b]$ is defined by $w_r(x, \delta, [a,b])_P := \sup_{0<h\leq\delta} \|\Delta_h^r(x,.)\|_{L_P[a,b]}, \delta \geq 0$

**Theorem1.1**: Let $x \in L^2{}_P(I)$, be 3- monotone function, and $m \in N$, set $t_i = t_{m,i}: a + i^{-1}m|I|$, $i = 0,1, \dots, m$, then there exists a 3- monotone quadratic spline $\sigma_{2,m}(x;.)$ with knots $t_i$, $i = 1, \dots, m-1$ such that

(1.1) $x''(t_{i-1}) \leq \sigma''_{2,m}(x;t) \leq x''(t_i)$, $t \in (t_{i-1}, t_i)$, $i = 1, \dots, m$ and

**(1.2)** $\left\|x(.) - \sigma_{2,m}(x;.)\right\|_P \leq Cm^{-2}|I|^2 w(x'', m^{-1}|I|)_P$ **and**

**(1.3)** $\left\|x'(.) - \sigma'_{2,m}(x;.)\right\|_P \leq \frac{7}{2} Cm^{-1}|I| w(x'', m^{-1}|I|)_P$ **and**

**(1.4)** $\left\|x''(.) - \sigma''_{2,m}(x;.)\right\|_P \leq w(x'', m^{-1}|I|)_P$.

## 1. Proof of the main result

First let us prove



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**Lemma2.1** : Let J=[a,b] and $m \in N$, and set $t_i = t_{m,i}: a + i^{-1}m$, i=0,1,…,m then for every function $x$ such that $x \in L_P$ and $x \in L^2{}_P(J)$, $\|x'\|_P \le 2m|J|^{-1}(b-a)^{1/P} \max_{0\le i\le m}|x(t_i)| + \frac{1}{2}m|J|^{-1}\|x"\|_P$.

**Proof: Let** $t_{k-1} \le t \le t_k$, $1 \le k \le m$, by Taylor`s formula $x(t) + x'(t)(t_j - t) = x(t_j) - \int_t^{t_j} x"(t)(t_j - \tau)d\tau$, j=k-1, k solving this system of linear equation for $x'(t)$, we obtain $x'(t) = (t_k - t_{k-1})^{-1}\sum_{j=k-1}^{k}(-1)^{k-j}\left(x(t_j) - \int_t^{t_j} x"(t)(t_j-\tau)d\tau\right)$

Hence

$(\int_a^b |x'(t)|^P dt)^{\frac{1}{P}} \le m|J|^{-1}\sum_{j=k-1}^{k}(-1)^{k-j}|x(t_j)|\,(\int_a^b |1|^P dt)^{\frac{1}{P}} + m|J|^{-1}[\int_a^b \left|\sum_{j=k-1}^{k}\int_t^{t_j}|x"(t)|(t_j-\tau)\right|^P dt]^{\frac{1}{P}}$

**and** $\|x"\|_P \le 2m|J|^{-1}(b-a)^{1/P}\max_{0\le i\le m}|x(t_i)| + \frac{1}{2}m|J|^{-1}\|x"\|_P$. This completes the proof.

**The proof of theorem 1.1**

First observe that (4) follows immediately from ( ) and we do not have to prove it separately , and that with no loss of generality we may assume that I=[0,1]. Note that if $w(x",h)_P = 0$ for some h > 0, then x is a quadratic polynomial and there is nothing and there is nothing to prove. thus , we assume that $w(x",h)_P > 0$, $0 < h \le 1$, also if m = 1, then we may take the quadratic polynomial $\frac{c}{2}t^2$, where $x"(0) \le c \le x"(1)$, thus we assume **m>1.**

First assume that $x(0) = x'(0) = x"(0)$ .Then x" is nonnegative and non decreasing . . Let $c = (c_1, \dots, c_m)$, and denote by $\sigma_2(.;c)$ the quadratic spline defined by

$\sigma"_2(t;c) = c_i\ t \in (t_{i-1}, t_i)$, $i = 1,, \dots, m$

$\sigma'_2(t;c) = \int_0^t \sigma"_2(\tau;c)\,d\tau$ **and** $\sigma_2(t;c) = \int_0^t \sigma'_2(\tau;c)\,d\tau \quad t \in [0,1]$

It follows that $\sigma_2(.;c)$ is linear in c . Let $e^{(1)} = (1,0,\dots,0), e^{(2)} = (0,1,\dots,0), \dots, e^{(m)} = (0,0,\dots,1)$, denote the usual unit vectors in $R^m$. Then it is easy to see that $\sigma'_2(t;e^{(i)}) = (t-t_{i-1})_+ - (t-t_i)_+$

$$= \begin{cases} 0, & 0 \le t \le t_{i-1} \\ t - t_{i-1}, & t_{i-1} < t < t_i \\ m^{-1}, & t_i \le t \le 1 \end{cases}$$

**(2.1)** $\sigma_2(t;e^{(i)}) = \frac{1}{2}((t-t_{i-1})_+^2 - (t-t_i)_+^2)$

$$= \begin{cases} 0, & 0 \le t \le t_{i-1} \\ \frac{1}{2}(t-t_{i-1})^2, & t_{i-1} < t < t_i \\ \frac{1}{2}m^{-2} + m^{-2}(t-t_i), & t_i \le t \le 1 \end{cases}$$

**For** $c \in R^m$**, set**

$\delta_2(x,t,c) = x(t) - \sigma_2(t,c), t \in [0,1]$

**Where**

**(2.2)** $\delta'_2(x,0,c) = \delta_2(x,0,c) = 0$

**Now let** $c_i^* = x"(t_i)$, $i = 0,1,\dots,m$ it follows that if $c = (c_1,\dots,c_m)$ is so that $c_{i-1}^* \le c_i \le c_i^*$, $i = 0,1,\dots,m$ then

**(2.3)** $\left\|\delta"_2(x,\dots,c)\right\|_P \le w(x",m^{-1})_P$

We are going to construct the required $\sigma_{2,m}(x;.)$ in stages step 1 .Let

$c_*^{(0)} = (c_1^*,\dots,c_m^*)$ **and** $c_*^{(i)} = c_*^{(0)} - \sum_{i=1}^{i}(c_l^* - c_{l-1}^*)e_1^l$, $1 \le i \le m$

That is

$$(2.4)\ c_*^i = \begin{cases} (c_1^*,\dots,c_m^*) & i = 0 \\ (c_0^*,\dots,c_{i-1}^*,c_{i+1}^*,\dots,c_m^*) & 1 \le i \le m-1 \\ (c_0^*,\dots,c_{m-1}^*) & i = m \end{cases}$$

The function $\delta'_2(x,\dots,c_*^i)$, $0 \le i \le m$, are increasing in the weak sense, i.e. non decreasing in $[0,t_i]$ and decreasing in the weak sense i.e non increasing in $[0,t_i]$. (we will continue to use increasing and decreasing in the weak sense without mentioning the latter), since

$\delta"_2(x,t,c_*^{(i)}) = x"(t) - c_{k-1}^* = x"(t) - x"(t_{k-1}) \ge 0$, $t \in (t_{k-1},t_k)$, $1 \le k \le i$

$\delta"_2(x,t,c_*^{(i)}) = x"(t) - c_k^* = x"(t) - x"(t_k) \le 0$, $t \in (t_{k-1},t_k)$, $i+1 \le k \le 1$

By (2.2) $\delta'_2(x,0,c_*^i) = 0$, $0 \le i \le m$, **hence** $\delta'_2(x,.,c_*^i) \ge 0$ and in particular $\delta_2(x,.,c_*^i)$ is increasing in $[0,t_i]$. Also for all $0 \le i \le m$

**(2.5)** $\delta'_2(x,t,c_*^i) = \delta'_2(x,t,c_*^{i+1})$, $t \in [0,t_i]$

$\delta'_2(x,t,c_*^i) = \delta'_2(x,t,c_*^{i+1})$, $t \in [t_i,1]$

**Let** $\tau_P(c) = (\int_0^1 |\delta_2(x,t,c)^P|dt)^{\frac{1}{P}} = \|\delta_2(x,t,c)\|_P$ Then by the above discussion, $\tau_P(c_*^i) \ge t_i$, $1 \le i \le m$. If $\tau_P(c_*^i) < 1$, then clearly $\delta'_2(x,\tau_P(c_*^i),c_*^i) = 0$, and since $\delta'_2(x,\dots,c_*^i)$ is decreasing in $[t_i,1]$, it follows that it is



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non negative in $[0,\tau_P(c_*^i)]$ and non positive in $[\tau_P(c_*^i),1]$, i.e, $\delta_2(x,t,c_*^i)$ is increasing in $[0,\tau_P(c_*^i)]$ and decreasing in $[\tau_P(c_*^i),1]$. Also, it is readily seen that for every $0 \le i \le m$

$\delta_2(x,t,c_*^i) = \delta_2(x,t,c_*^{i+1})$, $t \in [0,t_i]$

$\delta_2(x,t,c_*^i) \le \delta_2(x,t,c_*^{i+1})$, $t \in [t_i,1]$

**(2.6)**

We are ready to begin the construction. If

$$\delta_2(x,\tau_P(c_*^m),c_*^m) = \delta_2(x,1,c_*^m) \le \frac{1}{2}m^{-2}w(x",m^{-1}) \le \frac{1}{2}Cm^{-2}w(x",m^{-1})_P$$

Then we take $c^\wedge = (c_1^\wedge,\dots,c_m^\wedge) = c_*^m$ and set $\sigma_{2,m}(x,t) = \sigma_2(t,c^\wedge)$, $t \in [0,1]$. Then (1.1) and (1.2) hold, and as we will see towards the end of the proof, (1.3) will follow other wise

$$\delta_2(x,1,c_*^m) = \delta_2(x,\tau_P(c_*^m),c_*^m) > \frac{1}{2}m^{-2}w(x",m^{-1})_P \ge \frac{1}{2}Cm^{-2}w(x",m^{-1})_P$$

**Since** $\delta_2(x,.,c_*^0) \le 0$ then for some i, $0 \le i \le m-1$, **we have**

$$\delta_2(x,\tau_P(c_*^i),c_*^i) \le \frac{1}{2}Cm^{-2}w(x",m^{-1})_P < \delta_2(x,\tau_P(c_*^{i+1}),c_*^{i+1})$$

**Denote** $c_\varepsilon^i = c_*^i - \varepsilon e^{i+1}$, $\varepsilon \in R$. **Then**

$\delta_2^k(x,t,c_\varepsilon^i) = \delta_2^k(x,t,c_*^i) + \varepsilon\sigma_2^k(t,e^{i+1})$ $0 \le k \le,$ $t \in [0,1]$

**Hence for** $\varepsilon_i^* = c_{i+1}^* - c_i^*$

$\delta_2\left(x,t,c_{\varepsilon_i^*}^i\right) = \delta_2(x,t,c_*^{i+1})$, $t \in [0,1]$

**While** $\delta_2\left(x,t,c_0^i\right) = \delta_2(x,t,c_*^i)$, $t \in [0,1]$

By virtue of (1) ,(5) & (6) we conclude that for $0 \le \varepsilon \le \varepsilon_i^*$.

**(2.7)** $\delta_2'(x,t,c_*^i) = \delta_2'(x,t,c_\varepsilon^i) = \delta_2'(x,t,c_*^{i+1})$, $t \in [0,t_i]$

$\delta_2'(x,t,c_*^i) \le \delta_2'(x,t,c_\varepsilon^i) \le \delta_2'(x,t,c_*^{i+1})$, $t \in [t_i,1]$

**And**

**(2.8)** $\delta_2'(x,t,c_*^i) = \delta_2'(x,t,c_\varepsilon^i) = \delta_2'(x,t,c_*^{i+1})$, $t \in [0,t_i]$

$\delta_2(x,t,c_*^i) \le \delta_2(x,t,c_\varepsilon^i) \le \delta_2(x,t,c_*^{i+1})$, $t \in [t_i,1]$

Note that all the above vectors $c_\varepsilon^i$, are admissible, by continue there exists an $0 \le \varepsilon \le \varepsilon_i^*$ such that $\delta_2'(x,\tau_P(c_*^i),c_*^i) = 0$

**(2.9)** $\left(\int_a^b\left|\delta_2(x,t,c_\varepsilon^i)^P\right|dt\right)^{\frac{1}{P}} = \delta_2(x,\tau_P(c_\varepsilon^i),c_\varepsilon^i) \le \frac{1}{2}Cm^{-2}w(x",m^{-1})_P$

Next we show that $\delta_2(x,t,c_\varepsilon^i)$ can not be too small in $[0,\tau_P(c_\varepsilon^i)]$. To this end, in view (2.8). we only have to estimate $\delta_2(x,t,c_\varepsilon^i)$ from below in $[t_i,\tau_P(c_\varepsilon^i)]$. We claim that we only have to obtain an estimate in $[t_i,t_{i+1}]$. Indeed, our claim is self-evident if $\tau_P(c_\varepsilon^i) \le t_{i+1}$, and we may assume the opposite. Note that

$\delta_2'(x,t,c_\varepsilon^i) = \delta_2'(x,t,c_*^i)\varepsilon m^{-1}$, $t \in [t_{i+1},1]$

And this in turn implies that $\delta_2'(x,.,c_\varepsilon^i)$ is decreasing in that interval. Since $\delta_2'(x,\tau_P(c_\varepsilon^i),c_\varepsilon^i) \ge 0$, it follows that $\delta_2^i(x,t,c_\varepsilon^i) \ge 0$ for $t \in [t_{i+1},\tau_P(c_\varepsilon^i)]$, so that $\delta_2(x,t,c_\varepsilon^i)$ is increasing there. By Taylor's formula and (2.3)

$$\left(\int_a^b\left|\delta_2(x,t,c_\varepsilon^i)^P\right|dt\right)^{\frac{1}{P}} = \left(\int_a^b\left|x(t)-\sigma(c_\varepsilon^i,t)\right|^P dt\right)^{\frac{1}{P}}$$

$\delta_2'(x,t,c_\varepsilon^i) = \delta_2(x,t_i,c_\varepsilon^i) = \delta_2'(x,t_i,c_\varepsilon^i)(t-t_i) + \int_{t_i}^t \delta_2"(x,\theta,c_\varepsilon^i)(t-\theta)d\theta$

$\ge \left(\left|\int_{t_i}^t \delta_2"(x,\theta,c_\varepsilon^i)(t-\theta)d\theta\right|^P\right)^{\frac{1}{P}}$

$\ge -\int_{t_i}^t\left\|\delta_2"(x,\theta,c_\varepsilon^i)(t-\theta)d\theta\right\|_P$

$\ge \frac{-1}{2}m^{-2}\left\|\delta_2"(x,\theta,c_\varepsilon^i)\right\|_P$

$\ge \frac{-1}{2}m^{-2}w(x",m^{-1})_P \quad t_i \le t \le t_{i+1}$

**By (2.7), (2.8),** $\delta_2(x,t_i,c_\varepsilon^i) \ge 0$ **and** $\delta_2'(x,t_i,c_\varepsilon^i) \ge 0$ **hence**

**(2.10)** $\min_{i\in[0,\tau_P(c_\varepsilon^i)]} \delta_2(x,t,c_\varepsilon^i) \ge \frac{-1}{2}Cm^{-2}w(x",m^{-1})_P$

**Now,** $\tau_P(c_\varepsilon^i) \in [t_j,t_{j+1}]$ **where** $j \ge i$, **if** $\tau_P(c_\varepsilon^i) = t_{j+1}$. **Then we take**

$c^\wedge = (c_1^\wedge,\dots,c_m^\wedge) = c_\varepsilon^i$ **and set** $\sigma_{2,m}(x,t) = \sigma_2(t,c^\wedge)$, $t \in [0,1]$

Then 1.1 and 1.2 holds (the latter in $[0,t_{j+1}]$ also by (2.9)

(2.11) $x\left(t_{j+1}\right) - \sigma_{2,m}\left(x,t_{j+1}\right)\frac{1}{2}Cm^{-2}w(x",m^{-2})_P$ **and if j<m-1, then**

(2.12) $x'^{\left(t_{j+1}\right)} - \sigma'_{2,m}\left(x,t_{j+1}\right) = 0$



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Then $t_j \leq \tau_P(c_\varepsilon^i) \leq t_{j+1}$ we need so we fine –tuning, and we continue with step 1, we first note that
(2.13) $\delta_2'(x, t_j, c_\varepsilon^i) \geq 0$, $\varepsilon \geq 0$ for $j > i$,
we have explain it above , and if j=I then it follows immediately from (2.7) Set $c_{\varepsilon,\eta}^i = c_\varepsilon^i - \eta e^{j+1}$, $\eta \in R$.
It follows by 2.1 that $\delta_2'(x, t_j, c_{\varepsilon,\eta}^i) = \delta_2'(x, t_j, c_\varepsilon^i)$, **and** $\delta_2(x, t_j, c_{\varepsilon,\eta}^j) = \delta_2(x, t_j, c_\varepsilon^i)$.
**Also,** $\delta_2(x, t_{j+1}, c_{\varepsilon,\eta}^j)$ depends continuously on **η.**
**If** $j = i$ **then for** $\eta_i^* = \varepsilon_i^* - \varepsilon$, $c_{\varepsilon,\eta_i^*}^i = c_*^{i+1}$,
And $\delta_2(x, t_{j+1}, c_{\varepsilon,\eta}^j)$. **While** $\delta_2'(x, t, c_{\varepsilon,-\varepsilon}^i) = \delta_2'(x, t, c_*^i)$ $t \in [t_i, t_{i+1}]$. **For all** $-\varepsilon \leq \eta \leq \eta_i^*, c_{\varepsilon,\eta}^i$ **are admissible, and** $\delta_2'(x, t, c_*^i) \leq \delta_2'(x, t, c_{\varepsilon,\eta}^i) \leq \delta_2'(x, t, c_*^{i+1})$.
It follows from (2.7) and (2.8) that $\tau_P(c_*^i) \leq \tau_P(c_\varepsilon^i)$ hence $t_i \leq \tau_P(c_*^i) \leq t_{i+1}$ **since** $\delta_2'(x, \tau_P(c_*^i), c_*^i) = 0$ and $\delta_2(x, ., c_*^i)$, is decreasing in $[t_i, t_{i+1}]$, we see that $\delta_2'(x, t_{i+1}, c_*^i) \leq 0$. On the other hand $\delta_2'(x, t_{i+1}, c_*^{i+1}) \geq 0$. By continuity there exists $-\varepsilon \leq \eta \leq \eta_i^*$ such that
(2.14) $\delta_2'(x, t_{i+1}, c_{\varepsilon,\eta}^i) = 0$
Otherwise j > i. then for any $\eta \in R$, $\delta_2(x, t, c_{\varepsilon,\eta}^i) = \delta_2(x, t, c_\varepsilon^i)$ and $\delta_2(x, t, c_{\varepsilon,\eta}^j) = \delta_2(x, t, c_\varepsilon^i)$ $0 \leq t \leq t_j$.
Recall that $\delta_2(x, ., c_\varepsilon^i)$ is decreasing in $[t_{i+1}, 1]$
So that in particular $\delta_2'(x, t_{j+1}, c_{\varepsilon,0}^j) \leq \delta_2'(x, t_{j+1}, c_\varepsilon^i) \leq \delta_2'(x, \tau_P(c_*^i), c_*^i)$
On the other hand for $\eta_i^* = c_{j+1}^* - c_j^*$, it follows by (2.1)and (2.4) that

$$\delta_2''\left(x, t, c_{\varepsilon,\eta_j^*}^j\right) = \delta_2''(x, t, c_\varepsilon^i) + (c_{j+1}^* - c_j^*)\sigma_2''(t,)$$
$$= x''(t) - c_{j+1}^* + c_{j+1}^* + c_j^*$$
$$= x''(t) - c_j^*$$
$$\geq 0 \qquad t_j \leq t \leq t_{j+1}$$
$$\delta_2'(x, t, c_\varepsilon^j, \eta_i^*) = \delta_2'(x, t, c_\varepsilon^i) + \eta_j^* \, \sigma_2(t, e^{j+1})$$

Is increasing in $t \in [t_j, t_{j+1}]$. by virtue of (2.13) we conclude that $\delta_2'(x, t, c_\varepsilon^j, \eta_i^*) \geq 0$. Therefore there exists $0 \leq \eta \leq \eta_i^*$ such that
**(2.15)** $\delta_2(x, t_{j+1}, c_{\varepsilon,\eta}^i) = 0$
And $c_{\varepsilon,\eta}^i$ is admissible. Evidently $\delta_2(x, t, c_{\varepsilon,\eta}^i) = \delta_2(x, t, c_\varepsilon^i)$ $t \in [0, t_j]$
Thus we only have to estimate $\delta_2(x, ., c_{\varepsilon,\eta}^i)$, $t_j \leq t \leq t_{j-1}$
To this end

$$\delta_2(x, t, c_{\varepsilon,\eta}^j) = \delta_2(x, t, c_\varepsilon^i) + \eta \, \sigma_2(t, e^{j+1})$$
$$\leq \delta_2(x, \tau_P(c_\varepsilon^i), c_\varepsilon^i) + \eta_j^* \, \frac{1}{2}(t - t_j)^2$$
$$\leq \frac{1}{2} m^{-2} C w(x'', m^{-1})_P + \frac{1}{2} C m^{-2} w(x'', m^{-1})_P$$
$$= m^{-2} w(x'', m^{-1})$$

And by taylor's formula

$$\delta_2(x, t, c_{\varepsilon,\eta}^j) = \delta_2(x, t, c_\varepsilon^i) + \eta \, \sigma_2(t, e^{j+1})$$
$$\geq \delta_2(x, \tau_P(c_\varepsilon^i), c_\varepsilon^i) + \delta_2(x, \tau_P(c_\varepsilon^i), c_\varepsilon^i)(t - t_j) + \int_{\tau_P(c_\varepsilon^i)}^{t} \int_{(c_\varepsilon^i)}^{\tau} \delta_2''(x, \theta, c_\varepsilon^i) d\tau \, d\theta$$
$$\geq \delta_2(x, \tau_P(c_\varepsilon^i), c_\varepsilon^i) - \|\geq \delta_2(x, ., c_\varepsilon^i)\|_P \frac{1}{2} |t - \tau_P(c_\varepsilon^i)|^2$$
$$\geq \frac{1}{2} m^{-2} C w(x'', m^{-1})_P + \frac{1}{2} C m^{-2} w(x'', m^{-1})_P = 0$$

Here we have applied (2.9) and the fact that $\delta_2'(x, \tau_P(c_\varepsilon^i), c_\varepsilon^i) = 0$
In particular we have
(2.18) $0 \leq \delta_2(x, t_{m,j+1}, c_{\varepsilon,\eta}^j) \leq m^{-2} C w(x'', m^{-1})$
And combining with 2.9 and 2.10 we obtain that
$\min_{t \in [0, t_{j+1}]} \delta_2(x, t, c_{\varepsilon,\eta}^j) \geq \frac{1}{2} m^{-2} C w(x'', m^{-1})$,
$(\int_0^{t_{j-1}} |\delta_2(x, t, c_{\varepsilon,\eta}^j)|^P \, dt)^{\frac{1}{P}} \leq m^{-2} C w(x'', m^{-1})$,
Now let $c^\wedge = (c_1^\wedge, \dots, c_m^\wedge) = c_{\varepsilon,\eta}^j$ which, as we know, is admissible, and set $\sigma_{2,m}(x, t) = \sigma_2(t, c^\wedge)$, $t \in [0,1]$
Then (1.1) and (1.2) hold the latter in $[0, t_{j+1}]$, and by virtue of (2.11) and (2.12)and (1.15) and (1,18)



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$x(t_{j+1}) - \sigma_{2,m}(t_{j+1}, c^{\wedge}) = 0$,
**And**
**(2.19)** $0 \le (t_{j+1}) - \sigma_{2,m}(t_{j+1}, c^{\wedge}) \le m^{-2}Cw(x", m^{-1})$
We conclude step 1 by designating $j_1 = 1 + j$. *if* $j_1 < 1$, the $t_{j_1} < 1$, and we proceed to step 2, which is almost a mirror image of step 1.
**Step 2. Write** $t_j = t_{j_1} + i\ i = 0,1, \dots, m - j_1$, **and let** $c_*^{\wedge(0)} = (c_1^{\wedge}, \dots, c_{j_1}^{\wedge}, c_{j_1}^*, \dots, c_{m-1}^*)$, **for** $1 \le i \le m - j_1$, **set** $c_*^{\wedge(i)} = c_*^{\wedge(0)} + \sum_{l=1}^{i}(c_{j_1+l}^* - c_{j_1+l-1}^*)\, e^{(j_1+l)}$

$$= \begin{cases} (c_1^{\wedge}, \dots, c_{j_1}^{\wedge}, c_{j_1}^*, \dots, c_{m-1}^*) & ,i = 0 \\ (c_1^{\wedge}, \dots, c_{j_1}^{\wedge}, c_{j_1+1}^*, \dots, c_{j_1+1}^*, c_{j_1+1}^*, \dots, c_{m-1}^*) & ,1 \le i \le m - j_1 - 1 \\ (c_1^{\wedge}, \dots, c_{j_1}^{\wedge}, c_{j+1}^*, \dots, c_m^*) & i = m - j_1 \end{cases}$$

note that
(2.20) $c_*^{\wedge(m-j_1)} = c^{\wedge}$
By virtue of (2.12) and ( 2.15) we have
(2.21) $\sigma_2(x, t_0, c_*^{\wedge(i)}) = 0,\ 1 \le i \le m - j_1$
and it is readily seen that $\sigma_2(x,.,c_*^{\wedge(i)})$, $1 \le i \le m - j_1 - 1$ is decreasing in $[t_0, t_i]$ and increasing in $[t_i, 1]$, since $\delta_2^{"}(x, t, c_*^{\wedge(i)}) = x"(t) - c_{j_1+k}^* = x"(t) - x"(t_k) \le 0$, $t \in (t_{k-1}, t_k), 1 \le k \le i + 1$, **and** $\delta_2^{"}(x, t, c_*^{\wedge(i)}) = x"(t) - c_{j_1+k-1}^* = x"(t) - x"(t_{k-1}) \ge 0$, $t \in (t_{k-1}, t_k), i + 1 \le k \le m - j_1$, so by (2.21) , in particular, $\delta_2(x,.,c_*^{\wedge(0)})$ is nonnegative in $[t_i, 1]$, so that $\delta_2(x,.,c_*^{\wedge(0)})$ is increasing and by (2.18), nonnegative there, and $\delta_2(x,.,c_*^{\wedge(m-j_1)})$ is non positive in $[t_i, 1]$, so that $\delta_2(x,.,c_*^{\wedge(m-j_1)})$ is decreasing there. We proceed as in step 1, except that for an admissible $c \in R^m$,
we denote $\tau_{min}(c) = (\int_0^1 |\delta_2(x, \tau, c)|^P d\tau)^{\frac{1}{P}} = \min_{t_0 \le t \le 1} \delta_2(x, t, c)$
note that $\tau_{min}(c_*^{\wedge(i)}) \ge t_i$, $i = 0,1, \dots, m - j_1$, thus if

$$\delta_2(x, \tau_{min}(c_*^{\wedge(m-j_1)}), c_*^{\wedge(m-j_1)}) = \delta_2(x, 1, c_*^{\wedge(m-j_1)}) \ge \frac{1}{2}Cm^{-2}w(x", m^{-1})_P$$

Then we put $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_m) = c_*^{\wedge(m-j_1)}$, **and set** $\sigma_{2,m}(x; t) = \sigma_2(t; \tilde{c})$, $t \in [0,1]$
In view of 2.20 ,(1.1) and (1.2) are satisfied and step 2 is complete. otherwise

$$\delta_2(x, \tau_P(c_*^{\wedge(m-j_1)}), c_*^{\wedge(m-j_1)}) < \frac{-1}{2}Cm^{-2}w(x", m^{-1})_P$$

While as has been mentioned, $\delta_2(x,.,c_*^{\wedge(0)})$ is nonnegative in $[t_i, 1]$. Hence for some $0 \le i \le m - j_1$

$$\delta_2(x, \tau_P(c_*^{\wedge(i)}), c_*^{\wedge(i)}) \ge \frac{1}{2}m^{-2}w(x", m^{-1})_P > \delta_2(x, \tau_P(c_*^{\wedge(i+1)}), c_*^{\wedge(i)})$$

Repeating the considerations of step 1 in mirror image which we only out line below (leaving the details to the reader), there exists $c_{j_1+1}^* - c_{j_1+i+1}^* \le \varepsilon \le 0$, so the $c_\varepsilon^{\wedge(i)}$is admissible and
**(2.22)** $\delta_2(x, \tau_P(c_*^{\wedge(i)}), c_*^{\wedge(i)}) = \frac{-1}{2}m^{-2}Cw(x", m^{-1})_P$
As before $\delta_2(x,.,c_\varepsilon^{\wedge(i)})$ is decreasing in the interval $[t_i, \tau_P(\ c_\varepsilon^{\wedge(i)})]$, and in order to estimate it from above in $[t_i, \tau_P(\ c_\varepsilon^{\wedge(i)})]$, it suffices to estimate it in $[t_i, t_{i+1}]$.To this end we have by Taylor's formula
$\delta_2(x, t, c_\varepsilon^{\wedge(i)}) = \delta_2(x, t_i, c_\varepsilon^{\wedge(i)}) + \delta_2(x, t_i, c_\varepsilon^{\wedge(i)})(t - t_i) + \int_{t_i}^{t}\int_{t_i}^{\tau}\delta_2^{"}(x, \theta, c_\varepsilon^{\wedge(i)})d\tau\, d\theta$
$\le \delta_2(x, t_i, c_\varepsilon^{\wedge(i)}) + \|\delta_2^{"}(x,.,c^{\wedge})\|_P$
$\le cm^{-2}w(x", m^{-1})_P + \frac{1}{2}m^{-2}Cw(x", m^{-1})_P$
$= \frac{3}{2}m^{-2}Cw(x", m^{-1})_P$
where we have applied (2.19), and the fact $\delta_2'(x, t_i, c_\varepsilon^{\wedge(i)})$ for later reference we conclude that
(2.23) $(\int_a^b |\delta_2(x, t, c_\varepsilon^{\wedge(i)})|^P dt)^{\frac{1}{P}} \le \frac{3}{2}m^{-2}Cw(x", m^{-1})_P$
**Now**
$\tau_{min}(c_\varepsilon^{\wedge(i)}) \in [t_j, t_{j+1}], i \le j < m - j_1$, $\tau_{min}(c_\varepsilon^{(i)}) = t_{j+1}$
**Then we take** $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_m) = c_\varepsilon^{\wedge(i)}$, **and set** $\sigma_{2,m}(x; t) = \sigma_2(t; \tilde{c})$, $t \in [0,1]$
$\frac{-1}{2}m^{-2}cw(x", m^{-1}) \le x(t) - \sigma_2(t; \tilde{c}) \le \frac{3}{2}m^{-2}Cw(x", m^{-1})_P$, $0 \le t \le t_{j+1}$
And (2.24) $x(t_{j+1}) - \sigma_2(t_{j+1}; \tilde{c}) = \frac{-1}{2}m^{-2}Cw(x", m^{-1})_P$
Furthermore $x'^{(t_{j+1})} - \sigma_{2,m}'(t_{j+1}; \tilde{c}) = 0$



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And step 1 is complete. If on the other hand $\tau_{min}(c_\varepsilon^{\wedge(i)}) < t_{j+1}$, then we again need some fine – tuning and we continue with step 2. Just as in step 1. We find an $c^*_{j_1+j} - c^*_{j_1+j+1} - \varepsilon \geq \eta \geq -\varepsilon$ so that $c_{\varepsilon,\eta}^{\wedge(j)}$ is admissible and $\delta_2'(x, t_{j+1}, c_{\varepsilon,\eta}^{\wedge(j)}) = 0$ Compare with 2.14 & 2.15 again
$\delta_2(x,t,c_{\varepsilon,\eta}^{\wedge(j)}) = \delta_2(x,t,c_\varepsilon^{\wedge(i)}), t \in [t_0, t_j]$
So it suffices to estimate $\delta_2(x,.,c_{\varepsilon,\eta}^{\wedge(j)})$ **in** $t \in [t_j, t_{j+1}]$**.** To this end compare eith (2.16) & (2.17) we obtain by (2.22)

$$\delta_2(x,t,c_{\varepsilon,\eta}^{\wedge(j)}) = \delta_2(x,t,c_\varepsilon^{\wedge(i)}) + \eta\sigma_2(t, e^{j_1+j+1})$$
$$\geq \delta_2(x, \tau_{min}(c_\varepsilon^{\wedge(i)}), c_\varepsilon^{\wedge(i)}) + (c^*_{j_1+j} - c^*_{j_1+j+1})$$
$$\geq \frac{-1}{2} m^{-2} Cw(x", m^{-1})_P + \frac{-1}{2} m^{-2} Cw(x", m^{-1})_P$$
$$= m^{-2} Cw(x", m^{-1})_P$$
$$\delta_2(x,t,c_{\varepsilon,\eta}^{\wedge(i)}) \leq \delta_2(x, \tau_{\min}(c_\varepsilon^{\wedge(i)}), c_\varepsilon^{\wedge(i)}) + \delta_2'(x, \tau_{\min}(c_\varepsilon^{\wedge(i)}), c_\varepsilon^{\wedge(i)})$$
$$(t - \tau_{\min}(c_\varepsilon^{\wedge(i)})) + \int_{\tau_{\min}(c_\varepsilon^{\wedge(i)})}^{t} \int_{\tau_{\min}(c_\varepsilon^{\wedge(i)})}^{\tau} \delta_2^{"}(x, \theta, c_\varepsilon^{\wedge(i)}) d\tau\, d\theta$$
$$\leq \delta_2(x, \tau_{\min}(c_\varepsilon^{\wedge(i)}), c_\varepsilon^{\wedge(i)}) + (\left|\int^t \int^\tau \delta_2^{"}(x,.,c_\varepsilon^{\wedge(i)}) d\theta d\tau\right|^P)^{\frac{1}{P}}$$
$$\leq \delta_2(x, \tau_{\min}(c_\varepsilon^{\wedge(i)}), c_\varepsilon^{\wedge(i)}) + \frac{1}{2}\left\|\delta_2^{"}(x,.,c_\varepsilon^{\wedge(i)})\right\|_P (t - \tau_{\min}(c_\varepsilon^{\wedge(i)})^2$$

And
$\delta_2'(x, t_{j+1}, c_{\varepsilon,\eta}^{\wedge(i)}) = 0$
Here we have used the fact $\delta_2'(x, \tau_{\min}(c_\varepsilon^{\wedge(i)}), c_\varepsilon^{\wedge(i)}) = 0$
**(2.26)** $-m^{-2} Cw(x", m^{-1}) \leq \delta_2(x, t_{j+1}, c_{\varepsilon,\eta}^{\wedge(j)}) \leq 0$
**Now we write** $j_2 = j_2 + j + 1$**. denote** $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_m) = c_{\varepsilon,\eta}^{\wedge(j)}$**.**
Which is admissible, and set
$\sigma_{2,m}(x;t) = \sigma_2(t;\tilde{c})$**,** $t \in [0,1]$
Then it follows that (1.1)&(1.2) hold (the latter in $[0, t_{j_2}]$ **and** $x'(t_{j_2}) - \sigma_{2,m}^2(t;\tilde{c}) = 0$
Furthermore, by (2.24) and (2.26)
$-m^{-2} Cw(x", m^{-1}) \leq x(t_{j_2}) - \sigma_{2,m}(t_{j_2};\tilde{c}) \leq 0$
This completes step 2
If $j_2 = m$ then we are done. Other wise, $t_{m,j_2} < 1$, and we return to step 1, the only difference this time is that we have (2.27) (like (2.18)) indeed of $\delta_2(x,0,c)$. This accounts for the lower estimate of the right of $t_{j_2}$, **being** $-\frac{3}{2} m^{-2} Cw(x", m^{-1})$**,** just as in (2.23). However, the upper estimate of the newly constructed spline in that interval is $m^{-2} Cw(x", m^{-1})$, just as in (2.25) . We alternately repeat step 2&1 until we get to the end point . The upper and lower estimates each time we apply step 1, never exceed $m^{-2} Cw(x", m^{-1})$ and $-\frac{3}{2} m^{-2} Cw(x", m^{-1})$ respectively, and when we apply step 2 they never exceed $\frac{3}{2} m^{-2} Cw(x", m^{-1})$ **and** $-m^{-2} Cw(x", m^{-1})$ respectively. The construction in achieved in finitely many steps (at most m steps), then we obtain quadratic spline $\sigma_{2,m}(x,.)$ that satisfies (1.1)&(2.2)
In order to prove (1.3) we see that by virtue of lemma 1 .(1.2) and (1.4) yield
$\left\|x'(.) - \sigma_{2,m}'(x,.)\right\|_P \leq 2m\left\|x(.) - \sigma_{2,m}(x,.)\right\|_P + \frac{1}{2m}\left\|x^{"}(.) - \sigma_{2,m}^{"}(x,.)\right\|_P$
$\leq (2m\frac{3}{2}m^{-2} + \frac{1}{2}m^{-1}) Cw(x", m^{-1})_P$
$= \frac{7}{2} m^{-1} Cw(x", m^{-1})_P$**.**
The theorem under the additional assumption that $x(0) = x'(0) = x"(0) = 0$, for a general $x \in \Delta_+^3 L_P^2([0,1])$, we take $\tilde{x}(t) = x(t) - (0) - x'(0)t - \frac{1}{2}x"(0)t^2, t \in [0,1]$
This completes the proof.

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