A New Version of Gumbel Distribution Using Sine Technique Family: Properties, Parameter Estimation, and Data Analysis and Comparison with Fuzzy Data

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Abstract

In this paper, we propose a new version of the Gumbel Distribution using a sine technique family. We discuss the key properties of this distribution, such as the probability density function, the cumulative distribution function, the survival function, the hazard function, the cumulative hazard, and the moments. Additionally, we present a method for estimating the distribution's parameters. We then analyze a dataset using the original and generalized distributions, comparing the results and using goodness-of-fit measures to determine which distribution best fits the data. Finally, we provide conclusions based on our findings, with many examples and valid comparisons applied on fuzzy data.

Keywords: Continues Distributions; Statistical Properties; Gumbel distribution; Trigonometric Functions; Estimations Methods; Statistical Inference; fuzzy data; fuzzy analysis.

1. Introduction and Literatures Review

Numerous recent studies are concentrating on generalizing classical continuous distributions using the sine technique family. This approach enhances data set analysis. Therefore, we have opted for a modern technique known as the trigonometric function. This technique has been demonstrated in various forms of literature and is used to generalize one of the primary classical distributions utilized to study the extreme dataset. This distribution is known as the Gumbel Distribution (GD).

This study aims to introduce a new continuous distribution, the Sine Gumbel Distribution (SGD). The research delves into the statistical properties associated with the proposed distribution. First, we establish the cumulative distribution function (cdf) and the probability density function (pdf) of SGD. Then, we present and prove other statistical properties, such as the survival function, hazard function, moments, Rényi entropy, and more. The maximum likelihood estimation (MLE) method was used to estimate the parameters of the sine Gumbel distribution. Eventually, the estimated parameter values were then used to calculate goodness-of-fit measures and compared to the original distribution to determine which distribution best fits a dataset.

The Gumbel distribution is considered to be one of the most significant distributions for analyzing survival data and obtaining reliable statistical inferences [1]. As a result, many researchers have generalized this distribution using different statistical techniques. For example, the Beta Gumbel distribution [2], the exponentiated generalized Gumbel distribution [3], the Marshall-Olkin technique [4], and the Maxwell-Weibull distribution [5]. All of these generalizations have a significant impact on data analysis and statistical studies.

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In recent literature, many new techniques have been introduced to extend the range of continuous distributions. Some of these methods involve generalizations of existing techniques, such as the proportional hazard family that is used to extend the Lindley distribution [6], or the Marshall-Olkin technique demonstrated in [7]. Furthermore, there are some other studies that propose new techniques for this purpose, including the one presented in this paper and discussed in [8] and [9].

This paper is divided into four parts. In the first section, introduce the basic concepts associated with the statistical concepts of Gumbel distribution, the trigonometric function technique, and some of the fundamental concepts of some of the series expansion such as the binomial series. In the second part, we present the methodology and the main results related to this study. In the third part, discuss the estimation of the distribution parameters and demonstrate an application of the new distribution implemented on a real set of data through the goodness-of-fit measures. Finally, in the last part of this work, we present some conclusions related to our findings.

1.1. Background

In this part, we present information related to the main ideas of this study. It is well-known that a continuous random variable (r.v) Y has a pdf, and cdf of Gumbel distribution that can be written as follow

$$f_1(y) = \frac{1}{\beta} e^{-\frac{y-\alpha}{\beta} - e^{-\frac{y-\alpha}{\beta}}}, \quad -\infty < y < \infty, \quad -\infty < \alpha < \infty, \beta > 0 \tag{1}$$

$$\mathrm{F}_1(y) = e^{-e^{-\frac{y-\alpha}{\beta}}} \tag{2}$$

Also, a general formula of sine distribution family is recalled so that we can use in our proposed generalization of the desired distribution

A sine cdf, and pdf has the following forms

$$\mathrm{G}(y) = \sin\left[\frac{\pi}{2} F_1(y)\right] \tag{3}$$

$$\mathrm{g}(y) = \frac{\pi}{2} f_1(y) \cos\left[\frac{\pi}{2} F_1(y)\right] \tag{4}$$

In the context of conducting goodness-of-fit tests, we review the concept of Rényi entropy, which is articulated for a continuous random variable as follows:

$$I_R(\theta) = \frac{1}{1-\theta} \int_{-\infty}^{\infty} [f(y)]^{\theta} dy \tag{5}$$

The importance of Rényi entropy in statistics and information theory lies in its ability to provide a family of measures that go beyond the well-known Shannon entropy (which corresponds to the limit as θ approaches 1). Different values of θ yield different measures of information, and each has its own interpretation and use in various fields.

In order to facilitate the proofs that will be presented in the methodology section, it is important to keep in mind the following well-known concepts: the cosine series expansion, the exponential series expansion, the binomial series expansion, and the negative binomial series expansion. These concepts are defined by the following mathematical formulas

$$\cos(y) = \sum_{i=0}^{\infty} \frac{(-1)^i y^{2i}}{(2i)!} \tag{6}$$

$$e^y = \sum_{i=0}^{n} \frac{y^i}{i!} \tag{7}$$

$$(y_1 + y_2)^m = \sum_{i=0}^{m} \binom{m}{i} y_2^i \, y_1^{m-i} \tag{8}$$

$$(1 - y)^{-m} = \sum_{i=0}^{m} \binom{m+i-1}{i} y^i \tag{9}$$

Moreover, a gamma distribution pdf can be recalled, and it is known that any pdf holds the relation bellow:

$$1 = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha} e^{-\frac{y}{\beta}} dy \tag{10}$$

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1.2. Methodology and Properties

In this section, we introduce a new distribution model called the SGD, which is a hybrid of the GD and the sine distribution family. We will explain how this new distribution is derived and discuss its statistical properties, such as the survival function and hazard. Furthermore, we will present the derivations for the SGD's rth moment, the rth central moment, and the Rényi entropy. This will provide a comprehensive understanding of the SGD's statistical landscape.

After substituting the original cdf and pdf from equations (1), and (2) in equations (3) and (4), respectively, so we obtain the new pdf and cdf of the SGD as shown in equations (11), and (12), respectively.

$$G(y) = \sin\left[\frac{\pi}{2} e^{-e^{\frac{y-\alpha}{\beta}}}\right] \quad (11)$$

$$g(y) = \frac{\pi}{2\beta} e^{-\frac{y-\alpha}{\beta} - e^{-\frac{y-\alpha}{\beta}}} \cos\left[\frac{\pi}{2}\, e^{-e^{-\frac{y-\alpha}{\beta}}}\right] \quad (12)$$

where $-\infty < y < \infty$

By utilizing equations (11) and (12), we can obtain the survival and hazard functions of SGD directly

$$s(y) = 1 - \sin\left(\frac{\pi}{2}\, e^{-e^{-\frac{x-\alpha}{\beta}}}\right)$$

$$h(y) = \frac{\frac{\pi}{2\beta} e^{-\frac{y-\alpha}{\beta} - e^{-\frac{y-\alpha}{\beta}}} \cos\left[\frac{\pi}{2}\, e^{-e^{\frac{y-\alpha}{\beta}}}\right]}{1 - \sin\left(\frac{\pi}{2}\, e^{-e^{-\frac{x-\alpha}{\beta}}}\right)}$$

Also, we can conclude the formula of the cumulative hazard function of GSD as given in the equation below

$$H(y) = -log\left(1 - \sin\left(\frac{\pi}{2}\, e^{-e^{-\frac{x-\alpha}{\beta}}}\right)\right)$$

We present the properties like the rth moment about the origin, the rth central moment, the Rényi entropy, and others.

Theorem1 A r.v. Y from a SGD has the rth moment about origin

$$\mu_r' = E(Y^r) = \pi \sum_{i=0}^{r} \sum_{j,h=0}^{\infty} \binom{r}{i} \frac{\alpha^i \beta^{r-i} (-1)^{j+h} (-1)^i \left(\frac{\pi}{2}\right)^{2j} (1+2j)^h}{h!(2j)!} \frac{\Gamma(r-i+1)}{(1+h)^{r-i+1}}, \quad (13)$$

where $r = 1, \ldots, n$

Proof

It is known that the rth moment about origin is defined by

Then,

$$\mu_r' = E(Y^r) = \int_{-\infty}^{\infty} y^r \, \frac{\pi}{2\beta} e^{-\frac{y-\alpha}{\beta} - e^{-\frac{y-\alpha}{\beta}}} \cos\left[\frac{\pi}{2}\, e^{-e^{\frac{y-\alpha}{\beta}}}\right] dy$$

By using a changing of variable technique, suppose that

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$z = \frac{y-\alpha}{\beta} \rightarrow y = \beta z + \alpha$, and this implies that $y = \beta dz$,

Thus,

$$\mu_r' = E(Y^r) = \pi \int_0^{\infty} (\beta z + \alpha)^r e^{-z-e^{-z}} \cos\left[\frac{\pi}{2} e^{-e^{z}}\right] dz$$

Recalling the binomial expansion series

$$(\beta z + \alpha)^r = \sum_{i=0}^{r} \binom{r}{i} \alpha^i \beta^{r-i} z^{r-i}$$

and, by recalling the formula in equation (6) implies that

$$\cos\left(\frac{\pi}{2} e^{-e^{-z}}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left(\frac{\pi}{2} e^{-z}\right)^{2j}$$

Thus

$$\mu_r' = E(Y^r) = \pi \sum_{i=0}^{r} \sum_{j=0}^{\infty} \binom{r}{i} \frac{\alpha^i \beta^{r-i} (-1)^j \left(\frac{\pi}{2}\right)^{2j}}{(2j)!} \int_0^{\infty} z^{r-i+1-1} e^{-z} e^{-(2j+1)e^{-z}} dz$$

Also, by recalling the formula of exponential expansion series from equation (7), then

$$e^{-(2j+1)e^{-z}} = \sum_{h=0}^{\infty} \frac{(-1)^h}{h!} e^{-hz} (1+2j)^h$$

This implies that

$$\mu_r' = E(Y^r) = \frac{\pi}{\beta} \sum_{i=0}^{r} \sum_{j=0}^{\infty} \sum_{h=0}^{\infty} \binom{r}{i} \frac{\alpha^i \beta^{r-i} (-1)^{h+j} \left(\frac{\pi}{2}\right)^{2j} (1+2j)^h}{(2j)!\, h!} \times \int_0^{\infty} z^{r-i+1-1} e^{-(h+1)z} dz$$

A solution of the above integral with help of gamma distribution, that we have mentioned in equation (10), leads directly to the general formula of the rth moment in equation (13).

This completes the proof.

Theorem 2 A r.v. Y from SGD has the following rth central moment

$$E\left[\left(Y - E(Y)\right)^k\right] = \pi \sum_{i=0}^{k} \sum_{j=0}^{k-i} \sum_{h=0}^{\infty} \sum_{t=0}^{\infty} \frac{\binom{k}{i} \binom{k-i}{j} \mu^i \alpha^j \beta^{k-i-j} (-1)^{i+k+t} \left(\frac{\pi}{2}\right)^{2h}}{t!\,(2h)!} (1+2k)^h \frac{\Gamma(k-i-j+1)}{(1+h)^{k-i-j+1}},$$

where k=1,…,n

The proof is clear.

Theorem 3 The Rényi entropy of a r.v Y from SGD is given by the following formula

$$I_R(\theta) = \frac{1}{1-\theta} \log\left\{2\left(\frac{\pi}{2\beta}\right)^{\theta} \left[\sum_{i,j,k,h=0}^{\infty} \frac{\left(-\frac{1}{\beta}\right)^i (-1)^{j+h} \left(\frac{-j}{\beta}\right)^k \left(\frac{\pi}{2}\right)^{2h}}{i!\, j!\, k!\, (2h)!}\right]^{\theta} \sum_{t=0}^{\infty} \frac{(-2h\theta)^t \left(\frac{t}{\beta}\right)^{\theta(i+k)+1}}{t!} \Gamma(\theta(i+k) + 1)\right\}$$

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Proof

The proof follows by substituting the pdf in the formula of Rényi entropy in equation (5) and solving the desired integrals as shown in previous proofs

1.3. Estimation and Application

In this section, we demonstrate how we estimate the model parameters of the SGD using the Maximum Likelihood Estimation (MLE) method. Following this, we utilize the estimated parameter values to evaluate the goodness-of-fit measures and compare the results to determine the distribution that best fits the dataset.

Let Y_1,Y_2,…,Y_n be a random sample of size n from the SGD. Then the log-likelihood function of such distribution is given by

$$\ell = \log\left[L\left(\alpha,\beta,\underline{Y}\right)\right]$$

$$= \log\left[\prod_{i=1}^{n} f(y_i)\right] = \log\left[\prod_{i=1}^{n}\left(\frac{\pi}{2\beta}e^{-\frac{y_i-\alpha}{\beta}-e^{-\frac{y_i-\alpha}{\beta}}}\cos\left[\frac{\pi}{2}e^{-e^{-\frac{y_i-\alpha}{\beta}}}\right]\right)\right]$$

$$= n\log(\pi) - n\log(2) - n\log(\beta) - \sum_{i=1}^{n}\frac{y_i-\alpha}{\beta} - \sum_{i=1}^{n} e^{-\frac{y_i-\alpha}{\beta}} + \sum_{i=1}^{n}\log\left[\cos\left(\frac{\pi}{2}e^{-e^{-\frac{y_i-\alpha}{\beta}}}\right)\right]$$

Upon differentiating the log-likelihood above with respect to the model parameters vector (α, β), we obtain the following set of non-linear equations.

For simplicity, suppose that $w_i = \frac{y_{i-\hat{\alpha}}}{\hat{\beta}}, i = 1, \dots, n$

$$\frac{\partial\ell}{\partial\hat{\alpha}} = \frac{n}{\hat{\beta}} - \frac{1}{\hat{\beta}}\sum_{i=1}^{n} e^{-w_i} - \frac{\pi}{2\hat{\beta}}\sum_{i=1}^{n} e^{-w_i-e^{-w_i}}\, tan\left[\frac{\pi}{2}e^{-e^{-w_i}}\right] = 0$$

$$\frac{\partial\ell}{\partial\hat{\beta}} = \frac{-n}{\hat{\beta}} - \frac{1}{\hat{\beta}}\sum_{i=1}^{n} w_i + \frac{1}{\hat{\beta}}\sum_{i=1}^{n} w_i e^{-w_i} + \frac{\pi}{2\hat{\beta}}\sum_{i=1}^{n} w_i e^{-w_i-e^{-w_i}}\tan\left[\frac{\pi}{2}e^{-e^{-w_i}}\right] = 0$$

The non-linear equations above can only be solved by numerical methods. In our statues, we employed the Newton-Raphson iterative method to find the estimated values of each desired parameter.

On the other hand, the dataset comprises records of (102) patients diagnosed with meningitis, sourced from the Department of Public Health Division of Communicable Disease at the Najaf Health Department spanning the years (2015-2020). These years encompass the period from the patient's admission to the hospital until their recovery from the disease, measured in days. The dataset is shown as follow

4.12 8.23 0.12 3.23 0.21 2.11 0.10 6.22 1.18 4.23 0.20 2.10 0.17 9.23 1.18 0.22 0.22 2.11 0.17 0.12 0.27 2.12 0.18 4.11 0.15 0.18 6.21 2.15 0.12 4.16 0.11 6.00 1.19 2.16 0.23 4.16 0.10 1.16 1.19 2.15 0.90 4.22 0.14 0.18 1.19 2.18 1.21 4.22 0.19 0.17 1.21 2.18 1.21 5.19 0.11 0.19 1.23 5.10 0.23 6.12 0.16 0.90 0.90 5.12 1.24 3.23 0.12 0.20 1.11 0.16 1.90 1.17 0.10 0.19 4.12 0.17 1.93 6.19 0.12 0.20 5.12 1.95 0.15 1.11 0.13 0.21 1.12 1.99 0.17 1.13 0.13 0.21 1.12 2.06 1.16 1.14 0.13 0.21 1.12 2.11 1.17 1.18

We utilize the Newton-Raphson method with the provided dataset to compute the estimated model parameters, as detailed in Table 1. Note that all the required computations were accomplished by using MATLAB software.

Table 1: The values of the estimated parameters of original distribution and new distribution

Distribution	Estimated parameters
GD	$\hat{\alpha} = 4.097$ $\hat{\beta} = 9.928$
SGD	$\hat{\alpha} = 6.905$ $\hat{\beta} = 5.405$

By utilizing the above-estimated parameters of both the original distribution (GD) and the new distribution (SGD), we are able to calculate the estimated log-likelihood functions and assess the goodness-of-fit measures. This process presents a comparison between the distributions, aiding us in determining the most suitable one to fit our dataset, as presented in Table 2.

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Table 2: The estimated log-likelihood with the measures of goodness-of-fit

	$\hat{\ell}$	AIC	BIC	CAIC	HQIC
GD	-360	725	735	727.3	727.2
SGD	-356	716	726	718.7	718.6

The results presented in Table 2 demonstrate that the proposed distribution is better than the original one. This is evident from the lower measures-of-goodness values for the SGD distribution, which are somewhat better when compared to those of the original distribution.

Comparison based on fuzzy data

The first dataset is shown as follow

(4.12, 1.241.24) (8.23, 2.15) (0.12, 2.15) (3.23, 2.15) (0.21, 2.15) (2.11, 2.15) (0.10, 1.18) (6.22,) (1.18,) (4.23,) (0.20, 2.15) (2.10, 1.24) (0.17,) (9.23, 9.23) (1.18, 9.23) (0.22, 1.18) (0.22,) (2.11,) (0.17,) (0.12,) (0.27,) (2.12, 2.10) (0.18, 1.24) (4.11, 9.23) (0.15, 0.15) (0.18,) (6.21, 0.15) (2.15,) (0.12,) (4.16, 1.18) (0.11, 2.15) (6.00,) (1.19, 1.24) (2.16,) (0.23,) (4.16, 0.20) (0.10,) (1.16,) (1.19,) (2.15, 2.15) (0.90, 1.18) (4.22, 0.20) (0.14, 1.24) (0.18,) (1.19, 2.15) (2.18,) (1.21, 0.20) (4.22, 2.15) (0.19, 0.13) (0.17, 0.13) (1.21,) (2.18, 1.18) (1.21,) (5.19, 2.15) (0.11, 1.18) (0.19, 0.13) (1.23, 0.13) (5.10, 0.20) (0.23, 0.13) (6.12, 0.19) (0.16, 1.18) (0.90,) (0.90, 0.15) (5.12, 0.13) (1.24, 2.10) (3.23, 0.15) (0.12, 0.15) (0.20, 0.20) (1.11, (2.11) (0.16,) (1.90,) (1.17, 0.20) (1.13, (2.11) (0.13, (2.11) (0.21, 1.18) (1.12, 1.18) (2.06, 0.19) (1.16, 2.10) (1.14, 0.19) (0.13, 2.10) (0.21, 0.19) (1.12, (2.11) (2.11, 2.11) (1.17, 2.10) (1.18, 0.19)

We utilize the Newton-Raphson method with the provided dataset to compute the estimated model parameters, as detailed in Table 1. Note that all the required computations were accomplished by using MATLAB software.

Table 3: The values of the estimated parameters of fuzzy distribution and new distribution

Distribution	Estimated parameters
GD	$\hat{\alpha} = 6.217$ $\hat{\beta} = 8.928$
SGD	$\hat{\alpha} = 7.335$ $\hat{\beta} = 6.465$

By utilizing the above-estimated parameters of both the original distribution (GD) and the new distribution (SGD), we are able to calculate the estimated log-likelihood functions and assess the goodness-of-fit measures. This process presents a comparison between the distributions, aiding us in determining the most suitable one to fit our dataset, as presented in Table 2.

Table 4: The estimated log-likelihood with the measures of goodness-of-fit

	$\hat{\ell}$	AIC	BIC	CAIC	HQIC
GD	-360	746	744	716.3	744.2
SGD	-356	706	716	713.7	713.6

The second dataset is shown as follow

(4.12, 4.56) (8.23, 8.47) (0.12, 0.46) (3.23, 3.13) (0.21,0.88) (2.11,0) (0.10,0) (6.22, 3.23) (1.18, 3.23) (4.23, 0.88) (0.20, 0.5) (2.10, 0.5) (0.17,1) (9.23, 0.88) (1.18, 0.88) (0.22, 0.14) (0.22, 0.14) (2.11, 0.14) (0.17, 6.77) (0.12, 1.18) (0.27, 1.18) (2.12,3.12) (0.18,4.23) (4.11,1.45) (0.15,1) (0.18, 1.56) (6.21,2.32) (2.15,0.3) (0.12,0.3) (4.16, 0.5) (0.11, 3.23) (6.00,0) (1.19, 3.23) (2.16, 1.18) (0.23,0.5)

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(4.16, 1.18) (0.10,1) (1.16,1.55) (1.19,0.02) (2.15, 0.5) (0.90,0.,02) (4.22, 3.23) (0.14, 1.18) (0.18,0) (1.19,1) (2.18,1) (1.21,1) (4.22,0) (0.19,0) (0.17, 0.14) (1.21, 0.14) (2.18, 0.14) (1.21, 0.28) (5.19, 1.18) (0.11,0) (0.19,0.28) (1.23,0.49) (5.10,1) (0.23,1.34) (6.12,4.478) (0.16,1) (0.90,0) (0.90, 0.28) (5.12,0) (1.24, 0.28) (3.23,0) (0.12,1) (0.20,1) (1.11, 0.28) (0.16, 1.18) (1.90, 1.18) (1.17,1) (4.12 ,0.17) (1.93 ,6.19) (0.12 0.20) (5.12 1.95) (0.15 1.11) (0.13 0.21) (1.12 1.99) (1.13,1) (0.13,0) (0.21,1) (1.12,1.13) (2.06,2.345) (1.16,6.77) (1.14,0) (0.13,0.13) (0.21,0.13) (1.12,1) (2.11,1) (1.17,1) (1.18,0)

We utilize the Newton-Raphson method with the provided dataset to compute the estimated model parameters, as detailed in Table 1. Note that all the required computations were accomplished by using MATLAB software.

Table 3: The values of the estimated parameters of fuzzy distribution and new distribution

Distribution	Estimated parameters
GD	$\hat{\alpha} = 6.357$ $\hat{\beta} = 8.678$
SGD	$\hat{\alpha} = 7.6785$ $\hat{\beta} = 6.895$

By utilizing the above-estimated parameters of both the original distribution (GD) and the new distribution (SGD), we are able to calculate the estimated log-likelihood functions and assess the goodness-of-fit measures. This process presents a comparison between the distributions, aiding us in determining the most suitable one to fit our dataset, as presented in Table 2.

Table 4: The estimated log-likelihood with the measures of goodness-of-fit

	$\hat{\ell}$	**AIC**	**BIC**	**CAIC**	**HQIC**
GD	-360	745	743	713.2	743.9
SGD	-356	702	711	716.1	716.8

5. Conclusion

The utilization of trigonometric function techniques presents a valuable approach for extending various continuous distributions. Specifically, the SGD emerges as a potent tool for analyzing extreme-value data within this context. Notably, novel statistical properties distinct from those of the GD have been elucidated for the SGD. Through its application in practical data analysis, the SGD demonstrates its robustness and efficacy. Additionally, this avenue of exploration unveils several unresolved issues, including the potential for generalizing the GD through amalgamation with another continuous distribution.

References

[1] Cooray, Kahadawala. "Generalized gumbel distribution." Journal of Applied Statistics 37.1 (2010): 171-179.

[2] Nadarajah, Saralees, and Samuel Kotz. "The beta Gumbel distribution." Mathematical Problems in engineering 2004 (2004): 323-332.

[3] Andrade, Thiago, et al. "The exponentiated generalized Gumbel distribution." Revista Colombiana de Estadística 38.1 (2015): 123-143.

[4] Ghitany, M. E., E. K. Al-Hussaini, and R. A. Al-Jarallah. "Marshall–Olkin extended Weibull distribution and its application to censored data." Journal of Applied Statistics 32.10 (2005): 1025-1034.

Doi: https://doi.org/10.54216/IJNS.230327
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[5] Ishaq, Aliyu Ismail, and Alfred Adewole Abiodun. "The Maxwell–Weibull distribution in modeling lifetime datasets." Annals of Data Science 7.4 (2020): 639-662.
[6] Ahmed, AL-Adilee, et al. "GENERALIZED LINDLEY DISTRIBUTION USING PROPORTIONAL HAZARD FAMILY AND INFERENCE OF FAILURE TIME DATA." Nonlinear Functional Analysis and Applications (2023): 793-800.
[7] Marshall, Albert W., and Ingram Olkin. "A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families." Biometrika 84.3 (1997): 641-652.
[8] Kumar, Dinesh, Umesh Singh, and Sanjay Kumar Singh. "A new distribution using sine function-its application to bladder cancer patients data." Journal of Statistics Applications & Probability 4.3 (2015): 417.
[9] Chesneau, Christophe, and Farrukh Jamal. "The sine Kumaraswamy-G family of distributions." Journal of Mathematical Extension 15 (2020).

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