Neutrosophic ideals of several types in UP (BCC)-algebras

V. Rajam[1], N. Rajesh[2], Aiyared Iampan[3,*]
[1,2]Department of Mathematics, Rajah Serfoji Government College (affiliated to Bharathidasan University), Thanjavur 613005, Tamilnadu, India
[3]Department of Mathematics, School of Science, University of Phayao, 19 Moo 2, Tambon Mae Ka, Amphur Mueang, Phayao 56000, Thailand
Emails: rajamramv@gmail.com;nnrajesh_topology@yahoo.co.in; aiyared.ia@up.ac.th
* Corresponding author:aiyared.ia@up.ac.th

Abstract

Characterizations of $(\in, \in)$-neutrosophic ideals and $(q, \in \vee q)$-neutrosophic ideals are provided. Given special sets, so-called neutrosophic $\in$-subsets, neutrosophic q-subsets, and neutrosophic $(q, \in \vee q)$-subsets, conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets, and neutrosophic $(q, \in \vee q)$-subsets to be ideals are discussed.

Keywords: neutrosophic set; $(\in, \in)$-neutrosophic ideal; $(\in, q)$-neutrosophic ideal; $(q, \in)$-neutrosophic ideal; $(q, \in)$-neutrosophic ideal; $(q, \in \vee q)$-neutrosophic ideal; $(q, \in \vee q)$-neutrosophic ideal.

1 Introduction

The concept of fuzzy sets was proposed by Zadeh.[13] The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The notion of neutrosophic set theory developed by Smarandache[10,11] is a more general platform that extends the concepts of classic and fuzzy sets. Algebraic structures play a prominent role in mathematics, with wide-ranging applications in many disciplines, such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and so on. This provides sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy settings. UP-algebras are a class of logical algebras that were introduced by Iampan.[2] There is a deep relationship between UP-algebras and posets. Today UP-algebras have been studied by many authors and they have been applied to many branches of mathematics, such as group, functional analysis, probability theory, topology, fuzzy set theory, and so on. Various problems in system identification involve characteristics that are essentially non-probabilistic.[12] The notion of UP-algebras (see[2]) and the notion of BCC-algebras (see[5]) are the same, as shown by Jun et al.[4] in 2022. We shall refer to it as BCC rather than UP in this article out of respect for Komori, who initially described it in 1984. The neutrosophic set theory is also applied to several algebraic structures. The concept of neutrosophic points and several types of subalgebras (ideals) are introduced and studied in.[3,6–9]

In this paper, characterizations of $(\in, \in)$-neutrosophic ideals and $(q, \in \vee q)$-neutrosophic ideals are provided. Given special sets, so-called neutrosophic $\in$-subsets, neutrosophic q-subsets, and neutrosophic $(q, \in \vee q)$-subsets, conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets, and neutrosophic $(q, \in \vee q)$-subsets to be ideals are discussed.

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2 Preliminaries

The concept of BCC-algebras (see[5]) can be redefined without the condition (6) as follows:

An algebra $X = (X, *, 0)$ of type $(2, 0)$ is called a BCC-algebra (see[1]) if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y * z) * ((x * y) * (x * z)) = 0) \tag{1}$$
$$(\forall x \in X)(0 * x = x) \tag{2}$$
$$(\forall x \in X)(x * 0 = 0) \tag{3}$$
$$(\forall x, y \in X)(x * y = 0 = y * x \Rightarrow x = y) \tag{4}$$

After this, we assign X instead of a BCC-algebra $(X, *, 0)$ until otherwise specified.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0) \tag{5}$$

In X, the following assertions are valid (see[2]).

$$(\forall x \in X)(x \leq x) \tag{6}$$
$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{7}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow z * x \leq z * y) \tag{8}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow y * z \leq x * z) \tag{9}$$
$$(\forall x, y, z \in X)(x \leq y * x, \text{ in particular, } y * z \leq x * (y * z)) \tag{10}$$
$$(\forall x, y \in X)(y * x \leq x \Leftrightarrow x = y * x) \tag{11}$$
$$(\forall x, y \in X)(x \leq y * y) \tag{12}$$
$$(\forall a, x, y, z \in X)(x * (y * z) \leq x * ((a * y) * (a * z))) \tag{13}$$
$$(\forall a, x, y, z \in X)(((a * x) * (a * y)) * z \leq (x * y) * z) \tag{14}$$
$$(\forall x, y, z \in X)((x * y) * z \leq y * z) \tag{15}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z * y) \tag{16}$$
$$(\forall x, y, z \in X)((x * y) * z \leq x * (y * z)) \tag{17}$$
$$(\forall a, x, y, z \in X)((x * y) * z \leq y * (a * z)) \tag{18}$$

Definition 2.1. [2] A nonempty subset S of X is called a subalgebra of X if

$$(\forall x, y \in S)(x * y \in S). \tag{19}$$

Definition 2.2. [2] A nonempty subset S of X is called an ideal of X if

$$0 \in S, \tag{20}$$
$$(\forall x, y, z \in X)(x * (y * z), y \in S \Rightarrow x * z \in S). \tag{21}$$

Definition 2.3. [10] A neutrosophic set in a nonempty set X is defined to be a structure

$$A := \{(x, A_T(x), A_I(x), A_F(x)) : x \in X\}, \tag{22}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. The neutrosophic set in (22) is simply denoted by $A = (X, A_T, A_I, A_F)$.

Given a neutrosophic set $A = (X, A_T, A_I, A_F)$ in a nonempty set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:[3]

$$T_\in(A, \alpha) = \{x \in X : A_T(x) \geq \alpha\},$$

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$$\begin{aligned} I_{\in}(A,\beta) &= \{x \in X : A_I(x) \geq \beta\}, \\ F_{\in}(A,\gamma) &= \{x \in X : A_F(x) \leq \gamma\}, \\ T_q(A,\alpha) &= \{x \in X : A_T(x) + \alpha > 1\}, \\ I_q(A,\beta) &= \{x \in X : A_I(x) + \beta > 1\}, \\ F_q(A,\gamma) &= \{x \in X : A_F(x) + \gamma < 1\}, \\ T_{\in\vee q}(A,\alpha) &= \{x \in X : A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\}, \\ I_{\in\vee q}(A,\beta) &= \{x \in X : A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\}, \\ F_{\in\vee q}(A,\gamma) &= \{x \in X : A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}. \end{aligned}$$

We say $U_{\in}(A,\alpha)$, $I_{\in}(A,\beta)$, and $F_{\in}(A,\gamma)$ are neutrosophic $\in$-subsets of X, and $U_q(A,\alpha)$, $I_q(A,\beta)$, and $F_q(A,\gamma)$ are neutrosophic q-subsets of X and $U_{\in\vee q}(A,\alpha)$, $I_{\in\vee q}(A,\beta)$, and $F_{\in\vee q}(A,\gamma)$ are neutrosophic $\in\vee q$-subsets of X. For $\Phi \in \{\in, q, q \in \vee q\}$, the element of $T_\Phi(A,\alpha)$ (resp., $I_\Phi(A,\beta)$, $F_\Phi(A,\gamma)$) is called a neutrosophic T_Φ-point (resp., neutrosophic I_Φ-point, neutrosophic F_Φ-point) with value α (resp., β, γ). It is clear that

$$\begin{aligned} T_{\in\vee q}(A,\alpha) &= T_{\in}(A,\alpha) \cup T_q(A,\alpha), \\ I_{\in\vee q}(A,\beta) &= I_{\in}(A,\beta) \cup I_q(A,\beta), \\ F_{\in\vee q}(A,\gamma) &= F_{\in}(A,\gamma) \cup F_q(A,\gamma). \end{aligned}$$

3 Several types of neutrosophic ideals in BCC-algebras

In this section, we introduce the concepts of (Φ,Ψ)-neutrosophic subalgebras and (Φ,Ψ)-neutrosophic ideals of BCC-algebras, where $\Phi, \Psi \in \{\in, q, q \in \vee q\}$.

Definition 3.1. Given $\Phi, \Psi \in \{\in, q, q \in \vee q\}$, a neutrosophic set $A = (X, A_T, A_I, A_F)$ in X is called a (Φ,Ψ)-neutrosophic subalgebra of X if

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_\Phi(A,\alpha_x), y \in T_\Phi(A,\alpha_x) \Rightarrow x * y \in T_\Phi(A,\alpha_x) \\ x \in T_\Phi(A,\beta_x), y \in T_\Phi(A,\beta_x) \Rightarrow x * y \in T_\Phi(A,\beta_x) \\ x \in T_\Phi(A,\gamma_x), y \in T_\Phi(A,\gamma_x) \Rightarrow x * y \in T_\Phi(A,\gamma_x) \end{array} \right) \tag{23}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Definition 3.2. Given $\Phi, \Psi \in \{\in, q, q \in \vee q\}$, a neutrosophic set $A = (X, A_T, A_I, A_F)$ in X is called a (Φ,Ψ)-neutrosophic ideal of X if

$$(\forall x \in X) \left(\begin{array}{l} x \in T_\Phi(A,\alpha_x) \Rightarrow 0 \in T_\Phi(A,\alpha_x) \\ x \in T_\Phi(A,\beta_x) \Rightarrow 0 \in T_\Phi(A,\beta_x) \\ x \in T_\Phi(A,\gamma_x) \Rightarrow 0 \in T_\Phi(A,\gamma_x) \end{array} \right), \tag{24}$$

$$(\forall x, y, z \in X) \left(\begin{array}{l} x * (y * z) \in T_\Phi(A,\alpha_x), y \in T_\Phi(A,\alpha_x) \Rightarrow x * z \in T_\Phi(A,\alpha_x) \\ x * (y * z) \in T_\Phi(A,\beta_x), y \in T_\Phi(A,\beta_x) \Rightarrow x * z \in T_\Phi(A,\beta_x) \\ x * (y * z) \in T_\Phi(A,\gamma_x), y \in T_\Phi(A,\gamma_x) \Rightarrow x * z \in T_\Phi(A,\gamma_x) \end{array} \right) \tag{25}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Theorem 3.3. *A neutrosophic set $A = (X, A_T, A_I, A_F)$ in X is an $(\in,\in)$-neutrosophic ideal of X if and only if A satisfies*

$$(\forall x \in X) \left(\begin{array}{l} A_T(0) \geq A_T(x) \\ A_I(0) \geq A_I(x) \\ A_F(0) \leq A_F(x) \end{array} \right), \tag{26}$$

$$(\forall x, y, z \in X) \left(\begin{array}{l} A_T(x * z) \geq A_T(x * (y * z)) \wedge A_T(y) \\ A_I(x * z) \geq A_I(x * (y * z)) \wedge A_I(y) \\ A_F(x * z) \leq A_F(x * (y * z)) \vee A_F(y) \end{array} \right). \tag{27}$$

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Proof. Assume that the conditions (26) and (27) are valid, and let $x \in U_T^{\in}(A,\alpha)$, $a \in U_I^{\in}(A,\beta)$ and $u \in U_F^{\in}(A,\gamma)$ for any $x, a, u \in X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$. Hence, $0 \in U_T^{\in}(A,\alpha)$, $0 \in U_I^{\in}(A,\beta)$ and $0 \in U_F^{\in}(A,\gamma)$ and hence (26) is valid. Let $x, y, z, a, b, c, u, v, w \in X$ be such that $x * (y * z) \in U_T^{\in}(A,\alpha_x)$, $y \in U_T^{\in}(A,\alpha_y)$, $a * (b * c) \in U_I^{\in}(A,\beta_a)$, $b \in U_I^{\in}(A,\beta_b)$, $u * (v * w) \in U_F^{\in}(A,\gamma_u)$, and $v \in U_F^{\in}(A,\gamma_v)$ for all $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 1]$ and $\gamma_u, \gamma_v \in [0, 1)$. Then $A_T(x * (y * z)) \geq \alpha_x$, $A_T(y) \geq \alpha_y$, $A_I(a * (b * c)) \geq \beta_a$, $A_I(b) \geq \beta_b$, $A_F(u * (v * w)) \leq \gamma_u$, and $A_F(v) \leq \gamma_v$. Then by (27) that $A_T(x * z) \geq A_T(x * (y * z)) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y$, $A_I(a * c) \geq A_I(a * (b * c)) \wedge A_I(b) \geq ba \wedge \beta_b$, $A_F(u * w) \leq A_F(u * (v * w)) \vee A_F(v) \leq \gamma_u \vee \gamma_v$. Hence, $x * z \in U_T^{\in}(A,\alpha_x \wedge \alpha_y)$, $a * c \in U_I^{\in}(A,\beta_a \wedge \beta_b)$, and $u * w \in U_F^{\in}(A,\gamma_u \vee \gamma_v)$. Therefore, A is an $(\in, \in)$-neutrosophic ideal of X.

Conversely, let A be an $(\in, \in)$-neutrosophic ideal of X. If there exists $x_0 \in X$ such that $A_T(0) < A_T(x_0)$, then $x_0 \in U_T^{\in}(A,\alpha)$ and $0 \notin U_T^{\in}(A,\alpha)$, where $a = A_T(x_0)$. Which is a contradiction, and thus $A_T(0) \geq A_T(x)$ for all $x \in X$. Assume that $A_T(x_0 * z_0) < A_T(x_0 * (y_0 * z_0)) \wedge A_T(y_0)$ for some $x_0, y_0, z_0 \in X$. Taking $\alpha = A_T(x_0 * (y_0 * z_0)) \wedge A_T(y_0)$ implies that $x_0 * (y_0 * z_0) \in U_T^{\in}(A,\alpha)$ and $y_0 \in U_T^{\in}(A,\alpha)$; but $x_0 * z_0 \notin U_T^{\in}(A,\alpha)$. This is a contradiction, and thus $A_T(x * z) \geq A_T(x * (y * z)) \wedge A_T(y)$ for all $x, y, z \in X$. Similarly, we can verify that $A_I(x * z) \geq A_I(x * (y * z)) \wedge A_I(y)$ for all $x, y, z \in X$. Now, suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in U_F^{\in}(A,\gamma)$ and $0 \notin U_F^{\in}(A,\gamma)$ by taking $\gamma = A_F(a)$. This is impossible, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose there exist $a_0, b_0, z_0 \in X$ such that $A_F(a_0 * c_0) > A_F(a_0 * (b_0 * c_0)) \vee A_F(b_0)$ and take $\gamma = A_F(a_0 * (b_0 * c_0)) \vee A_F(b_0)$. Then $a_0 * (b_0 * c_0) \in U_F^{\in}(A,\gamma)$, $b_0 \in U_F^{\in}(A,\gamma)$, and $a_0 * c_0 \notin U_F^{\in}(A,\gamma)$, which is a contradiction. Thus, $A_F(x * z) \leq A_F(x * (y * z)) \vee A_F(y)$ for all $x, y, z \in X$. Therefore, A satisfies (26) and (27). □

Lemma 3.4. *Every* $(\in, \in)$*-neutrosophic ideal* $A = (X, A_T, A_I, A_F)$ *of* X *satisfies*

$$(\forall x, y \in X) \left(y \leq x \Rightarrow \begin{cases} A_T(x) \geq A_T(y), \\ A_I(x) \geq A_I(y), \\ A_F(x) \leq A_F(y)) \end{cases} \right). \tag{28}$$

Proof. Let A be an $(\in, \in)$-neutrosophic ideal of X. Let $x, y \in X$ be such that $y \leq x$. Then $y * x = 0$, and thus $A_T(x) = A_T(0 * x) \geq A_T(0 * (y * x)) \wedge A_T(y) = A_T(y * x) \wedge A_T(y) = A_T(0) \wedge A_T(y) = A_T(y)$, $A_I(x) = A_I(0 * x) \geq A_I(0 * (y * x)) \wedge A_I(y) = A_I(y * x) \wedge A_I(y) = A_I(0) \wedge A_I(y) = A_I(y)$, and $A_F(x) = A_F(0 * x) \leq A_F(0 * (y * x)) \vee A_F(y) = A_F(y * x) \vee A_F(y) = A_F(0) \vee A_F(y) = A_F(y)$. □

Theorem 3.5. *Every* $(\in, \in)$*-neutrosophic ideal* $A = (X, A_T, A_I, A_F)$ *of* X *satisfies*

$$(\forall x, y, z, w \in X) \left(x \leq w * (y * z) \Rightarrow \begin{cases} A_T(x * z) \geq A_T(w) \wedge A_T(y) \\ A_I(x * z) \geq A_I(w) \wedge A_I(y) \\ A_F(x * z) \leq A_F(w) \vee A_F(y) \end{cases} \right). \tag{29}$$

Proof. Let A be an $(\in, \in)$-neutrosophic ideal of X. Let $x, y, z, w \in X$ be such that $x \leq w * (y * z)$. Then $x * (w * (y * z)) = 0$. Also

$$\begin{aligned} A_T(x * z) &\geq A_T(x * (y * z)) \wedge A_T(y) \\ &\geq A_T(x * (w * (y * z))) \wedge A_T(w) \wedge A_T(x) \\ &= A_T(0) \wedge A_T(w) \wedge A_T(y) \\ &= A_T(w) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x * z) &\geq A_I(x * (y * z)) \wedge A_I(y) \\ &\geq A_I(x * (w * (y * z))) \wedge A_I(w) \wedge A_I(x) \\ &= A_I(0) \wedge A_I(w) \wedge A_I(y) \\ &= A_I(w) \wedge A_I(y), \end{aligned}$$

$$\begin{aligned} A_F(x * z) &\leq A_F(x * (y * z)) \vee A_F(y) \\ &\leq A_F(x * (w * (y * z))) \vee A_F(w) \vee A_F(x) \\ &= A_F(0) \vee A_F(w) \vee A_F(y) \\ &= A_F(w) \vee A_F(y). \end{aligned}$$

Therefore, (29) is proved. □

Theorem 3.6. *Every* $(\in, \in)$*-neutrosophic ideal* $A = (X, A_T, A_I, A_F)$ *of* X *satisfies*

$$(\forall x, y, z \in X) \left(x \leq (y * z) \Rightarrow \begin{cases} A_T(x * z) \geq A_T(y) \\ A_I(x * z) \geq A_I(y) \\ A_F(x * z) \leq A_F(y) \end{cases} \right). \tag{30}$$

Proof. Let A be an $(\in, \in)$-neutrosophic ideal of X. Let $x, y, z \in X$ be such that $x \leq y * z$. By Theorem 3.5, put $w = 0$. Then $x \leq 0 * (y * z)$. Hence,

$$A_T(x * z) \geq A_T(0) \wedge A_T(y) = A_T(y),$$

$$A_I(x * z) \geq A_I(0) \wedge A_I(y) = A_I(y),$$

$$A_F(x * z) \leq A_F(0) \vee A_F(y) = A_F(y).$$

Therefore, (30) is proved. □

Theorem 3.7. *A neutrosophic set* $A = (X, A_T, A_I, A_F)$ *in* X *is an* $(\in, \in)$*-neutrosophic ideal of* X *if and only if the nonempty neutrosophic* $\in$*-subsets* $T_\in(A, \alpha)$*,* $T_\in(A, \beta)$*, and* $T_\in(A, \gamma)$ *are ideals of* X *for all* $\alpha, \beta \in (0, 1]$ *and* $\gamma \in [0, 1)$*.*

Proof. Let A be an $(\in, \in)$-neutrosophic ideal of X and assume that $T_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then there exist $x, y, z \in X$ such that $x \in T_\in(A, \alpha)$, $y \in I_\in(A, \beta)$, and $z \in F_\in(A, \gamma)$. Hence, $0 \in T_\in(A, \alpha) \cap I_\in(A, \beta) \cap F_\in(A, \gamma)$. Let $x, y, z, a, b, c, u, v, w \in X$ be such that $x * (y * z) \in T_\in(A, \alpha)$, $y \in T_\in(A, \alpha)$, $a * (b * c) \in I_\in(A, \beta)$, $b \in I_\in(A, \beta)$, $u * (v * w) \in F_\in(A, \gamma)$, and $v \in F_\in(A, \gamma)$. Then $A_T(x*w) \geq A_T(x*(y*z)) \wedge A_T(y) \geq \alpha \wedge \alpha = \alpha$ $A_I(a*c) \geq A_I(a*(b*c)) \wedge A_I(b) \geq \beta \wedge \beta = \beta$ $A_F(u * w) \leq A_F(u * (v * w)) \vee A_F(v) \leq \gamma_1 \vee \gamma_2$, and so $x * z \in T_\in(A, \alpha)$, $a * c \in I_\in(A, \beta)$, and $u * w \in F_\in(A, \gamma)$. Hence, the nonempty neutrosophic $\in$-subsets $T_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.

Conversely, let A be a neutrosophic set in X for which $T_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are nonempty and are ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Assume that $A_T(0) < A_T(x)$, $A_I(0) < A_I(y)$, and $A_F(0) > A_F(z)$ for some $x, y, z \in X$. Then $x \in T_\in(A, A_T(x))$, $y \in I_\in(A, A_I(y))$, and $z \in F_\in(A, A_F(z))$, that is, $T_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are nonempty. But $0 \notin T_\in(A, A_T(x)) \cap I_\in(A, A_I(y)) \cap F_\in(A, \gamma A_F(z))$, which is a contradiction since $T_\in(A, A_T(x))$, $I_\in(A, A_I(y))$, and $F_\in(A, \gamma A_F(z))$ are ideals of X. Hence, $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose that $A_T(x * z) < A_T(x * (y * z)) \wedge A_T(y)$, $A_I(a * c) < A_I(a * (b * c)) \wedge A_I(b)$, and $A_F(u * w) > A_F(u * (v * w)) \vee A_F(v)$ for some $x, y, z, a, b, c, u, v, w \in X$. Taking $\alpha = A_T(x * (y * z)) \wedge A_T(y)$, $\beta = A_I(a * (b * c)) \wedge A_I(b)$, and $\gamma = A_F(u * (v * w)) \vee A_F(v)$ imply that $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, $x * (y * z) \in T_\in(A, \alpha)$, $y \in T_\in(A, \alpha)$, $a * (b * c) \in I_\in(A, \beta)$, $b \in I_\in(A, \beta)$, $u * (v * w) \in F_\in(A, \gamma)$, and $v \in F_\in(A, \gamma)$. But $x * z \notin T_\in(A, \alpha)$, $a * c \notin I_\in(A, \beta)$, and $u * w \notin F_\in(A, \gamma)$. This is a contradiction since $T_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are ideals of X. Thus, $A_T(x * z) \geq A_T(x * (y * z)) \wedge A_T(y)$, $A_I(x * z) \geq A_I(x * (y * z)) \wedge A_I(y)$, and $A_F(x * z) \leq A_F(x * (y * z)) \vee A_F(y)$ for all $x, y, z \in X$. Therefore, A is an $(\in, \in)$-neutrosophic ideal of X. □

Theorem 3.8. *Any ideal of* X *can be realized as level neutrosophic ideals of some* $(\in, \in)$*-neutrosophic ideal of* X*.*

Proof. Let I be an ideal of X and let $A = (X, A_T, A_I, A_F)$ be a neutrosophic set in X given as follows:

$$A_T : X \to [0, 1], x \mapsto \begin{cases} \alpha & \text{if } x \in I \\ 0 & \text{otherwise,} \end{cases}$$

$$A_I : X \to [0, 1], x \mapsto \begin{cases} \beta & \text{if } x \in I \\ 0 & \text{otherwise,} \end{cases}$$

$$A_F : X \to [0, 1], x \mapsto \begin{cases} \gamma & \text{if } x \in I \\ 1 & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then $T_\in(A, \alpha) = I$, $I_\in(A, \beta) = I$, and $F_\in(A, \gamma) = I$. Obviously, $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$, and $A_F(0) \leq A_F(x)$ for all $x \in X$. Let $x, y, z \in X$. If $x * (y * z) \in I$

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and $y \in I$, then $x * z \in I$. Hence, $A_T(x*(y*z)) = A_T(y) = A_T(x*z) = \alpha$, $A_I(x*(y*z)) = A_I(y) = A_I(x*z) = \beta$, and $A_F(x*(y*z)) = A_F(y) = A_F(x*z) = \gamma$, and so $A_T(x*z) \geq A_T(x*(y*z)) \wedge A_T(y)$, $A_I(x*z) \geq A_I(x*(y*z)) \wedge A_I(y)$, and $A_F(x*z) \leq A_F(x*(y*z)) \vee A_F(y)$. If $x*(y*z) \notin I$ and $y \notin I$, then $A_T(x*(y*z)) = A_T(y) = 0$, $A_I(x*(y*z)) = A_I(y) = 0$, and $A_F(x*(y*z)) = A_F(y) = 1$. Thus, $A_T(x*z) \geq A_T(x*(y*z)) \wedge A_T(y)$, $A_I(x*z) \geq A_I(x*(y*z)) \wedge A_I(y)$, and $A_F(x*z) \leq A_F(x*(y*z)) \vee A_F(y)$. If $x*(y*z) \in I$ and $y \notin I$, then $A_T(x*(y*z)) = \alpha$ and $A_T(y) = 0$, $A_I(x*(y*z)) = \beta$ and $A_I(y) = 0$, and $A_F(x*(y*z)) = \gamma$ and $A_F(y) = 1$. It follows that $A_T(x*z) \geq 0 = A_T(x*(y*z)) \wedge A_T(y)$, $A_I(x*z) \geq 0 = A_I(x*(y*z)) \wedge A_I(y)$, and $A_F(x*z) \leq 1 = A_F(x*(y*z)) \vee A_F(y)$. Similarly, if $x*(y*z) \notin I$ and $y \in I$, then $A_T(x*z) \geq A_T(x*(y*z)) \wedge A_T(y)$, $A_I(x*z) \geq A_I(x*(y*z)) \wedge A_I(y)$, and $A_F(x*z) \leq A_F(x*(y*z)) \vee A_F(y)$. Therefore, A is an $(\in,\in)$-neutrosophic ideal of X. □

Theorem 3.9. *Every $(\in,\in)$-neutrosophic ideal of X is an $(\in,\in)$-neutrosophic subalgebra.*

Proof. Let $A = (X, A_T, A_I, A_F)$ be an $(\in,\in)$-neutrosophic ideal of X. Let $x, y \in X$. Then $A_T(x*y) \geq A_T(x*(y*y)) \wedge A_T(y) = A_T(x*0) \wedge A_T(y) = A_T(0) \wedge A_T(y) = A_T(y) = A_T(x) \wedge A_T(y)$, $A_I(x*y) \geq A_I(x*(y*y)) \wedge A_I(y) = A_I(x*0) \wedge A_I(y) = A_I(0) \wedge A_I(y) = A_I(y) = A_I(x) \wedge A_I(y)$, and $A_F(x*y) \leq A_F(x*(y*y)) \vee A_F(y) = A_F(x*0) \vee A_F(y) = A_F(0) \vee A_F(y) = A_F(y) = A_F(x) \vee A_F(y)$. Therefore, A is an $(\in,\in)$-neutrosophic subalgebra of X. □

The following example shows that the converse of Theorem 3.9 is not true in general.

Example 3.10. Consider a BCC-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	0
3	0	1	2	0

Let $A = (X, A_T, A_I, A_F)$ be a neutrosophic set in X that is given by:

X	A_T	A_I	A_F
0	0.7	0.9	0.2
1	0.7	0.6	0.2
2	0.3	0.6	0.8
3	0.7	0.4	0.2

It is routine to verify that A is an $(\in,\in)$-neutrosophic subalgebra of X but not an $(\in,\in)$-neutrosophic ideal of X.

A mapping $f : X \to Y$ of BCC-algebras is called a homomorphism if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCC-algebras, then $f(0_X) = 0_Y$. Given a homomorphism $f : X \to Y$ of BCC-algebras and a neutrosophic set $A = (Y, A_T, A_I, A_F)$ in Y, we define a neutrosophic set $A^f = (X, A_T^f, A_I^f, A_F^f)$ in X, which is called the induced neutrosophic set, as follows: $A_T^f : X \to [0,1], x \mapsto A_T(f(x))$, $A_I^f : X \to [0,1], x \to A_I(f(x))$, and $A_F^f : X \to [0,1], x \to A_F(f(x))$.

Theorem 3.11. *Let $f : X \to Y$ be a homomorphism of BCC-algebras. If $A = (Y, A_T, A_I, A_F)$ is an $(\in,\in)$-neutrosophic ideal of Y, then the induced neutrosophic set $A^f = (X, A_T^f, A_I^f, A_F^f)$ in X is an $(\in,\in)$-neutrosophic ideal of X.*

Proof. Assume that $A = (Y, A_T, A_I, A_F)$ is an $(\in,\in)$-neutrosophic ideal of Y. For any $x \in X$, we have

$$A_T^f(x) = A_T(f(x)) \leq A_T(0_Y) = A_T(f(0_X)) = A_T^f(0_X),$$

$$A_I^f(x) = A_I(f(x)) \leq A_I(0_Y) = A_I(f(0_X)) = A_I^f(0_X),$$

$$A_F^f(x) = A_F(f(x)) \geq A_F(0_Y) = A_F(f(0_X)) = A_F^f(0_X).$$

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Let $x, y, z \in X$. Then

$$\begin{aligned} A^f_T(x*(y*z)) \wedge A^f_T(y) &= A_T(f(x*(y*z))) \wedge A_T(f(y)) \\ &= A_T(f(x)*(f(y)*f(z)) \wedge A_T(f(y)) \\ &\leq A_T(f(x)*f(z)) \\ &= A_T(f(x*z)) \\ &= A^f_T(x*z), \end{aligned}$$

$$\begin{aligned} A^f_I(x*(y*z)) \wedge A^f_I(y) &= A_I(f(x*(y*z))) \wedge A_I(f(y)) \\ &= A_I(f(x)*(f(y)*f(z)) \wedge A_I(f(y)) \\ &\leq A_I(f(x)*f(z)) \\ &= A_I(f(x*z)) \\ &= A^f_I(x*z), \end{aligned}$$

$$\begin{aligned} A^f_F(x*(y*z)) \vee A^f_F(y) &= A_F(f(x*(y*z))) \vee A_F(f(y)) \\ &= A_F(f(x)*(f(y)*f(z)) \vee A_F(f(y)) \\ &\geq A_F(f(x)*f(z)) \\ &= A_F(f(x*z)) \\ &= A^f_F(x*z). \end{aligned}$$

Therefore, $A^f = (X, A^f_T, A^f_I, A^f_F)$ is an $(\in, \in)$-neutrosophic ideal of X. □

Theorem 3.12. *Let $f : X \to Y$ be an onto homomorphism of BCC-algebras and let $A = (Y, A_T, A_I, A_F)$ be a neutrosophic set in Y. If the induced neutrosophic set $A^f = (X, A^f_T, A^f_I, A^f_F)$ in X is an $(\in, \in)$-neutrosophic ideal of X, then A is an $(\in, \in)$-neutrosophic ideal of Y.*

Proof. Assume that the induced neutrosophic set $A^f = (X, A^f_T, A^f_I, A^f_F)$ in X is an $(\in, \in)$-neutrosophic ideal of X. For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$ since f is onto. Then

$$A_T(x) = A_T(f(a)) = A^f_T(a) \leq A^f_T(0_X) = A_T(f(0_X)) = A_T(0_Y),$$

$$A_I(x) = A_I(f(a)) = A^f_I(a) \leq A^f_I(0_X) = A_I(f(0_X)) = A_I(0_Y),$$

$$A_F(x) = A_F(f(a)) = A^f_F(a) \geq A^f_F(0_X) = A_F(f(0_X)) = A_F(0_Y).$$

Let $x, y, z \in Y$. Then $f(a) = x$, $f(b) = y$, and $f(c) = z$ for some $a, b, c \in X$. Thus,

$$\begin{aligned} A_T(x*z) &= A_T(f(a)*f(c)) \\ &= A_T(f(a*c)) \\ &= A^f_T(a*c) \\ &\geq A^f_T(a*(b*c)) \wedge A^f_T(b) \\ &= A_T(f(a*(b*c))) \wedge A_T(f(b)) \\ &= A_T(f(a)*(f(b)*f(c))) \wedge A_T(f(b)) \\ &= A_T(x*(y*z)) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x*z) &= A_I(f(a)*f(c)) \\ &= A_I(f(a*c)) \\ &= A^f_I(a*c) \\ &\geq A^f_I(a*(b*c)) \wedge A^f_I(b) \\ &= A_I(f(a*(b*c))) \wedge A_I(f(b)) \\ &= A_I(f(a)*(f(b)*f(c))) \wedge A_I(f(b)) \\ &= A_I(x*(y*z)) \wedge A_I(y), \end{aligned}$$

$$\begin{aligned} A_F(x*z) &= A_F(f(a)*f(c)) \\ &= A_F(f(a*c)) \\ &= A^f_F(a*c) \\ &\leq A^f_F(a*(b*c)) \vee A^f_F(b) \\ &= A_F(f(a*(b*c))) \vee A_F(f(b)) \\ &= A_F(f(a)*(f(b)*f(c))) \vee A_F(f(b)) \\ &= A_F(x*(y*z)) \vee A_F(y). \end{aligned}$$

Therefore, A is an $(\in, \in)$-neutrosophic ideal of Y. □

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Theorem 3.13. *For a neutrosophic set $A = (X, A_T, A_I, A_F)$ in X, the following are equivalent.*

1. *The nonempty neutrosophic $\in$-subsets $T_\in(A, \alpha)$, $T_\in(A, \beta)$, and $T_\in(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.*
2. *A satisfies the following:*

$$(\forall x \in X) \begin{pmatrix} A_T(0) \vee 0.5 \geq A_T(x) \\ A_I(0) \vee 0.5 \geq A_I(x) \\ A_F(0) \wedge 0.5 \leq A_F(x) \end{pmatrix}, \tag{31}$$

$$(\forall x, y, z \in X) \begin{pmatrix} A_T(x * z) \vee 0.5 \geq A_T(x * (y * z)) \wedge A_T(y) \\ A_I(x * z) \vee 0.5 \geq A_I(x * (y * z)) \wedge A_I(y) \\ A_F(x * z) \wedge 0.5 \leq A_F(x * (y * z)) \vee A_F(y) \end{pmatrix}. \tag{32}$$

Proof. Assume that the nonempty neutrosophic $\in$-subsets $T_\in(A, \alpha)$, $T_\in(A, \beta)$, and $T_\in(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. If there exist $a, b \in X$ such that $A_T(0) \vee 0.5 < A_T(a)$ and $A_I(0) \vee 0.5 < A_I(b)$, respectively, then $\alpha_a = A_T(a) \in (0.5, 1]$ and $\beta_b = A_I(b) \in (0.5, 1]$, and thus $a \in T_\in(A, \alpha_a)$ and $b \in I_\in(A, \beta_b)$. Then $A_T(0) < A_T(a)$ and $A_I(0) < A_I(b)$, which imply that $0 \notin T_\in(A, \alpha_a)$ and $0 \notin T_\in(A, \beta_b)$. This is a contradiction, and so we get $A_T(0) \vee 0.5 \geq A_T(x)$ and $A_I(0) \vee 0.5 \geq A_I(x)$ for all $x \in X$. If $A_F(0) \wedge 0.5 > A_F(x)$ for some $x \in X$, then $A_F(x) \in [0, 0.5)$. Since $F_\in(A, A_F(x))$ is an ideal of X, we have $0 \in F_\in(A, A_F(x))$ and so $A_F(0) \leq A_F(x)$. This is a contradiction, and so $A_F(0) \wedge 0.5 \leq A_F(x)$ for all $x \in X$. Suppose that $A_T(x * z) \vee 0.5 < A_T(x * (y * z)) \wedge A_T(y)$ for some $x, y, z \in X$ and take $\alpha = A_T(x * (y * z)) \wedge A_T(y)$. Then $\alpha \in (0.5, 1]$ and $x * (y * z), y \in T_\in(A, \alpha)$. But $x * z \notin T_\in(A, \alpha)$ since $A_T(x * z) < \alpha$, which is a contradiction. If $A_I(a * c) \vee 0.5 < A_I(a * (b * c)) \wedge A_I(b)$ for some $a, b, c \in X$, then $a * (b * c), b \in I_\in(A, \beta)$ and $a * c \notin I_\in(A, \beta)$ where $\beta = A_I(a * (b * c)) \wedge A_I(b)$. This is a contradiction. Assume that there exist $x, y, z \in X$ such that $A_F(x * z) \wedge 0.5 > A_F(x * (y * z)) \vee A_F(y) = \gamma$. Then $\gamma \in [0, 0.5)$, $x * (y * z), y \in F_\in(A, \gamma)$, but $x * z \notin F_\in(A, \gamma)$. This is a contradiction.

Conversely, let A be a neutrosophic set in X satisfying the conditions (31) and (32). Let $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ be such that $T_\in(A, \alpha) \neq \emptyset$, $T_\in(A, \beta) \neq \emptyset$, and $T_\in(A, \gamma) \neq \emptyset$. For any $x \in T_\in(A, \alpha)$, $y \in I_\in(A, \beta)$, and $z \in F_\in(A, \gamma)$, we have $A_T(0) \vee 0.5 \geq A_T(x) \geq \alpha > 0.5$, $A_I(0) \vee 0.5 \geq A_I(x) \geq \beta > 0.5$, and $A_F(0) \wedge 0.5 \leq A_F(x) \leq \gamma < 0.5$, and thus $A_T(0) \geq \alpha$, $A_I(0) \geq \beta$, and $A_F(0) \leq \gamma$. Hence, $0 \in T_\in(A, \alpha)$, $0 \in I_\in(A, \beta)$, and $0 \in F_\in(A, \gamma)$. Let $x, y, z, a, b, c, u, v, w \in X$ be such that $x * (y * z) \in T_\in(A, \alpha)$, $y \in T_\in(A, \beta)$, $a * (b * c) \in I_\in(A, \beta)$, $b \in I_\in(A, \gamma)$, $u * (v * w) \in F_\in(A, \gamma)$, and $v \in F_\in(A, \gamma)$. Then

$$A_T(x * z) \vee 0.5 > A_T(x * (y * z)) \wedge A_T(y) \geq \alpha > 0.5,$$

$$A_I(a * c) \vee 0.5 > A_I(a * (b * c)) \wedge A_I(b) \geq \beta > 0.5,$$

$$A_F(u * w) \wedge 0.5 < A_F(u * (v * w)) \vee A_F(y) \leq \gamma < 0.5,$$

and so that $A_T(x * z) \geq \alpha$, $A_I(a * c) \geq \beta$, and $A_F(u * w) \leq \gamma$, that is, $x * z \in T_\in(A, \alpha)$, $a * c \in I_2(A, \beta)$, and $u * w \in F_\in(A, \gamma)$. Therefore, the nonempty neutrosophic $\in$-subsets $T_\in(A, \alpha)$, $T_\in(A, \beta)$, and $T_\in(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. □

Theorem 3.14. *For a neutrosophic set $A = (X, A_T, A_I, A_F)$ in X, the following are equivalent.*

1. *A is an $(\in, \in \vee q)$-neutrosophic ideal of X,*
2. *A satisfies the following assertions:*

$$(\forall x \in X) \begin{pmatrix} A_T(0) \geq A_T(x) \wedge 0.5 \\ A_I(0) \geq A_I(x) \wedge 0.5 \\ A_F(0) \leq A_F(x) \vee 0.5 \end{pmatrix}, \tag{33}$$

$$(\forall x, y, z \in X) \begin{pmatrix} A_T(x * z) \geq \bigwedge\{A_T(x * (y * z)), A_T(y), 0.5\} \\ A_I(x * z) \geq \bigwedge\{A_I(x * (y * z)), A_I(y), 0.5\} \\ A_F(x * z) \leq \bigvee\{A_F(x * (y * z)), A_F(y), 0.5\} \end{pmatrix}. \tag{34}$$

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Proof. Suppose that A is an $(\in, \in \vee q)$-neutrosophic ideal of X. Let $x \in X$ and assume that $A_T(x) < 0.5$. If $A_T(0) < A_T(x)$, then $A_T(0) < \alpha_x < A_T(x)$ for some $\alpha_x \in (0, 0.5)$. It follows that $x \in T_\in(A, \alpha_x)$ and $0 \notin T_\in(A, \alpha_x)$. Also, $A_T(0) + \alpha_x < 1$, that is, $0 \notin T_q(A, \alpha_x)$. Hence, $0 \notin T_{\in\vee q}(A, \alpha_x)$ which is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Now, if $A_T(x) > 0.5$, then $x \in T_\in(A, 0.5)$ and thus $0 \in T_{\in\vee q}(A, 0.5)$. If $A_T(0) < 0.5$, then $A_T(0) + 0.5 < 1$, that is, $0 \notin T_q(A, 0.5)$. This is a contradiction, and thus $A_T(0) \geq 0.5$. Consequently, $A_T(0) \geq A_T(x) \wedge 0.5$ for all $x \in X$. Similarly, we know that $A_I(0) \geq A_I(x) \wedge 0.5$ for all $x \in X$. Assume that there exists $z \in X$ such that $A_F(0) > A_F(z) \vee 0.5$. Then $A_F(0) > \gamma_z > A_F(z) \vee 0.5$ for some $\gamma_z \in (0, 1)$, which implies that $\gamma_z > 0.5$, $z \in F_\in(A, \gamma_z)$, and $0 \notin F_\in(A, \gamma_z)$. Since $A_F(0) + \gamma_z > 1$, we have $0 \notin Fq(A, \gamma_z)$. This is impossible, and so $A_F(0) \leq A_F(x) \vee 0.5$ for all $x \in X$. Suppose that there exist $a, b, c \in X$ such that $A_T(a*c) < \bigwedge\{A_T(a*(b*c)), A_T(b), 0.5\}$. Then $A_T(a*c) < \alpha \leq \bigwedge\{A_T(a*(b*c)), A_T(b), 0.5\}$ for some $\alpha \in (0, 1)$. It follows that $a*(b*c) \in T_\in(A, \alpha)$, $b \in T_\in(A, \alpha)$, and $a*c \notin T_\in(A, \gamma)$. Since $\alpha \leq 0.5$, we have $A_T(a*c) + \alpha < 2\alpha \leq 1$ and so $a*c \notin T_q(A, \alpha)$. This is a contradiction, and therefore $A_T(x*z) \geq \alpha \bigwedge\{A_T(x*(y*z)), A_T(y), 0.5\}$ for all $x, y, z \in X$. Let $x, y, z \in X$ and suppose that $A_I(x*(y*z)) \wedge A_I(y) < 0.5$. Then $A_I(x*z) > A_I(x*(y*z)) \wedge A_I(y)$. If not, then $A_I(x*z) < \beta < A_I(x*(y*z)) \wedge A_I(y)$ for some $\beta \in (0, 0.5)$. It follows that $x*(y*z) \in I_\in(A, \beta)$, $y \in I_\in(A, \beta)$ but $x*z \notin I_{\in\vee q}(A, \beta)$, which is a contradiction. Hence, $A_I(x*z) \geq A_I(x*y) \wedge A_I(y)$ whenever $A_I(x*y) \wedge A_I(y) < 0.5$. If $A_I(x*(y*z)) \wedge A_I(y) > 0.5$, then $x*(y*z) \in I_\in(A, 0.5)$ and $y \in I_\in(A, 0.5)$, which implies that $x*z \in I_{\in\vee q}(A, 0.5)$. Therefore, $A_I(x*z) > 0.5$ because if $A_I(x*z) < 0.5$, then $A_I(x*z) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction. Hence, $A_I(x*z) \geq \bigwedge\{A_I(x*(y*z)), A_I(y), 0.5\}$ for all $x, y, z \in X$. Now, suppose that $A_F(x*z) > \bigvee\{A_F(x*(y*z)), A_F(y), 0.5\}$ for some $x, y, z \in X$. Then there exists $\gamma \in (0, 1)$ such that $A_F(x*z) > \gamma > \bigvee\{A_F(x*(y*z)), A_F(y), 0.5\}$. Thus, $\gamma > 0.5$, $x*(y*z) \in F_\in(A, \gamma)$ and $y \in F_\in(A, \gamma)$. Then $x*z \in F_{\in\vee q}(A, \gamma)$. Since $A_F(x*z) > \gamma$ and $A_F(x*z) + \gamma > 2\gamma > 1$, we have $x*z \notin F_{\in\vee q}(A, \gamma)$, which is a contradiction. Therefore, $A_F(x*z) \leq \bigvee\{A_F(x*(y*z)), A_F(y), 0.5\}$ for all $x, y, z \in X$.

Conversely, let A be a neutrosophic set in X satisfying the conditions (33) and (34). For any $x, y, z \in X$, let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A, \alpha)$, $y \in I_\in(A, \beta)$, and $z \in F_\in(A, \gamma)$. Then $A_T(x) \geq \alpha$, $A_I(y) \geq \beta$, and $A_F(z) \leq \gamma$. Suppose that $A_T(0) < \alpha$, $A_I(0) < \beta$, and $A_F(0) > \gamma$. If $A_T(x) < 0.5$, then $A_T(0) > A_T(x) \wedge 0.5 = A_T(x) \geq \alpha$, which is a contradiction. Hence, we know that $A_T(x) > 0.5$ and so $A_T(0) + \alpha > 2A_T(0) > 2(A_T(x) \wedge 0.5) = 1$. Hence, $0 \in T_q(A, \alpha) \subseteq T_{\in\vee q}(A, \alpha)$. We can verify that $0 \in I_{\in\vee q}(A, \alpha)$ in a similar way. If $A_F(x) > 0.5$, then $A_F(0) \leq A_F(x) \vee 0.5 = A_F(x) \leq \gamma$, which is a contradiction. Thus, $A_F(x) \leq 0.5$ and so $A_F(0) + \gamma < 2A_F(0) \leq 2(A_F(x) \vee 0.5) = 1$. Hence, $0 \in Fq(A, \gamma) \subseteq F_{\in\vee q}(A, \gamma)$. For any $x, y, z, a, b, c, u, v, w \in X$, let $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 1]$ and $\gamma_u, \gamma_v \in [0, 1)$ be such that $x*(y*z) \in T_\in(A, \alpha_x)$, $y \in T_\in(A, \alpha_y)$, $a*(b*c) \in I_\in(A, \beta_a)$, $b \in I_\in(A, \beta_b)$, $u*(v*w) \in F_\in(A, \gamma_u)$, and $v \in F_\in(A, \gamma_v)$. Then $A_T(x*(y*z)) > \alpha_x$, $A_T(y) > \alpha_y$, $A_I(a*(b*c)) > \beta_a$, $A_I(b) > \beta_b$, $A_F(u*(v*w)) \leq \beta_u$, and $A_F(v) \leq \beta_v$. Suppose that $A_T(x*z) < \alpha_x \wedge \alpha_y$. If $A_T(x*(y*z)) \wedge A_T(y) < 0.5$, then $A_T(x*z) > \bigwedge\{A_T(x*(y*z)), A_T(y), 0.5\} = A_T(x*(y*z)) \wedge A_T(y) > \alpha_x \wedge \alpha_y$, which is a contradiction. Hence, $A_T(x*(y*z)) \wedge A_T(y) \geq 0.5$, and so $A_T(x*z) + (\alpha_x \wedge \alpha_y) > 2A_T(x*z) \geq 2(\bigwedge\{A_T(x*(y*z)), A_T(y), 0.5\}) = 1$. This induces $x*z \in T_q(A, \gamma_x \wedge \alpha_y) \subseteq T_{\in\vee q}(A, \gamma_x \wedge \alpha_y)$. Similarly, we have $a*c \in I_{\in\vee q}(A, \beta_a \wedge \beta_b)$. Assume that $A_F(u*w) > \gamma_u \vee \gamma_v$, that is, $u*w \notin F_\in(A, \gamma_u \vee \gamma_v)$. If $A_F(u*(v*w)) \vee A_F(v) > 0.5$, then $A_F(u*w) \leq \bigvee\{A_F(u*(v*w)), A_F(v), 0.5\} = A_F(u*(v*w)) \vee A_F(v) \leq \gamma_u \vee \gamma_v$, which is a contradiction. Hence, $A_F(u*(v*w)) \vee A_F(v) \leq 0.5$, and so $A_F(u*w) + (\gamma_u \vee \gamma - v) < 2A_F(u*w) \leq 2(\bigvee\{A_F(u*(v*w)), A_F(v), 0.5\}) = 1$. This induces $u*w \in F_q(A, \gamma_u \vee \gamma_v) \subseteq F_{\in\vee q}(A, \gamma_u \vee \gamma_v)$. Consequently, A is an $(\in, \in \vee q)$-neutrosophic ideal of X. □

Proposition 3.15. *Every $(\in, \in \vee q)$-neutrosophic ideal $A = (X, A_T, A_I, A_F)$ of X satisfies*

$$(\forall x, y, z \in X) \left(x*y \leq z \Rightarrow \left\{ \begin{array}{l} A_T(x) \geq \bigwedge\{A_T(y), A_T(z), 0.5\} \\ A_I(x) \geq \bigwedge\{A_I(y), A_I(z), 0.5\} \\ A_F(x) \leq \bigvee\{A_F(y), A_F(z), 0.5\} \end{array} \right. \right). \tag{35}$$

Proof. Assume that $A = (X, A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic ideal of X. Let $x, y, z \in X$ be such that $x*(y*z) \leq z$. Then $(x*(y*z))*z = 0$, which implies that $A_T(x*(y*z)) \geq \bigwedge\{A_T((x*(y*z))*z), A_T(z), 0.5\} = \bigwedge\{A_T(0), A_T(z), 0.5\} > A_T(z) \wedge 0.5$, $A_I(x*(y*z)) \geq \bigwedge\{A_I((x*(y*z))*z), A_I(z), 0.5\} = \bigwedge\{A_I(0), A_I(z), 0.5\} > A_I(z) \wedge 0.5$, and $A_F(x*(y*z)) \leq \bigvee\{A_F((x*(y*z))*z), A_F(z), 0.5\} = \bigvee\{A_F(0), A_F(z), 0.5\} > A_F(z) \vee 0.5$. It follows that $A_T(x) \geq \bigwedge\{A_T(x*(y*z)), A_T(y), 0.5\} = \bigwedge\{A_T(y), A_T(z), 0.5\}$, $A_I(x) \geq \bigwedge\{A_I(x*(y*z)), A_I(y), 0.5\} = \bigwedge\{A_I(y), A_I(z), 0.5\}$, and $A_F(x) \leq \bigvee\{A_F(x*(y*z)), A_F(y), 0.5\} = \bigvee\{A_F(y), A_F(z), 0.5\}$. □

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Theorem 3.16. *A neutrosophic set $A = (X, A_T, A_I, A_F)$ in X is an $(\in, \in \vee q)$-neutrosophic ideal of X if and only if the nonempty neutrosophic $\in$-subsets $T_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Proof. Assume that A is an $(\in, \in \vee q)$-neutrosophic ideal of X and let $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ be such that $T_\in(A, \alpha) \neq \emptyset$, $I_\in(A, \beta) \neq \emptyset$, and $F_\in(A, \gamma) \neq \emptyset$. Then $A_T(0) > A_T(x) \wedge 0.5$, $A_I(0) > A_I(y) \wedge 0.5$, and and $A_F(0) \leq A_F(z) \vee 0.5$ for all $x \in T_\in(A, \alpha)$, $y \in I_\in(A, \beta)$, and $z \in F_\in(A, \gamma)$. Hence, $A_T(0) \geq \alpha \wedge 0.5 = \alpha$, $A_I(0) \geq \beta \wedge 0.5 = \beta$, and $A_F(0) \leq \gamma \vee 0.5 = \gamma$, that is, $0 \in T_\in(A, \alpha)$, $0 \in I_\in(A, \beta)$, and $0 \in F_\in(A, \gamma)$. Now, let $x, y, z, a, b, c, u, v, w \in X$ be such that $x * (y * z) \in T_\in(A, \alpha)$, $y \in T_\in(A, \alpha)$, $a * (b * c) \in I_\in(A, \beta)$, $b \in I_\in(A, \beta)$, $u * (v * w) \in F_\in(A, \gamma)$, and $v \in F_\in(A, \gamma)$ for $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Then $A_T(x * (y * z)) \geq \alpha$, $A_T(y) \geq \alpha$, $A_I(a * b) \geq \beta$, $A_I(b) \geq \beta$, $A_F(u * (v * w)) \leq \gamma$, and $A_F(v) \leq \gamma$. Thus

$$A_T(x * z) \geq \bigwedge\{A_T(x * (y * z)), A_T(y), 0.5\} \geq \alpha \wedge 0.5 = \alpha,$$

$$A_I(a * c) \geq \bigwedge\{A_I(a * (b * c)), A_I(b), 0.5\} \geq \beta \wedge 0.5 = \beta,$$

$$A_F(u * w) \leq \bigvee\{A_F(u * (v * w)), A_F(v), 0.5\} \leq \gamma \vee 0.5 = \gamma,$$

and so that $x * z \in T_\in(A, \alpha)$, $a * c \in I_\in(A, \beta)$, and $u * w \in F_\in(A, \gamma)$. Hence, $T_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.

Conversely, assume that the nonempty neutrosophic $\in$-subsets $T_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. If there are $x, y, z \in X$ such that $A_T(0) < A_T(x) \wedge 0.5$, $A_I(0) < A_I(y) \wedge 0.5$, and $A_F(0) > A_F(z) \vee 0.5$, then $A_T(0) < \alpha_x \leq A_T(x) \wedge 0.5$, $A_I(0) < \beta_y \leq A_I(y) \wedge 0.5$, and $A_F(0) > \gamma_z > A_F(z) \vee 0.5$ for some $\alpha_x, \beta_y \in (0, 0.5]$ and $\gamma_z \in [0, 0.5)$. Hence, $0 \notin T_\in(A, \alpha_x)$, $0 \notin I_\in(A, \alpha_y)$, and $0 \notin F_\in(A, \alpha_z)$, which is a contradiction. Therefore, $A_T(0) \geq A_T(x) \wedge 0.5$, $A_I(0) \geq A_I(x) \wedge 0.5$, and $A_F(0) \leq A_F(x) \vee 0.5$ for all $x \in X$. Assume that there exist $x, y, z, a, b, c, u, v, w \in X$ such that $A_T(x * z) < \bigwedge\{A_T(x * (y * z)), A_T(y), 0.5\}$, $A_I(a * c) < \bigwedge\{A_I(a * b), A_I(b), 0.5\}$, and $A_F(u * w) > \bigvee\{A_F(u * (v * w)), A_F(v), 0.5\}$. Taking $\alpha = \frac{1}{2}\left(A_T(x * z) + \bigwedge\{A_T(x * (y * z)), A_T(y), 0.5\}\right)$ implies that $\alpha \in (0, 0.5)$ and $A_T(x * z) < \alpha < \bigwedge\{A_T(x * (y * z)), A_T(y), 0.5\}$. Then $x * (y * z) \in T_\in(A, \alpha)$ and $y \in T_\in(A, \alpha)$, but $x * z \notin T_\in(A, \alpha)$. This is a contradiction. If $\beta = \bigwedge\{A_I(a * (b * c)), A_I(b), 0.5\}$, then $\beta \in (0, 0.5]$, $a * (b * c) \in I_\in(A, \beta)$, and $b \in I_\in(A, \beta)$. But $A_I(a * c) < \beta$ implies $a * c \notin I_\in(A, \beta)$, which is a contradiction. Taking $\gamma = \{A_F(u * (v * w)), A_F(v), 0.5\}$ induces $\gamma \in [0.5, 1)$, $u * (v * w) \in F_\in(A, \gamma)$, and $v \in F_\in(A, \gamma)$. Since $A_F(u * w) > \gamma$, we have $u * w \notin F_\in(A, \gamma)$, which is a contradiction. Therefore, $A_T(x * z) \geq \bigwedge\{A_T(x * (y * z)), A_T(y), 0.5\}$, $A_I(x * z) \geq \bigwedge\{A_I(x * (y * z)), A_I(y), 0.5\}$, and $A_F(x * z) \leq \bigvee\{A_F(x * (y * z)), A_F(y), 0.5\}$ for all $x, y, z \in X$. Hence, A is an $(\in, \in \vee q)$-neutrosophic ideal of X. □

We note that every $(\in, \in)$-neutrosophic ideal is an $(\in, \in \vee q)$-neutrosophic ideal. But an $(\in, \in \vee q)$-neutrosophic ideal may not be an $(\in, \in)$-neutrosophic ideal as seen in the following example.

Example 3.17. Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	1	1
2	0	0	0	1
3	0	0	1	0

Let $A = (X, A_T, A_I, A_F)$ be a neutrosophic set in X that is given by:

X	A_T	A_I	A_F
0	1	1	0
1	0.2	0.3	1
2	0.3	0.6	1
3	0.5	0.4	0.5

It is routine to verify that A is an $(\in, \in \vee q)$-neutrosophic ideal of X but not an $(\in, \in)$-neutrosophic ideal of X.

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We provide some conditions for an $(\in, \in \vee q)$-neutrosophic ideal to be an $(\in, \in)$-neutrosophic ideal.

Theorem 3.18. *Let $A = (X, A_T, A_I, A_F)$ be an $(\in, \in \vee q)$-neutrosophic ideal of X such that $A_T(x) < 0.5$, $A_I(x) < 0.5$, and $A_F(x) > 0.5$ for all $x \in X$. Then A is an $(\in, \in)$-neutrosophic ideal of X.*

Proof. Let $x, y, z \in X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A, \alpha)$, $y \in I_\in(A, \beta)$, and $z \in F_\in(A, \gamma)$. Then $A_T(x) \geq \alpha$, $A_I(y) \geq \beta$, and $A_F(z) \leq \gamma$, which imply that $A_T(0) \geq A_T(x) \wedge 0.5 = A_T(x) \geq \alpha$, $A_I(0) \geq A_I(y) \wedge 0.5 = A_I(y) \geq \beta$, and $A_F(0) \leq A_F(z) \vee 0.5 = A_F(z) \leq \gamma$. It follows that $0 \in T_\in(A, \alpha)$, $0 \in I_\in(A, \beta)$, and $0 \in F_\in(A, \gamma)$. For any $x, y, z, a, b, c, u, v, w \in X$, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1]$ and $\gamma_1, \gamma_2 \in [0, 1)$ be such that $x*(y*z) \in T_\in(A, \alpha_1)$, $y \in T_\in(A, \alpha_2)$, $a*(b*c) \in I_\in(A, \beta_1)$, $b \in I_\in(A, \beta_2)$, $u*(v*w) \in F_\in(A, \gamma_1)$, and $v \in F_\in(A, \gamma_2)$. Then $A_T(x*(y*z)) \geq \alpha_1$, $A_T(y) \geq \alpha_2$, $A_I(a*(b*c)) \geq \beta_1$, $A_I(b) \geq \beta_2$, $A_F(u*(v*w)) \leq \gamma_1$, and $A_F(v) \leq \gamma_2$. Thus $A_T(x*z) \geq \bigwedge\{A_T(x*(y*z)), A_T(y), 0.5\} = A_T(x*(y*z)) \wedge A_T(y) \geq \alpha_1 \wedge \alpha_2$, $A_I(a*c) \geq \bigwedge\{A_I(a*(b*c)), A_I(b), 0.5\} = A_I(a*(b*c)) \wedge A_I(b) \geq \beta_1 \wedge \beta_2$, and $A_F(u*w) \leq \bigvee\{A_F(u*(v*w)), A_F(v), 0.5\} = A_F(u*(v*w)) \vee A_F(v) \leq \gamma_1 \vee \gamma_2$. Hence, $x*z \in T_\in(A, \alpha_1 \wedge \alpha_2)$, $a*c \in I_\in(A, \beta_1 \wedge \beta_2)$, and $u*w \in F_\in(A, \gamma_1 \vee \gamma_2)$. Therefore, A is an $(\in, \in)$-neutrosophic ideal of X. □

Theorem 3.19. *Every $(\in \vee q, \in \vee q)$-neutrosophic ideal of X is an $(\in, \in \vee q)$-neutrosophic ideal.*

Proof. Let $A = (X, A_T, A_I, A_F)$ be an $(\in \vee q, \in \vee q)$-neutrosophic ideal of X. Let $x, y, z \in X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A, \alpha)$, $y \in I_\in(A, \beta)$, and $z \in F_\in(A, \gamma)$. Then $x \in T_{\in\vee q}(A, \alpha)$, $y \in I_{\in\vee q}(A, \beta)$, and $z \in F_{\in\vee q}(A, \gamma)$. It follows that $0 \in T_{\in\vee q}(A, \alpha)$, $0 \in I_{\in\vee q}(A, \beta)$, and $0 \in F_{\in\vee q}(A, \gamma)$. For any $x, y, z, a, b, c, u, v, w \in X$, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1]$ and $\gamma_1, \gamma_2 \in [0, 1)$ be such that $x*(y*z) \in T_\in(A, \alpha_1)$, $y \in T_\in(A, \alpha_2)$, $a*(b*c) \in I_\in(A, \beta_1)$, $b \in I_\in(A, \beta_2)$, $u*(v*w) \in F_\in(A, \gamma_1)$, and $v \in F_\in(A, \gamma_2)$. Then $x*(y*z) \in T_{\in\vee q}(A, \alpha_1)$, $y \in T_{\in\vee q}(A, \alpha_2)$, $a*(b*c) \in I_{\in\vee q}(A, \beta_1)$, $b \in I_{\in\vee q}(A, \beta_2)$, $u*(v*w) \in F_{\in\vee q}(A, \gamma_1)$, and $v \in F_{\in\vee q}(A, \gamma_2)$. It follows from (25) that $x*z \in T_{\in\vee q}(A, \alpha)$, $a*c \in I_{\in\vee q}(A, \beta)$, and $u*w \in F_{\in\vee q}(A, \gamma)$. Therefore, A is an $(\in, \in \vee q)$-neutrosophic ideal of X. □

The converse of Theorem 3.19 is not true in general as seen in the following example.

Example 3.20. From Example 3.17, it is routine to verify that A is an $(\in, \in \vee q)$-neutrosophic ideal of X but not an $(\in \vee q, \in \vee q)$-neutrosophic ideal of X.

Theorem 3.21. *For a neutrosophic set $A = (X, A_T, A_I, A_F)$ in X, if the nonempty neutrosophic $\in \vee q$-subsets $T_{\in\vee q}(A, \alpha)$, $I_{\in\vee q}(A, \beta)$, and $F_{\in\vee q}(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then $A = (X, A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic ideal of X.*

Proof. Let $A = (X, A_T, A_I, A_F)$ be a neutrosophic set in X such that the nonempty neutrosophic $\in \vee q$-subsets $T_{\in\vee q}(A, \alpha)$, $I_{\in\vee q}(A, \beta)$, and $F_{\in\vee q}(A, \gamma)$ are ideals of X for all α, β $(0, 0.5]$ and $\gamma \in [0.5, 1)$. Assume that $A_T(0) < A_T(x) \wedge 0.5 = \alpha_x$, $A_I(0) < A_I(y) \wedge 0.5 = \beta_y$, and $A_F(0) > A_F(z) \vee 0.5 = \gamma_z$ for some $x, y, z \in X$. Then $\alpha_x, \beta_y \in (0, 0.5]$, $\gamma_z \in [0.5, 1)$, $x \in T_\in(A, \alpha_x) \subseteq T_{\in\vee q}(A, \alpha_x)$, $y \in I_\in(A, \beta_y) \subseteq I_{\in\vee q}(A, \beta_y)$, $\gamma_z \in F_\in(A, \gamma_z) \subseteq F_{\in\vee q}(A, \gamma_z)$, $0 \notin T_\in(A, \alpha_x)$, $0 \notin I_\in(A, \beta_y)$, and $0 \notin F_\in(A, \gamma_z)$. Also, since $A_T(0) + \alpha_x < 2\alpha_x \leq 1$, that is, $0 \notin T_q(A, \alpha_x)$, $A_I(0) + \beta_y < 2\beta_y \leq 1$, that is, $0 \notin I_q(A, \beta_y)$, and $A_F(0) + \gamma_z > 2\gamma_z > 1$, that is, $0 \notin Fq(A, \gamma_z)$, we have $0 \notin T_{\in\vee q}(A, \alpha_x)$, $0 \notin I_{\in\vee q}(A, \beta_y)$, and $0 \notin F_{\in\vee q}(A, \gamma_z)$. This is a contradiction. Assume that there exist $x, y, z, a, b, c, u, v, w \in X$ such that $A_T(x*z) < \bigwedge\{A_T(x*(y*z)), A_T(y), 0.5\} = \alpha$, $A_I(a*c) < \bigwedge\{A_I(a*(b*c)), A_I(b), 0.5\} = \beta$, and $A_F(u*w) > \bigvee\{A_F(u*(v*w)), A_F(v), 0.5\} = \gamma$. Then $\alpha, \beta \in (0, 0.5]$, $\gamma \in [0.5, 1)$, $x*z \notin T_\in(A, \alpha)$, $a*c \notin I_\in(A, \beta)$, $u*w \notin F_\in(A, \gamma)$, and $x*(y*z) \in T_\in(A, \alpha) \subseteq T_{\in\vee q}(A, \alpha)$, $y \in T_\in(A, \alpha) \subseteq T_{\in\vee q}(A, \alpha)$, $a*(b*c) \in I_\in(A, \beta) \subseteq I_{\in\vee q}(A, \beta)$, $b \in I_\in(A, \beta) \subseteq I_{\in\vee q}(A, \beta)$, $u*(v*w) \in F_\in(A, \gamma) \subseteq F_{\in\vee q}(A, \gamma)$, $v \in F_\in(A, \gamma) \subseteq F_{\in\vee q}(A, \gamma)$. Since $T_{\in\vee q}(A, \alpha)$, $I_{\in\vee q}(A, \beta)$, and $F_{\in\vee q}(A, \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, implies that $x*z \in T_\in(A, \alpha) \subseteq T_{\in\vee q}(A, \alpha)$, $a*c \in I_\in(A, \beta) \subseteq I_{\in\vee q}(A, \beta)$, and $u*w \in F_\in(A, \gamma) \subseteq F_{\in\vee q}(A, \gamma)$. On the other hand, $A_T(x*z) + \alpha < 2\alpha \leq 1$, $A_I(a*c) + \beta < 2\beta \leq 1$, and $A_F(u*w) + \gamma > 2\gamma > 1$, that is, $x*z \notin T_q(A, \alpha)$, $a*c \notin I_q(A, \beta)$, and $u*w \notin F_q(A, \gamma)$. Hence, $x*z \notin T_{\in\vee q}(A, \alpha)$, $a*c \notin I_{\in\vee q}(A, \beta)$, and $u*w \notin F_{\in\vee q}(A, \gamma)$, which is a contradiction. Hence, A is an $(\in, \in \vee q)$-neutrosophic ideal of X. □

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Theorem 3.22. *For a subset J of X, let $A = (X, A_T, A_I, A_F)$ be a neutrosophic set in X such that*

$$(\forall x \in X)(A_T(0) \geq A_T(x), A_I(0) \geq A_I(x), A_F(0) \leq A_F(x)) \tag{36}$$

$$(\forall x \in J)(A_T(x) \geq 0.5, A_I(x) \geq 0.5, A_F(x) \leq 0.5), \tag{37}$$

$$(\forall x \in X \backslash J)(A_T(x) = 0, A_I(x) = 0, A_F(x) = 1). \tag{38}$$

If J is an ideal of X, then A is a $(q, \in \vee q)$-neutrosophic ideal of X.

Proof. Assume that J is an ideal of X. Let $x, y, z \in X$, $\alpha, \beta \in (0, 1]$, and $\gamma \in [0, 1)$ be such that $x \in T_q(A, \alpha)$, $y \in I_q(A, \beta)$, and $z \in F_q(A, \gamma)$. Then $A_T(x) + \alpha > 1$, $A_I(y) + \beta > 1$, and $A_F(z) + \gamma < 1$. Then $A_T(0) + \alpha \geq A_T(x) + \alpha > 1$, $A_I(0) + \beta \geq A_I(y) + \beta > 1$, and $A_F(0) + \gamma \leq A_F(z) + \gamma < 1$, that is, $0 \in T_q(A, \alpha) \subseteq T_{\in\vee q}(A, \alpha)$, $0 \in I_q(A, \beta) \subseteq I_{\in\vee q}(A, \beta)$, and $0 \in F_q(A, \gamma) \subseteq F_{\in\vee q}(A, \gamma)$. For any $x, y, z, a, b, c, u, v, w \in X$, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1]$ and $\gamma_1, \gamma_2 \in [0, 1)$ be such that $x * (y * z) \in T_q(A, \alpha_1)$, $y \in T_q(A, \alpha_2)$, $a * (b * c) \in I_q(A, \beta_1)$, $b \in I_q(A, \beta_2)$, $u * (v * w) \in F_q(A, \gamma_1)$, and $v \in F_q(A, \gamma_2)$. Then $A_T(0) + \alpha \geq A_T(x) + \alpha > 1$, $A_I(0) + \beta \geq A_I(x) + \beta > 1$, and $A_F(0) + \gamma \leq A_F(x) + \gamma < 1$, that is, $A_I(a * (b * c)) \geq \beta_1$, $A_I(b) \geq \beta_2$, $A_F(u * (v * w)) \leq \gamma_1$, and $A_F(v) \leq \gamma_2$. Then $A_T(x * (y * z)) + \alpha_1 > 1$, $A_T(y) + \alpha_2 > 1$, $A_I(a * (b * c)) + \beta_1 > 1$, $A_I(b) + \beta_2 > 1$, $A_F(u * (v * w)) + \gamma_1 < 1$, and $A_F(v) + \gamma_2 < 1$. If $x * (y * z) \notin J$ or $y \notin J$ (resp., $a * (b * c) \notin J$ or $b \notin J$), then $A_T(x * (y * z)) = 0$ or $A_T(y) = 0$ (resp., $A_I(a * (b * c)) = 0$ or $A_I(b) = 0$). It follows that $A_T(x * (y * z)) + \alpha_1 = \alpha_1 < 1$ or $A_T(y) + \alpha_2 = \alpha_2 \leq 1$ (resp., $A_I(a * (b * c)) + \beta_1 = \beta_1 \leq 1$ or $A_I(b) + \beta_2 = \beta_2 \leq 1$). This is a contradiction, and so $x * (y * z) \in J$ and $y \in J$ (resp., $a * (b * c) \in J$ and $b \in J$). If $u * (v * w) \notin J$ or $v \notin J$, then $A_F(u * (v * w)) = 1$ or $A_F(v) = 1$. Hence, $A_F(u * (v * w)) + \gamma_1 = 1 + \gamma_1 > 1$ or $A_F(v) + \gamma_2 = 1 + \gamma_2 > 1$, which is a contradiction. Thus, $u * (v * w) \in J$ and $v \in J$. Since J is an ideal of X, we get $x * z \in J$, $a * c \in J$, and $u * w \in J$. Thus, $A_T(x * z) \geq 0.5$, $A_I(a * c) \geq 0.5$, and $A_F(u * w) \leq 0.5$. If $\alpha_1 > 0.5$ or $\alpha_2 > 0.5$ (resp., $\beta_1 < 0.5$ or $\beta_2 < 0.5$), then $A_T(x * z) > 0.5 > \alpha_1 \wedge \alpha_2$ (resp., $A_I(a * c) > 0.5 > \beta_1 \wedge \beta_2$), that is, $x * z \in T_\in(A, \alpha_1 \wedge \alpha_2)$ (resp., $a * c \in I_\in(A, \beta_1 \wedge \beta_2)$). If $\beta_1 > 0.5$ and $\beta_2 > 0.5$ (resp., $\beta_1 > 0.5$ and $\beta_2 > 0.5$), then $A_T(x * z) + (\alpha_1 \wedge \alpha_2) > 0.5 + 0.5 = 1$ (resp., $A_I(a * c) + (\beta_1 \wedge \beta_2) > 0.5 + 0.5 = 1$), that is, $x * z \in T_q(A, \alpha_1 \wedge \alpha_2)$ (resp., $a * c \in I_q(A, \beta_1 \wedge \beta_2)$). Therefore, $x * z \in T_{\in\vee q}(A, \alpha_1 \wedge \alpha_2)$ (resp., $a * c \in I_{\in\vee q}(A, \beta_1 \wedge \beta_2)$). Also, if $\gamma_1 > 0.5$ or $\gamma_2 > 0.5$, then $A_F(u * w) \leq 0.5 \leq \gamma_1 \vee \gamma_2$ and so $u * w \in F_\in(A, \gamma_1 \vee \gamma_2) \subseteq F_{\in\vee q}(A, \gamma_1 \vee \gamma_2)$. If $\gamma_1 < 0.5$ and $\gamma_2 < 0.5$, then $A_F(u * w) + (\gamma_1 \vee \gamma_2) < 0.5 + 0.5 = 1$ and thus $u * w \in F_q(A, \gamma_1 \vee \gamma_2) \subseteq F_{\in\vee q}(A, \gamma_1 \vee \gamma_2)$. Consequently, A is an $(\in, \in \vee q)$-neutrosophic ideal of X. □

Theorem 3.23. *Every $(q, \in \vee q)$-neutrosophic ideal of X is an $(\in, \in \vee q)$-neutrosophic ideal.*

Proof. Let $A = (X, A_T, A_I, A_F)$ be a $(q, \in \vee q)$-neutrosophic ideal of X. For any $x, y, z \in X$, let $\alpha_x, \beta_y \in (0, 1]$ and $\gamma_z \in [0, 1)$ be such that $x \in T_\in(A, \alpha_x)$, $y \in I_\in(A, \beta_y)$, and $z \in F_\in(A, \gamma_z)$. Then $A_T(x) \geq \alpha_x$, $A_I(y) \geq \beta_y$, and $A_F(z) \leq \gamma_z$. Suppose $0 \notin T_{\in\vee q}(A, \alpha_x)$, $0 \notin I_{\in\vee q}(A, \beta_y)$, and $0 \notin F_{\in\vee q}(A, \gamma_z)$. Then $A_T(0) < \alpha_x$, $A_I(0) < \beta_y$, $A_F(0) > \gamma_z$, $A_T(0) + \alpha_x \leq 1$, $A_I(0) + \beta_y \leq 1$, and $A_F(0) + \gamma_z > 1$. Thus $A_T(0) < 0.5$, $A_I(0) < 0.5$, and $A_F(0) > 0.5$. Hence, $A_T(0) < \alpha_x \wedge 0.5$, $A_I(0) < \beta \vee \wedge 0.5$, and $A_F(0) > \gamma_z \vee 0.5$, and so

$$1 - A_T(0) > 1 - (\alpha_x \wedge 0.5) = (1 - \alpha_x) \vee 0.5 \geq (1 - A_T(x)) \vee 0.5,$$

$$1 - A_I(0) > 1 - (\beta_y \wedge 0.5) = (1 - \beta_y) \vee 0.5 \geq (1 - A_I(x)) \vee 0.5,$$

$$1 - A_F(0) < 1 - (\gamma_z \wedge 0.5) = (1 - \gamma_z) \wedge 0.5 \leq (1 - A_F(x)) \wedge 0.5.$$

Hence, there exist $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ such that

$$1 - A_T(0) \geq \alpha > (1 - A_T(x)) \vee 0.5,$$

$$1 - A_I(0) \geq \beta > (1 - A_I(x)) \vee 0.5,$$

$$1 - A_F(0) \leq \gamma < (1 - A_F(x)) \wedge 0.5.$$

The above conditions induce $A_T(x) + \alpha > 1$, $A_I(x) + \beta > 1$, and $A_F(x) + \gamma < 1$, that is, $x \in T_q(A, \alpha)$, $y \in I_q(A, \beta)$, and $z \in F_q(A, \gamma)$. Since A is a $(q, \in \vee q)$-neutrosophic ideal of X, we have $0 \in T_{\in\vee q}(A, \alpha)$, $0 \in I_{\in\vee q}(A, \beta)$, and $0 \in F_{\in\vee q}(A, \gamma)$. But $A_T(0) + \alpha \leq 1$ and $A_T(0) \leq 1 - \alpha < 0.5 < \alpha$, $A_I(0) + \beta \leq 1$ and $A_I(0) \leq 1 - \beta < 0.5 < \beta$, $A_I(0) + \alpha \leq 1$ and $A_I(0) \leq 1 - \beta < 0.5 < \beta$, and $A_F(0) + \gamma > 1$ and $A_F(0) >$

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$1-\gamma > 0.5 > \gamma$, that is, $0 \notin T_{\in\vee q}(A,\alpha)$, $0 \notin I_{\in\vee q}(A,\beta)$, and $0 \notin F_{\in\vee q}(A,\gamma)$. This is a contradiction, and so $0 \in T_{\in\vee q}(A,\alpha_x)$, $0 \in I_{\in\vee q}(A,\beta_y)$, and $0 \in F_{\in\vee q}(A,\gamma_z)$. For any $x,y,z,a,b,c,u,v,w \in X$, let $\alpha_1,\alpha_2,\beta_1,\beta_2 \in (0,1]$ and $\gamma_1,\gamma_2 \in [0,1)$ be such that $x*(y*z) \in T_{\in}(A,\alpha_1)$, $y \in T_{\in}(A,\alpha_2)$, $a*(b*c) \in I_{\in}(A,\beta_1)$, $b \in I_{\in}(A,\beta_2)$, $u*(v*w) \in F_{\in}(A,\gamma_1)$, and $v \in F_{\in}(A,\gamma_2)$. Then $A_T(x*(y*z)) \geq \alpha_1$, $A_T(y) \geq \alpha_2$, $A_I(a*(b*c)) \geq \beta_1$, $A_I(b) \geq \beta_2$, $A_F(u*(v*w)) \leq \gamma_1$, and $A_F(v) \leq \gamma_2$. Suppose $x*z \notin T_{\in\vee q}(A,\alpha_1\wedge\alpha_2)$, $a*c \notin I_{\in\vee q}(A,\beta_1\wedge\beta_2)$, and $u*w \notin F_{\in\vee q}(A,\gamma_1\vee\gamma_2)$. Then $A_T(x*z) < \alpha_1\wedge\alpha_2$, $A_I(a*c) < \beta_1\wedge\beta_2$, $A_F(u*w) > \gamma_1\vee\gamma_2$, $A_T(x*z)+(\alpha_1\wedge\alpha_2) \leq 1$, $A_I(a*c)+(\beta_1\wedge\beta_2) \leq 1$, and $A_F(u*w)+(\gamma_1\vee\gamma_2) \geq 1$. It follows that $A_T(x*z) < 0.5$, $A_I(a*c) < 0.5$, and $A_F(u*w) > 0.5$. So we have $A_T(x*z) < \bigwedge\{\alpha_1,\alpha_2,0.5\}$, $A_I(a*c) < \bigwedge\{\beta_1,\beta_2,0.5\}$, and $A_F(u*w) < \bigvee\{\gamma_1,\gamma_2,0.5\}$ and thus

$$\begin{aligned} 1-A_T(x*z) &> 1-\bigwedge\{\alpha_1,\alpha_2,0.5\} \\ &= \bigvee\{1-\alpha_1,1-\alpha_2,0.5\} \\ &\geq \bigvee\{1-A_T(x*(y*z)),1-A_T(y),0.5\}, \end{aligned}$$

$$\begin{aligned} 1-A_I(a*c) &> 1-\bigwedge\{\beta_1,\beta_2,0.5\} \\ &= \bigvee\{1-\beta_1,1-\beta_2,0.5\} \\ &\geq \bigvee\{1-A_I(a*(b*c)),1-A_I(b),0.5\}, \end{aligned}$$

$$\begin{aligned} 1-A_F(u*w) &< 1-\bigvee\{\gamma_1,\gamma_2,0.5\} \\ &= \bigwedge\{1-\gamma_1,1-\gamma_2,0.5\} \\ &\leq \bigwedge\{1-A_F(u*(v*w)),1-A_F(v),0.5\}. \end{aligned}$$

Therefore, there exist $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$ such that

$$1-A_T(x*z) \geq \alpha > \bigvee\{1-A_T(x*(y*z)),1-A_T(y),0.5\},$$

$$1-A_I(a*c) \geq \beta > \bigvee\{1-A_I(a*(b*c)),1-A_I(b),0.5\},$$

$$1-A_F(u*w) \leq \gamma < \bigwedge\{1-A_F(u*(v*w)),1-A_F(v),0.5\}.$$

It follows that $A_T(x*(y*z))+\alpha > 1$ and $A_T(y)+\alpha > 1$, that is, $x*(y*z) \in T_q(A,\alpha)$ and $y \in T_q(A,\alpha)$, $A_I(a*(b*c))+\beta > 1$ and $A_I(b)+\beta > 1$, that is, $a*(b*c) \in I_q(A,\beta)$ and $b \in I_q(A,\beta)$, $A_F(u*(v*w))+\gamma < 1$ and $A_F(v)+\gamma < 1$, that is, $u*(v*w) \in F_q(A,\gamma)$ and $v \in F_q(A,\gamma)$, $A_T(x*z)+\alpha \leq 1$ and $A_T(x*z) \leq 1-\alpha < \alpha$, that is, $x*z \notin T_{\in\vee q}(A,\alpha)$, $A_I(a*c)+\alpha \leq 1$ and $A_I(a*c) \leq 1-\beta < \beta$, that is, $a*c \notin I_{\in\vee q}(A,\beta)$, $A_F(u*w)+\gamma > 1$ and $A_F(u*w) > 1-\gamma > \gamma$, that is, $u*w \notin F_{\in\vee q}(A,\gamma)$. It is a contradiction because A is a $(q,\in\vee q)$-neutrosophic ideal of X. Therefore, $x*z \in T_{\in\vee q}(A,\alpha_1\wedge\alpha_2)$, $a*c \in I_{\in\vee q}(A,\beta_1\wedge\beta_2)$, and $u*w \in F_{\in\vee q}(A,\gamma_1\vee\gamma_2)$. Consequently, A is an $(\in,\in\vee q)$-neutrosophic ideal of X. □

4 Conclusion

Neutrosophic ideals $(\in,\in)$ and $(q,\in\vee q)$-neutrosophic ideals are analyzed. We have discussed the criteria for neutrosophic $\in$-subsets, neutrosophic q-subsets, and neutrosophic $(q,\in\vee q)$-subsets to be ideals.

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