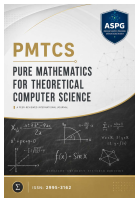
Pure Mathematics for Theoretical Computer Science

♦ ISSN: 2995-3162 Vol: 03, No. 01, 2024 ♦ PP. 01–07

Article DOI: https://doi.org/10.54216/PMTCS.030101



FULL LENGTH ARTICLE

On the Approximation of 3-Monotone Functions in L_P Spaces

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Received: June 19, 2023 Revised: September 18, 2023 Accepted: December 14, 2023 ⋆ Corresponding author

ABSTRACT

The main goal of this study is to obtain a direct theorem for the simultaneous approximation of 3-monotone functions in L_P by splines in the space L_P. Also, we have illustrated many examples to clarify the validity of our work.

Keywords: approximation theory ▪ functional space ▪ monotone ▪ L_P space

1. INTRODUCTION AND BASIC DEFINITIONS

A function $x:[a,b]\to\mathbb{R}$ is said to be k-monotone, $k\geq 1$, on $[a,b]$ if and only if, for all choices of $k+1$ distinct points $x_0,x_1,\ldots,x_k$ in $[a,b]$, the inequality

$$x[x_0,x_1,\ldots,x_k]>0$$

holds, where

$$x[x_0,x_1,\ldots,x_k]=\sum_{j=0}^{k}\frac{f(x_j)}{\omega'(x_j)}$$

denotes the kth divided difference of x at $x_0,x_1,\ldots,x_k$, and

$$\omega(y)=\prod_{j=0}^{k}(y-x_j).$$

Note that 1-monotone and 2-monotone functions are just nondecreasing and convex functions, respectively. We denote the class of all k-monotone functions on $[a,b]$ by Δ^k. If $x\in C^k[a,b]$, then $x\in\Delta^k$ if and only if

$$x^{(k)}(y)\geq 0,\qquad y\in[a,b].$$

In recent years the first author has many papers dealing with the degree of approximation of nonnegative, monotone, and convex functions by algebraic polynomials and splines that are similarly nonnegative, monotone, and convex, the so-called positive, monotone, and convex approximation (see [2, 1]). Also, she has papers dealing with the degree of approximation of functions that change their positivity, monotonicity, or convexity finitely many times in $[-1,1]$, by polynomials and splines that have the same changes at exactly the same points. This is the copositive, comonotone, and coconvex approximation (see [2, 3, 4]). On the other hand, very little is known on the degree of approximation of 3-monotone functions by 3-monotone polynomials and splines. We are aware of only one paper [5] by Konovalov and Leviatan on the uniform Jackson estimates of 3-monotone functions by 3-monotone splines. In this paper we will obtain the degree of simultaneous approximation of such a function in L_P.

We need a few notations. Let $\|\cdot\|_P$, $P>0$, denote the L_P quasi-norm. The rth symmetric difference of x is given by

$$\begin{aligned}\Delta_h^r(x,y,[a,b]) &:= \Delta_h^r(f,x)\\ &:=\begin{cases}\displaystyle\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}x\left(y-\frac{rh}{2}+ih\right), \\ \qquad\qquad y\pm\dfrac{rh}{2}\in[a,b],\\ 0,\quad \text{o.w.}\end{cases}\end{aligned}$$

Then the rth usual modulus of smoothness of $x\in L_P[a,b]$ is

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defined by

$$\omega_r(x,\delta,[a,b])_P := \sup_{0<h\le\delta} \|\Delta_h^r(x,\cdot)\|_{L_P[a,b]}, \qquad \delta \ge 0.$$

Theorem 1.1. Let $x \in L_P^2(I)$ be a 3-monotone function, and let $m \in \mathbb{N}$. Set

$$t_i = t_{m,i} := a + im^{-1}|I|, \qquad i = 0,1,\dots,m.$$

Then there exists a 3-monotone quadratic spline $\sigma_{2,m}(x;\cdot)$, with knots t_i, $i = 1,\dots,m-1$, such that

$$x''(t_{i-1}) \le \sigma_{2,m}''(x;t) \le x''(t_i), \quad t \in (t_{i-1},t_i), \quad i = 1,\dots,m. \tag{1.1}$$

$$\|x(\cdot) - \sigma_{2,m}(x;\cdot)\|_P \le Cm^{-2}|I|^2 \times \omega(x'',m^{-1}|I|)_P. \tag{1.2}$$

$$\left\|x'(\cdot) - \sigma_{2,m}'(x;\cdot)\right\|_P \le \frac{7}{2}Cm^{-1}|I| \times \omega(x'',m^{-1}|I|)_P. \tag{1.3}$$

$$\left\|x''(\cdot) - \sigma_{2,m}''(x;\cdot)\right\|_P \le \omega(x'',m^{-1}|I|)_P. \tag{1.4}$$

2. PROOF OF THE MAIN RESULT

First let us prove the following lemma.

Lemma 2.1. Let $J = [a,b]$, $m \in \mathbb{N}$, and set

$$t_i = t_{m,i} := a + im^{-1}|J|, \qquad i = 0,1,\dots,m.$$

Then, for every function x such that $x \in L_P$ and $x \in L_P^2(J)$,

$$\left\|x'\right\|_P \le 2m|J|^{-1}(b-a)^{1/P} \max_{0\le i\le m} |x(t_i)| + \frac{1}{2}m|J|^{-1}\left\|x''\right\|_P.$$

Proof. Let $t_{k-1} \le t \le t_k$, $1 \le k \le m$. By Taylor's formula,

$$x(t) + x'(t)(t_j - t) = x(t_j) - \int_t^{t_j} x''(\tau)(t_j - \tau)\,\mathrm{d}\tau, \qquad j = k-1, k.$$

Solving this system of linear equations for $x'(t)$, we obtain

$$x'(t) = (t_k - t_{k-1})^{-1} \sum_{j=k-1}^{k} (-1)^{k-j} \times \left(x(t_j) - \int_t^{t_j} x''(\tau)(t_j-\tau)\,\mathrm{d}\tau\right).$$

Hence

$$\left(\int_a^b |x'(t)|^P\,\mathrm{d}t\right)^{1/P} \le m|J|^{-1} \sum_{j=k-1}^{k} (-1)^{k-j}|x(t_j)| \times \left(\int_a^b 1^P\,\mathrm{d}t\right)^{1/P} + m|J|^{-1}\left[\int_a^b \left|\sum_{j=k-1}^{k}\int_t^{t_j} |x''(\tau)| \times (t_j-\tau)\,\mathrm{d}\tau\right|^P \mathrm{d}t\right]^{1/P}.$$

Therefore,

$$\left\|x'\right\|_P \le 2m|J|^{-1}(b-a)^{1/P} \max_{0\le i\le m} |x(t_i)| + \frac{1}{2}m|J|^{-1}\left\|x''\right\|_P.$$

This completes the proof.

2.1 Proof of Theorem 1.1

First observe that (1.4) follows immediately from (1.1), and we do not have to prove it separately. Without loss of generality, we may assume that $I = [0,1]$. Note that if $\omega(x'',h)_P = 0$ for some $h > 0$, then x is a quadratic polynomial and there is nothing to prove. Thus, we assume that $\omega(x'',h)_P > 0$, $0 < h \le 1$. Also, if $m = 1$, then we may take the quadratic polynomial $ct^2/2$, where $x''(0) \le c \le x''(1)$; thus, we assume $m > 1$.

First assume that

$$x(0) = x'(0) = x''(0) = 0.$$

Then x'' is nonnegative and non-decreasing. Let $c = (c_1,\dots,c_m)$, and denote by $\sigma_2(\cdot;c)$ the quadratic spline defined by

$$\sigma_2''(t;c) = c_i, \qquad t \in (t_{i-1},t_i), \quad i = 1,\dots,m,$$
$$\sigma_2'(t;c) = \int_0^t \sigma_2''(\tau;c)\,\mathrm{d}\tau,$$
$$\sigma_2(t;c) = \int_0^t \sigma_2'(\tau;c)\,\mathrm{d}\tau, \quad t \in [0,1].$$

It follows that $\sigma_2(\cdot;c)$ is linear in c. Let

$$e^{(1)} = (1,0,\dots,0), \quad e^{(2)} = (0,1,\dots,0), \quad \dots, \quad e^{(m)} = (0,0,\dots,1)$$

denote the usual unit vectors in $\mathbb{R}^m$. Then it is easy to see that

$$\sigma_2'(t;e^{(i)}) = (t-t_{i-1})_+ - (t-t_i)_+ = \begin{cases} 0, & 0 \le t \le t_{i-1}, \\ t - t_{i-1}, & t_{i-1} < t < t_i, \\ m^{-1}, & t_i \le t \le 1. \end{cases}$$

Moreover,

$$\sigma_2(t;e^{(i)}) = \frac{1}{2}\left((t-t_{i-1})_+^2 - (t-t_i)_+^2\right) = \begin{cases} 0, & 0 \le t \le t_{i-1}, \\ \frac{1}{2}(t-t_{i-1})^2, & t_{i-1} < t < t_i, \\ \frac{1}{2}m^{-2} + m^{-2}(t-t_i), & t_i \le t \le 1. \end{cases} \tag{2.1}$$

For $c \in \mathbb{R}^m$, set

$$\delta_2(x,t,c) = x(t) - \sigma_2(t,c), \qquad t \in [0,1].$$

Then

$$\delta_2'(x,0,c) = \delta_2(x,0,c) = 0. \tag{2.2}$$

Now let

$$c_i^* = x''(t_i), \qquad i = 0,1,\dots,m.$$

It follows that, if $c = (c_1,\dots,c_m)$ is such that

$$c_{i-1}^* \le c_i \le c_i^*, \qquad i = 1,\dots,m,$$

then

$$\left\|\delta_2''(x,\cdot,c)\right\|_P \le \omega(x'',m^{-1})_P. \tag{2.3}$$

We are going to construct the required $\sigma_{2,m}(x;\cdot)$ in stages.

Step 1. Let

$$c_*^{(0)} = (c_1^*,\dots,c_m^*),$$
$$c_*^{(i)} = c_*^{(0)} - \sum_{\ell=1}^{i}(c_\ell^* - c_{\ell-1}^*)e^{(\ell)}, \qquad 1 \le i \le m.$$

That is,

$$c_*^{(i)} = \begin{cases} (c_1^*, \dots, c_m^*), & i = 0, \\ (c_0^*, \dots, c_{i-1}^*, c_{i+1}^*, \dots, c_m^*), & 1 \le i \le m-1, \\ (c_0^*, \dots, c_{m-1}^*), & i = m. \end{cases} \quad (2.4)$$

The functions $\delta_2'(x,\cdot,c_*^{(i)})$, $0 \le i \le m$, are increasing in the weak sense (i.e., non-decreasing) in $[0,t_i]$ and decreasing in the weak sense (i.e., non-increasing) in $[t_i,1]$. We will continue to use increasing and decreasing in the weak sense without mentioning the latter. Indeed,

$$\begin{aligned} \delta_2''(x,t,c_*^{(i)}) &= x''(t) - c_{k-1}^* \\ &= x''(t) - x''(t_{k-1}) \ge 0, \\ &\quad t \in (t_{k-1},t_k), \quad 1 \le k \le i, \end{aligned}$$

$$\begin{aligned} \delta_2''(x,t,c_*^{(i)}) &= x''(t) - c_k^* \\ &= x''(t) - x''(t_k) \le 0, \\ &\quad t \in (t_{k-1},t_k), \quad i+1 \le k \le m. \end{aligned}$$

By (2.2), $\delta_2'(x,0,c_*^{(i)}) = 0$, $0 \le i \le m$; hence $\delta_2'(x,\cdot,c_*^{(i)}) \ge 0$, and in particular $\delta_2(x,\cdot,c_*^{(i)})$ is increasing in $[0,t_i]$. Also, for all $0 \le i \le m-1$,

$$\begin{aligned} \delta_2'(x,t,c_*^{(i)}) &= \delta_2'(x,t,c_*^{(i+1)}), \quad t \in [0,t_i], \\ \delta_2'(x,t,c_*^{(i)}) &\le \delta_2'(x,t,c_*^{(i+1)}), \quad t \in [t_i,1]. \end{aligned} \quad (2.5)$$

Let

$$\tau_P(c) = \left(\int_0^1 |\delta_2(x,t,c)|^P \, \mathrm{d}t \right)^{1/P} = \|\delta_2(x,\cdot,c)\|_P .$$

Then, by the above discussion, $\tau_P(c_*^{(i)}) \ge t_i$, $1 \le i \le m$. If $\tau_P(c_*^{(i)}) < 1$, then clearly

$$\delta_2'(x, \tau_P(c_*^{(i)}), c_*^{(i)}) = 0.$$

Since $\delta_2'(x,\cdot,c_*^{(i)})$ is decreasing in $[t_i,1]$, it follows that it is nonnegative in $[0,\tau_P(c_*^{(i)})]$ and nonpositive in $[\tau_P(c_*^{(i)}),1]$. Thus, $\delta_2(x,t,c_*^{(i)})$ is increasing in $[0,\tau_P(c_*^{(i)})]$ and decreasing in $[\tau_P(c_*^{(i)}),1]$. It is also readily seen that, for every $0 \le i \le m-1$,

$$\begin{aligned} \delta_2(x,t,c_*^{(i)}) &= \delta_2(x,t,c_*^{(i+1)}), \quad t \in [0,t_i], \\ \delta_2(x,t,c_*^{(i)}) &\le \delta_2(x,t,c_*^{(i+1)}), \quad t \in [t_i,1]. \end{aligned} \quad (2.6)$$

We are ready to begin the construction. If

$$\begin{aligned} \delta_2(x, \tau_P(c_*^{(m)}), c_*^{(m)}) &= \delta_2(x,1,c_*^{(m)}) \\ &\le \frac{1}{2} m^{-2} \omega(x'', m^{-1}) \\ &\le \frac{1}{2} C m^{-2} \omega(x'', m^{-1})_P, \end{aligned}$$

then we take $\widehat{c} = (\widehat{c}_1, \dots, \widehat{c}_m) = c_*^{(m)}$ and set

$$\sigma_{2,m}(x,t) = \sigma_2(t,\widehat{c}), \qquad t \in [0,1].$$

Then (1.1) and (1.2) hold, and, as we will see toward the end of the proof, (1.3) follows. Otherwise,

$$\begin{aligned} \delta_2(x,1,c_*^{(m)}) &= \delta_2(x, \tau_P(c_*^{(m)}), c_*^{(m)}) \\ &> \frac{1}{2} m^{-2} \omega(x'', m^{-1})_P \ge \frac{1}{2} C m^{-2} \omega(x'', m^{-1})_P. \end{aligned}$$

Since $\delta_2(x,\cdot,c_*^{(0)}) \le 0$, for some i, $0 \le i \le m-1$, we have

$$\begin{aligned} \delta_2(x, \tau_P(c_*^{(i)}), c_*^{(i)}) &\le \frac{1}{2} C m^{-2} \omega(x'', m^{-1})_P \\ &< \delta_2(x, \tau_P(c_*^{(i+1)}), c_*^{(i+1)}). \end{aligned}$$

Denote

$$c_\varepsilon^i = c_*^{(i)} - \varepsilon e^{(i+1)}, \qquad \varepsilon \in \mathbb{R}.$$

Then

$$\begin{aligned} \delta_2^{(k)}(x,t,c_\varepsilon^i) &= \delta_2^{(k)}(x,t,c_*^{(i)}) \\ &\quad + \varepsilon \sigma_2^{(k)}(t, e^{(i+1)}), \\ &\qquad 0 \le k \le 2, \quad t \in [0,1]. \end{aligned}$$

Hence, for $\varepsilon_i^* = c_{i+1}^* - c_i^*$,

$$\delta_2(x,t,c_{\varepsilon_i^*}^i) = \delta_2(x,t,c_*^{(i+1)}), \qquad t \in [0,1],$$

while

$$\delta_2(x,t,c_0^i) = \delta_2(x,t,c_*^{(i)}), \qquad t \in [0,1].$$

By virtue of (2.1), (2.5), and (2.6), for $0 \le \varepsilon \le \varepsilon_i^*$,

$$\begin{aligned} \delta_2'(x,t,c_*^{(i)}) &= \delta_2'(x,t,c_\varepsilon^i) = \delta_2'(x,t,c_*^{(i+1)}), \quad t \in [0,t_i], \\ \delta_2'(x,t,c_*^{(i)}) &\le \delta_2'(x,t,c_\varepsilon^i) \le \delta_2'(x,t,c_*^{(i+1)}), \quad t \in [t_i,1]. \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \delta_2(x,t,c_*^{(i)}) &= \delta_2(x,t,c_\varepsilon^i) = \delta_2(x,t,c_*^{(i+1)}), \quad t \in [0,t_i], \\ \delta_2(x,t,c_*^{(i)}) &\le \delta_2(x,t,c_\varepsilon^i) \le \delta_2(x,t,c_*^{(i+1)}), \quad t \in [t_i,1]. \end{aligned} \quad (2.8)$$

All the above vectors c_ε^i are admissible. By continuity, there exists $0 \le \varepsilon \le \varepsilon_i^*$ such that

$$\delta_2'(x, \tau_P(c_\varepsilon^i), c_\varepsilon^i) = 0$$

and

$$\begin{aligned} \left(\int_a^b |\delta_2(x,t,c_\varepsilon^i)|^P \, \mathrm{d}t \right)^{1/P} &= \delta_2(x, \tau_P(c_\varepsilon^i), c_\varepsilon^i) \\ &\le \frac{1}{2} C m^{-2} \omega(x'', m^{-1})_P. \end{aligned} \quad (2.9)$$

Next we show that $\delta_2(x,t,c_\varepsilon^i)$ cannot be too small in $[0,\tau_P(c_\varepsilon^i)]$. In view of (2.8), we only have to estimate it from below in $[t_i,\tau_P(c_\varepsilon^i)]$. We claim that it is enough to obtain an estimate in $[t_i,t_{i+1}]$. Indeed, this is self-evident if $\tau_P(c_\varepsilon^i) \le t_{i+1}$, and we may assume the opposite. Note that

$$\delta_2'(x,t,c_\varepsilon^i) = \delta_2'(x,t,c_*^{(i)}) + \varepsilon m^{-1}, \qquad t \in [t_{i+1},1].$$

This implies that $\delta_2'(x,\cdot,c_\varepsilon^i)$ is decreasing in that interval.

Since

$$\delta_2'(x,\tau_P(c_\varepsilon^i),c_\varepsilon^i)\ge 0,$$

it follows that $\delta_2'(x,t,c_\varepsilon^i)\ge 0$ for $t\in[t_{i+1},\tau_P(c_\varepsilon^i)]$, so that $\delta_2(x,t,c_\varepsilon^i)$ is increasing there.

By Taylor's formula and (2.3),

$$\begin{aligned}\delta_2(x,t,c_\varepsilon^i)&=\delta_2(x,t_i,c_\varepsilon^i)+\delta_2'(x,t_i,c_\varepsilon^i)(t-t_i)\\&\quad+\int_{t_i}^t\delta_2''(x,\theta,c_\varepsilon^i)(t-\theta)\,\mathrm{d}\theta\\&\ge-\int_{t_i}^t\left\|\delta_2''(x,\theta,c_\varepsilon^i)(t-\theta)\right\|_P\,\mathrm{d}\theta\\&\ge-\frac{1}{2}m^{-2}\left\|\delta_2''(x,\cdot,c_\varepsilon^i)\right\|_P\\&\ge-\frac{1}{2}m^{-2}\omega(x'',m^{-1})_P,\qquad t_i\le t\le t_{i+1}.\end{aligned}$$

By (2.7) and (2.8), $\delta_2(x,t_i,c_\varepsilon^i)\ge 0$ and $\delta_2'(x,t_i,c_\varepsilon^i)\ge 0$; hence

$$\min_{t\in[0,\tau_P(c_\varepsilon^i)]}\delta_2(x,t,c_\varepsilon^i)\ge-\frac{1}{2}Cm^{-2}\omega(x'',m^{-1})_P.\tag{2.10}$$

Now $\tau_P(c_\varepsilon^i)\in[t_j,t_{j+1}]$, where $j\ge i$. If $\tau_P(c_\varepsilon^i)=t_{j+1}$, then we take

$$\widehat{c}=(\widehat{c}_1,\dots,\widehat{c}_m)=c_\varepsilon^i,\qquad\sigma_{2,m}(x,t)=\sigma_2(t,\widehat{c}),\quad t\in[0,1].$$

Then (1.1) and (1.2) hold, the latter in $[0,t_{j+1}]$. Also, by (2.9),

$$x(t_{j+1})-\sigma_{2,m}(x,t_{j+1})\le\frac{1}{2}Cm^{-2}\omega(x'',m^{-2})_P,\tag{2.11}$$

and, if $j<m-1$,

$$x'(t_{j+1})-\sigma_{2,m}'(x,t_{j+1})=0.\tag{2.12}$$

Suppose now that $t_j\le\tau_P(c_\varepsilon^i)\le t_{j+1}$; we need some fine-tuning and continue with Step 1. First note that

$$\delta_2'(x,t_j,c_\varepsilon^i)\ge 0,\qquad\varepsilon\ge 0,\quad j>i.\tag{2.13}$$

This was explained above, and if $j=i$, it follows immediately from (2.7). Set

$$c_{\varepsilon,\eta}^i=c_\varepsilon^i-\eta e^{(j+1)},\qquad\eta\in\mathbb{R}.$$

It follows from (2.1) that

$$\begin{aligned}\delta_2'(x,t_j,c_{\varepsilon,\eta}^i)&=\delta_2'(x,t_j,c_\varepsilon^i),\\\delta_2(x,t_j,c_{\varepsilon,\eta}^i)&=\delta_2(x,t_j,c_\varepsilon^i).\end{aligned}$$

Also, $\delta_2(x,t_{j+1},c_{\varepsilon,\eta}^i)$ depends continuously on η.

If $j=i$, then for $\eta_i^*=\varepsilon_i^*-\varepsilon$,

$$c_{\varepsilon,\eta_i^*}^i=c_*^{(i+1)},$$

while

$$\delta_2'(x,t,c_{\varepsilon,-\varepsilon}^i)=\delta_2'(x,t,c_*^{(i)}),\qquad t\in[t_i,t_{i+1}].$$

For all $-\varepsilon\le\eta\le\eta_i^*$, the vectors $c_{\varepsilon,\eta}^i$ are admissible and

$$\delta_2'(x,t,c_*^{(i)})\le\delta_2'(x,t,c_{\varepsilon,\eta}^i)\le\delta_2'(x,t,c_*^{(i+1)}).$$

It follows from (2.7) and (2.8) that

$$\tau_P(c_*^{(i)})\le\tau_P(c_\varepsilon^i),\qquad t_i\le\tau_P(c_*^{(i)})\le t_{i+1}.$$

Since $\delta_2'(x,\tau_P(c_*^{(i)}),c_*^{(i)})=0$ and $\delta_2'(x,\cdot,c_*^{(i)})$ is decreasing in $[t_i,t_{i+1}]$, we see that

$$\delta_2'(x,t_{i+1},c_*^{(i)})\le 0.$$

On the other hand, $\delta_2'(x,t_{i+1},c_*^{(i+1)})\ge 0$. By continuity there exists $-\varepsilon\le\eta\le\eta_i^*$ such that

$$\delta_2'(x,t_{i+1},c_{\varepsilon,\eta}^i)=0.\tag{2.14}$$

Otherwise $j>i$. Then, for any $\eta\in\mathbb{R}$,

$$\delta_2(x,t,c_{\varepsilon,\eta}^j)=\delta_2(x,t,c_\varepsilon^i),\qquad 0\le t\le t_j.$$

Recall that $\delta_2(x,\cdot,c_\varepsilon^i)$ is decreasing in $[t_{i+1},1]$. Therefore,

$$\delta_2'(x,t_{j+1},c_{\varepsilon,0}^j)\le\delta_2'(x,t_{j+1},c_\varepsilon^i)\le\delta_2'(x,\tau_P(c_\varepsilon^i),c_\varepsilon^i)=0.$$

On the other hand, for $\eta_j^*=c_{j+1}^*-c_j^*$, it follows from (2.1) and (2.4) that

$$\begin{aligned}\delta_2''(x,t,c_{\varepsilon,\eta_j^*}^j)&=\delta_2''(x,t,c_\varepsilon^i)+(c_{j+1}^*-c_j^*)\sigma_2''(t,e^{(j+1)})\\&=x''(t)-c_{j+1}^*+c_{j+1}^*-c_j^*\\&=x''(t)-c_j^*\ge 0,\qquad t_j\le t\le t_{j+1}.\end{aligned}$$

Also,

$$\delta_2'(x,t,c_{\varepsilon,\eta_j^*}^j)=\delta_2'(x,t,c_\varepsilon^i)+\eta_j^*\sigma_2'(t,e^{(j+1)})$$

is increasing in $t\in[t_j,t_{j+1}]$. By virtue of (2.13), we conclude that it is nonnegative. Therefore, there exists $0\le\eta\le\eta_j^*$ such that

$$\delta_2'(x,t_{j+1},c_{\varepsilon,\eta}^j)=0.\tag{2.15}$$

The vector $c_{\varepsilon,\eta}^j$ is admissible. Evidently,

$$\delta_2(x,t,c_{\varepsilon,\eta}^j)=\delta_2(x,t,c_\varepsilon^i),\qquad t\in[0,t_j].$$

Thus, we only have to estimate $\delta_2(x,\cdot,c_{\varepsilon,\eta}^j)$ for $t_j\le t\le t_{j+1}$. To this end,

$$\begin{aligned}\delta_2(x,t,c_{\varepsilon,\eta}^j)&=\delta_2(x,t,c_\varepsilon^i)+\eta\sigma_2(t,e^{(j+1)})\\&\le\delta_2(x,\tau_P(c_\varepsilon^i),c_\varepsilon^i)+\eta_j^*\frac{1}{2}(t-t_j)^2\\&\le\frac{1}{2}m^{-2}C\omega(x'',m^{-1})_P+\frac{1}{2}Cm^{-2}\omega(x'',m^{-1})_P\\&=m^{-2}C\omega(x'',m^{-1})_P.\end{aligned}$$

By Taylor's formula,

$$\begin{aligned}\delta_2(x,t,c^j_{\varepsilon,\eta}) &\geq \delta_2(x,\tau_P(c^i_\varepsilon),c^i_\varepsilon)\\ &\quad+\delta_2'(x,\tau_P(c^i_\varepsilon),c^i_\varepsilon)(t-t_j)\\ &\quad+\int_{\tau_P(c^i_\varepsilon)}^{t}\int_{\tau_P(c^i_\varepsilon)}^{\tau}\delta_2''(x,\theta,c^i_\varepsilon)\,\mathrm{d}\theta\,\mathrm{d}\tau\\ &\geq \delta_2(x,\tau_P(c^i_\varepsilon),c^i_\varepsilon)-\frac{1}{2}\left\|\delta_2''(x,\cdot,c^i_\varepsilon)\right\|_P\\ &\qquad\times|t-\tau_P(c^i_\varepsilon)|^2\\ &\geq -m^{-2}C\omega(x'',m^{-1})_P.\end{aligned}$$

Here we have applied (2.9) and the fact that

$$\delta_2'(x,\tau_P(c^i_\varepsilon),c^i_\varepsilon)=0.$$

In particular, we have

$$0\leq\delta_2(x,t_{m,j+1},c^j_{\varepsilon,\eta})\leq m^{-2}C\omega(x'',m^{-1}). \tag{2.18}$$

Combining (2.9) and (2.10), we obtain

$$\min_{t\in[0,t_{j+1}]}\delta_2(x,t,c^j_{\varepsilon,\eta})\geq-\frac{1}{2}m^{-2}C\omega(x'',m^{-1}),$$

$$\left(\int_0^{t_{j+1}}|\delta_2(x,t,c^j_{\varepsilon,\eta})|^P\,\mathrm{d}t\right)^{1/P}\leq m^{-2}C\omega(x'',m^{-1}).$$

Now let

$$\widehat{c}=(\widehat{c}_1,\dots,\widehat{c}_m)=c^j_{\varepsilon,\eta},\qquad\sigma_{2,m}(x,t)=\sigma_2(t,\widehat{c}),\quad t\in[0,1].$$

Then (1.1) and (1.2) hold, the latter in $[0,t_{j+1}]$. By virtue of (2.11), (2.12), (2.15), and (2.18),

$$x(t_{j+1})-\sigma_{2,m}(t_{j+1},\widehat{c})=0,$$

and

$$0\leq x(t_{j+1})-\sigma_{2,m}(t_{j+1},\widehat{c})\leq m^{-2}C\omega(x'',m^{-1}). \tag{2.19}$$

We conclude Step 1 by designating $j_1=1+j$. If $j_1<m$, then $t_{j_1}<1$, and we proceed to Step 2, which is almost a mirror image of Step 1.

Step 2. Write $t_i=t_{j_1+i}$, $i=0,1,\dots,m-j_1$, and let

$$\widehat{c}^{(0)}_*=(\widehat{c}_1,\dots,\widehat{c}_{j_1},c^*_{j_1},\dots,c^*_{m-1}).$$

For $1\leq i\leq m-j_1$, set

$$\widehat{c}^{(i)}_*=\widehat{c}^{(0)}_*+\sum_{\ell=1}^{i}(c^*_{j_1+\ell}-c^*_{j_1+\ell-1})e^{(j_1+\ell)}.$$

Equivalently,

$$\begin{aligned}\widehat{c}^{(0)}_*&=(\widehat{c}_1,\dots,\widehat{c}_{j_1},c^*_{j_1},\dots,c^*_{m-1}),\\ \widehat{c}^{(i)}_*&=(\widehat{c}_1,\dots,\widehat{c}_{j_1},c^*_{j_1+1},\dots,\\ &\qquad c^*_{j_1+i},c^*_{j_1+i},\dots,c^*_{m-1}),\\ &\qquad 1\leq i\leq m-j_1-1,\\ \widehat{c}^{(m-j_1)}_*&=(\widehat{c}_1,\dots,\widehat{c}_{j_1},c^*_{j_1+1},\dots,c^*_m).\end{aligned}$$

Note that

$$\widehat{c}^{(m-j_1)}_*=\widehat{c}. \tag{2.20}$$

By virtue of (2.12) and (2.15), we have

$$\sigma_2(x,t_0,\widehat{c}^{(i)}_*)=0,\qquad 1\leq i\leq m-j_1. \tag{2.21}$$

It is readily seen that $\sigma_2(x,\cdot,\widehat{c}^{(i)}_*)$, $1\leq i\leq m-j_1-1$, is decreasing in $[t_0,t_i]$ and increasing in $[t_i,1]$, since

$$\begin{aligned}\delta_2''(x,t,\widehat{c}^{(i)}_*)&=x''(t)-c^*_{j_1+k}\\ &=x''(t)-x''(t_k)\leq 0,\\ &\quad t\in(t_{k-1},t_k),\quad 1\leq k\leq i+1,\end{aligned}$$

$$\begin{aligned}\delta_2''(x,t,\widehat{c}^{(i)}_*)&=x''(t)-c^*_{j_1+k-1}\\ &=x''(t)-x''(t_{k-1})\geq 0,\\ &\quad t\in(t_{k-1},t_k),\quad i+1\leq k\leq m-j_1.\end{aligned}$$

Thus, by (2.21), $\delta_2(x,\cdot,\widehat{c}^{(0)}_*)$ is nonnegative in $[t_i,1]$, so that it is increasing and, by (2.18), nonnegative there. Also, $\delta_2(x,\cdot,\widehat{c}^{(m-j_1)}_*)$ is nonpositive in $[t_i,1]$, so it is decreasing there. We proceed as in Step 1, except that, for an admissible $c\in\mathbb{R}^m$, we denote

$$\tau_{\min}(c)=\left(\int_0^1|\delta_2(x,\tau,c)|^P\,\mathrm{d}\tau\right)^{1/P}=\min_{t_0\leq t\leq 1}\delta_2(x,t,c).$$

Note that $\tau_{\min}(\widehat{c}^{(i)}_*)\geq t_i$, $i=0,1,\dots,m-j_1$. Thus, if

$$\begin{aligned}\delta_2(x,\tau_{\min}(\widehat{c}^{(m-j_1)}_*),\widehat{c}^{(m-j_1)}_*)&=\delta_2(x,1,\widehat{c}^{(m-j_1)}_*)\\ &\geq\frac{1}{2}Cm^{-2}\omega(x'',m^{-1})_P,\end{aligned}$$

then we put

$$\begin{aligned}\widetilde{c}&=(\widetilde{c}_1,\dots,\widetilde{c}_m)=\widehat{c}^{(m-j_1)}_*,\\ \sigma_{2,m}(x;t)&=\sigma_2(t;\widetilde{c}),\quad t\in[0,1].\end{aligned}$$

In view of (2.20), (1.1) and (1.2) are satisfied, and Step 2 is complete. Otherwise,

$$\delta_2(x,\tau_P(\widehat{c}^{(m-j_1)}_*),\widehat{c}^{(m-j_1)}_*)<-\frac{1}{2}Cm^{-2}\omega(x'',m^{-1})_P.$$

While, as mentioned, $\delta_2(x,\cdot,\widehat{c}^{(0)}_*)$ is nonnegative in $[t_i,1]$. Hence, for some $0\leq i\leq m-j_1$,

$$\begin{aligned}\delta_2(x,\tau_P(\widehat{c}^{(i)}_*),\widehat{c}^{(i)}_*)&\geq\frac{1}{2}m^{-2}\omega(x'',m^{-1})_P\\ &>\delta_2(x,\tau_P(\widehat{c}^{(i+1)}_*),\widehat{c}^{(i)}_*).\end{aligned}$$

Repeating the considerations of Step 1 in mirror image, which we only outline below, there exists

$$c^*_{j_1+1}-c^*_{j_1+i+1}\leq\varepsilon\leq 0$$

such that $\widehat{c}^{(i)}_\varepsilon$ is admissible and

$$\delta_2(x,\tau_P(\widehat{c}^{(i)}_*),\widehat{c}^{(i)}_*)=-\frac{1}{2}m^{-2}C\omega(x'',m^{-1})_P. \tag{2.22}$$

As before, $\delta_2(x,\cdot,\widehat{c}^{(i)}_\varepsilon)$ is decreasing in the interval

$[t_i, \tau_P(\widehat{c}_\varepsilon^{(i)})]$, and, in order to estimate it from above in that interval, it suffices to estimate it in $[t_i, t_{i+1}]$. By Taylor's formula,

$$\begin{aligned}\delta_2(x,t,\widehat{c}_\varepsilon^{(i)}) &= \delta_2(x,t_i,\widehat{c}_\varepsilon^{(i)}) + \delta_2'(x,t_i,\widehat{c}_\varepsilon^{(i)})(t-t_i)\\ &\quad + \int_{t_i}^{t}\int_{t_i}^{\tau} \delta_2''(x,\theta,\widehat{c}_\varepsilon^{(i)})\,\mathrm{d}\theta\,\mathrm{d}\tau\\ &\le \delta_2(x,t_i,\widehat{c}_\varepsilon^{(i)}) + \left\|\delta_2''(x,\cdot,\widehat{c})\right\|_P\\ &\le Cm^{-2}\omega(x'',m^{-1})_P + \frac{1}{2}m^{-2}C\omega(x'',m^{-1})_P\\ &= \frac{3}{2}m^{-2}C\omega(x'',m^{-1})_P.\end{aligned}$$

Here we have applied (2.19). For later reference, we conclude that

$$\left(\int_a^b |\delta_2(x,t,\widehat{c}_\varepsilon^{(i)})|^P\,\mathrm{d}t\right)^{1/P} \le \frac{3}{2}m^{-2}C\omega(x'',m^{-1})_P. \quad (2.23)$$

Now $\tau_{\min}(\widehat{c}_\varepsilon^{(i)}) \in [t_j, t_{j+1}]$, $i \le j < m - j_1$. If $\tau_{\min}(\widehat{c}_\varepsilon^{(i)}) = t_{j+1}$, then we take

$$\widetilde{c} = (\widetilde{c}_1,\dots,\widetilde{c}_m) = \widehat{c}_\varepsilon^{(i)}, \qquad \sigma_{2,m}(x;t) = \sigma_2(t;\widetilde{c}), \quad t \in [0,1].$$

Then

$$\begin{aligned}-\frac{1}{2}m^{-2}C\omega(x'',m^{-1}) &\le x(t) - \sigma_2(t;\widetilde{c})\\ &\le \frac{3}{2}m^{-2}C\omega(x'',m^{-1})_P,\\ &\qquad 0 \le t \le t_{j+1}.\end{aligned}$$

and

$$x(t_{j+1}) - \sigma_2(t_{j+1};\widetilde{c}) = -\frac{1}{2}m^{-2}C\omega(x'',m^{-1})_P. \quad (2.24)$$

Furthermore,

$$x'(t_{j+1}) - \sigma_{2,m}'(t_{j+1};\widetilde{c}) = 0.$$

Step 2 is complete. If, on the other hand, $\tau_{\min}(\widehat{c}_\varepsilon^{(i)}) < t_{j+1}$, then we again need some fine-tuning and continue with Step 2. Just as in Step 1, we find

$$c^*_{j_1+j} - c^*_{j_1+j+1} - \varepsilon \ge \eta \ge -\varepsilon$$

such that $\widehat{c}_{\varepsilon,\eta}^{(j)}$ is admissible and

$$\delta_2'(x,t_{j+1},\widehat{c}_{\varepsilon,\eta}^{(j)}) = 0.$$

Compare with (2.14) and (2.15). Again,

$$\delta_2(x,t,\widehat{c}_{\varepsilon,\eta}^{(j)}) = \delta_2(x,t,\widehat{c}_\varepsilon^{(i)}), \qquad t \in [t_0,t_j].$$

It therefore suffices to estimate $\delta_2(x,\cdot,\widehat{c}_{\varepsilon,\eta}^{(j)})$ for $t \in [t_j,t_{j+1}]$. Comparing with the preceding estimates and using (2.22), we obtain

$$\begin{aligned}\delta_2(x,t,\widehat{c}_{\varepsilon,\eta}^{(j)}) &= \delta_2(x,t,\widehat{c}_\varepsilon^{(i)}) + \eta\,\sigma_2(t,e^{(j_1+j+1)})\\ &\ge \delta_2(x,\tau_{\min}(\widehat{c}_\varepsilon^{(i)}),\widehat{c}_\varepsilon^{(i)}) + (c^*_{j_1+j} - c^*_{j_1+j+1})\\ &\ge -m^{-2}C\omega(x'',m^{-1})_P.\end{aligned}$$

Also,

$$\begin{aligned}\delta_2(x,t,\widehat{c}_{\varepsilon,\eta}^{(j)}) &\le \delta_2(x,\tau_{\min}(\widehat{c}_\varepsilon^{(i)}),\widehat{c}_\varepsilon^{(i)})\\ &\quad + \delta_2'(x,\tau_{\min}(\widehat{c}_\varepsilon^{(i)}),\widehat{c}_\varepsilon^{(i)})\big(t - \tau_{\min}(\widehat{c}_\varepsilon^{(i)})\big)\\ &\quad + \int_{\tau_{\min}(\widehat{c}_\varepsilon^{(i)})}^{t}\int_{\tau_{\min}(\widehat{c}_\varepsilon^{(i)})}^{\tau} \delta_2''(x,\theta,\widehat{c}_\varepsilon^{(i)})\,\mathrm{d}\theta\,\mathrm{d}\tau\\ &\le \delta_2(x,\tau_{\min}(\widehat{c}_\varepsilon^{(i)}),\widehat{c}_\varepsilon^{(i)})\\ &\quad + \frac{1}{2}\left\|\delta_2''(x,\cdot,\widehat{c}_\varepsilon^{(i)})\right\|_P \big(t - \tau_{\min}(\widehat{c}_\varepsilon^{(i)})\big)^2.\end{aligned}$$

Moreover,

$$\delta_2'(x,t_{j+1},\widehat{c}_{\varepsilon,\eta}^{(i)}) = 0.$$

Here we have used the fact that

$$\delta_2'(x,\tau_{\min}(\widehat{c}_\varepsilon^{(i)}),\widehat{c}_\varepsilon^{(i)}) = 0.$$

Consequently,

$$-m^{-2}C\omega(x'',m^{-1}) \le \delta_2(x,t_{j+1},\widehat{c}_{\varepsilon,\eta}^{(j)}) \le 0. \quad (2.26)$$

Now write $j_2 = j_1 + j + 1$, and denote

$$\widetilde{c} = (\widetilde{c}_1,\dots,\widetilde{c}_m) = \widehat{c}_{\varepsilon,\eta}^{(j)}.$$

This vector is admissible, and we set

$$\sigma_{2,m}(x;t) = \sigma_2(t;\widetilde{c}), \qquad t \in [0,1].$$

Then (1.1) and (1.2) hold, the latter in $[0,t_{j_2}]$, and

$$x'(t_{j_2}) - \sigma_{2,m}'(t_{j_2};\widetilde{c}) = 0.$$

Furthermore, by (2.24) and (2.26),

$$-m^{-2}C\omega(x'',m^{-1}) \le x(t_{j_2}) - \sigma_{2,m}(t_{j_2};\widetilde{c}) \le 0.$$

This completes Step 2.

If $j_2 = m$, then we are done. Otherwise, $t_{m,j_2} < 1$, and we return to Step 1. The only difference this time is that we have (2.27), like (2.18), instead of $\delta_2(x,0,c)$. This accounts for the lower estimate to the right of t_{j_2}, namely

$$-\frac{3}{2}m^{-2}C\omega(x'',m^{-1}),$$

just as in (2.23). However, the upper estimate of the newly constructed spline in that interval is

$$m^{-2}C\omega(x'',m^{-1}),$$

just as in (2.25). We alternately repeat Steps 2 and 1 until we get to the endpoint. The upper and lower estimates each time we apply Step 1 never exceed

$$m^{-2}C\omega(x'',m^{-1}) \quad \text{and} \quad -\frac{3}{2}m^{-2}C\omega(x'',m^{-1}),$$

respectively, and when we apply Step 2 they never exceed

$$\frac{3}{2}m^{-2}C\omega(x'',m^{-1}) \quad \text{and} \quad -m^{-2}C\omega(x'',m^{-1}),$$

respectively. The construction is achieved in finitely many steps (at most m steps). We thus obtain a quadratic spline $\sigma_{2,m}(x,\cdot)$ that satisfies (1.1) and (1.2).

In order to prove (1.3), we see that by virtue of Lemma 2.1, (1.2), and (1.4),

$$\begin{aligned}\left\|x'(\cdot)-\sigma'_{2,m}(x,\cdot)\right\|_P &\le 2m\left\|x(\cdot)-\sigma_{2,m}(x,\cdot)\right\|_P \\ &\quad+\frac{1}{2m}\left\|x''(\cdot)-\sigma''_{2,m}(x,\cdot)\right\|_P \\ &\le \left(2m\frac{3}{2}m^{-2}+\frac{1}{2}m^{-1}\right)C\omega(x'',m^{-1})_P \\ &= \frac{7}{2}m^{-1}C\omega(x'',m^{-1})_P.\end{aligned}$$

The theorem has now been proved under the additional assumption that $x(0)=x'(0)=x''(0)=0$. For a general $x\in\Delta_+^3L_P^2([0,1])$, we take

$$\widetilde{x}(t)=x(t)-x(0)-x'(0)t-\frac{1}{2}x''(0)t^2, \qquad t\in[0,1].$$

This completes the proof.

REFERENCES

[1] A. Hussain and E. Samir, "Orders of coapproximation of functions by algebraic polynomials," *Alqqadisia Journal for Pure and Sciences*, vol. 1, no. 9, pp. 338–345, 2004.

[2] E. Samir, *On the Constrained and Unconstrained Approximation*. Ph.D. thesis, Baghdad University, College of Education (Ibn-AlHaitham), 2004.

[3] A. Hussain and E. Samir, "Weak copositive approximation and Whitney theorem in L_P, $P<1$," *Jordan Journal of Applied Science–Natural Sciences*, vol. 2, no. 8, pp. 51–58, 2006.

[4] E. Samir, "Almost copositive polynomial approximation in $L_P[-1,1]$, $P<1$," in *First Conference on Mathematical Sciences*, Zarqa Private University, Jordan, pp. 52–57, 2006.

[5] "Estimates for Comonotone Polynomial Approximation in L_P," in *Second Conference on Mathematical Sciences*, Zarqa Private University, Jordan, 2008.

[6] V. Konovalov and D. Leviatan, "Estimates on the approximation of 3-monotone functions by 3-monotone quadratic splines," *East Journal of Approximation*, pp. 333–349, 2001.