



Neutrosophic $\mathcal{N}$ -structures over Hilbert algebras

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Abstract

The notions of neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals of Hilbert algebras are introduced, and several properties are investigated. Conditions for neutrosophic $\mathcal{N}$ -structures to be neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals of Hilbert algebras are provided. The Cartesian product of neutrosophic $\mathcal{N}$ -structures is also supplied. Finally, we also find the property of the homomorphic pre-image of neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals.

Keywords: Hilbert algebra; neutrosophic $\mathcal{N}$ -subalgebra; neutrosophic $\mathcal{N}$ -ideal; homomorphic pre-image.

1 Introduction

Zadeh¹⁹ introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov¹ introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache proposed the term “neutrosophic” because “neutrosophic” etymologically comes from “neutrosophic” [French neuter, Latin neuter, neutral, and Greek sophia, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between “fuzzy/intuitionistic” logic/set and “neutrosophic” logic/set, that is, the included middle component, that is, the neutral/indeterminate/unknown part (besides the truth/membership and falsehood/non-membership components that both appear in fuzzy logic/set). Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). The concept of the neutrosophic set developed by Smarandache^{16,17} is a more general platform that extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set, and interval-valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various parts (refer to the site <http://fs.gallup.unm.edu/neutrosophy.htm>). Diego⁵ proved that Hilbert algebras form a locally finite variety. Hilbert algebras were treated by Busneag^{2,3} and Jun⁹ and some of their filters forming deductive systems were recognized. Dudek⁶ considered the fuzzification of subalgebras and deductive systems in Hilbert algebras.

The negative structure of sets is constantly being defined and studied. Jun et al.¹⁰ introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$ -structures in 2009. Jun et al.^{11,18} considered neutrosophic $\mathcal{N}$ -structures applied to BCK/BCI-algebras and neutrosophic commutative $\mathcal{N}$ -ideals in BCK-algebras in 2017. Jun et al.¹² studied neutrosophic positive implicative $\mathcal{N}$ -ideals in BCK-algebras in 2018. Rangsuk et al.¹⁵ introduced the notions of (special) neutrosophic $\mathcal{N}$ -UP-subalgebras, (special) neutrosophic $\mathcal{N}$ -near UP-filters, (special) neutrosophic $\mathcal{N}$ -UP-filters, (special) neutrosophic $\mathcal{N}$ -UP-ideals, and (special) neutrosophic $\mathcal{N}$ -strong UP-ideals of UP-algebras in 2019.

In this paper, the notions of neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals of Hilbert algebras are introduced, and several properties are investigated. Conditions for neutrosophic $\mathcal{N}$ -structures to be neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals of Hilbert algebras are provided. The Cartesian product of neutrosophic $\mathcal{N}$ -structures is also supplied. Finally, we also find the property of the homomorphic pre-image of neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals.

2 Preliminaries

Before we begin our study, we will give the definition of a Hilbert algebra.

Definition 2.1.⁵ A Hilbert algebra is a triplet with the formula $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:

- (1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
- (2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
- (3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

In,⁶ the following conclusion was established.

Lemma 2.2. Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then

- (1) $(\forall x \in X)(x \cdot x = 1)$,
- (2) $(\forall x \in X)(1 \cdot x = x)$,
- (3) $(\forall x \in X)(x \cdot 1 = 1)$,
- (4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,
- (5) $(\forall x, y, z \in X)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.3.²⁰ A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a *subalgebra* of X if $x \cdot y \in D$ for all $x, y \in D$.

Definition 2.4.^{4,7} A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an *ideal* of X if the following conditions hold:

- (1) $1 \in D$,
- (2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
- (3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

A *fuzzy set*¹⁹ in a nonempty set X is defined to be a function $\mu : X \rightarrow [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.5.¹⁴ A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 2.6.⁸ A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy ideal* of X if the following conditions hold:

- (1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
- (2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,
- (3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

Definition 2.7.¹ A neutrosophic set in a nonempty set H is defined to be a structure

$$A := \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in H\}, \quad (1)$$

where $T_A : H \rightarrow [0, 1]$ is a truth membership function, $I_A : H \rightarrow [0, 1]$ is an indeterminate membership function, and $F_A : H \rightarrow [0, 1]$ is a false membership function. The neutrosophic set in (1) is simply denoted by $A = (X, T_A, I_A, F_A)$.

Definition 2.8.¹³ We denote the family of all functions from a nonempty set X to the closed interval $[-1, 0]$ of the real line by $\mathcal{F}(X, [-1, 0])$. An element of $\mathcal{F}(X, [-1, 0])$ is called a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$ -function on X). An ordered pair of a nonempty set X and an $\mathcal{N}$ -function on X is called an $\mathcal{N}$ -fuzzy structure. A neutrosophic $\mathcal{N}$ -structure $X_{\mathcal{N}}$ over a nonempty universe of discourse X is defined to be the structure $(X, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$, where $T_{\mathcal{N}}, I_{\mathcal{N}}$, and $F_{\mathcal{N}}$ are $\mathcal{N}$ -functions on X which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function on X , respectively.

For the sake of simplicity, we will use the notation $X_{\mathcal{N}}$ instead of the neutrosophic $\mathcal{N}$ -structure $(X, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$.¹⁰

Definition 2.9.¹⁵ Let $X_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$ -structure over a nonempty set X . The neutrosophic $\mathcal{N}$ -structure $\overline{X_{\mathcal{N}}} = (X, \overline{T_{\mathcal{N}}}, \overline{I_{\mathcal{N}}}, \overline{F_{\mathcal{N}}})$ defined by

$$(\forall x \in X) \begin{pmatrix} \overline{T_{\mathcal{N}}}(x) = -1 - T_{\mathcal{N}}(x) \\ \overline{I_{\mathcal{N}}}(x) = -1 - I_{\mathcal{N}}(x) \\ \overline{F_{\mathcal{N}}}(x) = -1 - F_{\mathcal{N}}(x) \end{pmatrix} \quad (2)$$

is called the *complement* of $X_{\mathcal{N}}$ in X .

3 Neutrosophic $\mathcal{N}$ -fuzzy subalgebras and ideals of Hilbert algebras

In what follows, let X denote a Hilbert algebra $(X, \cdot, 1)$ unless otherwise specified.

Definition 3.1. A neutrosophic $\mathcal{N}$ -structure $X_{\mathcal{N}}$ over X is called a neutrosophic $\mathcal{N}$ -fuzzy subalgebra of X if

$$(\forall x, y \in X) \begin{pmatrix} T_{\mathcal{N}}(x \cdot y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} \\ I_{\mathcal{N}}(x \cdot y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} \\ F_{\mathcal{N}}(x \cdot y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} \end{pmatrix}. \quad (3)$$

Example 3.2. Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define a neutrosophic $\mathcal{N}$ -structure $X_{\mathcal{N}}$ over X as follows:

X	1	x	y	z	0
$T_{\mathcal{N}}$	-1	-0.8	-0.8	-0.7	-0.4
$I_{\mathcal{N}}$	-0.3	-0.5	-0.7	-0.3	-0.6
$F_{\mathcal{N}}$	-1	-0.8	-0.8	-0.7	-0.4

Then $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of X .

Proposition 3.3. Every neutrosophic $\mathcal{N}$ -subalgebra of X satisfies

$$(\forall x \in X) \begin{pmatrix} T_{\mathcal{N}}(1) \leq T_{\mathcal{N}}(x) \\ I_{\mathcal{N}}(1) \geq I_{\mathcal{N}}(x) \\ F_{\mathcal{N}}(1) \leq F_{\mathcal{N}}(x) \end{pmatrix}. \quad (4)$$

Proof. For any $x \in X$, we have

$$\begin{aligned} T_{\mathcal{N}}(1) &= T_{\mathcal{N}}(x \cdot x) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(x)\} = T_{\mathcal{N}}(x), \\ I_{\mathcal{N}}(1) &= I_{\mathcal{N}}(x \cdot x) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(x)\} = I_{\mathcal{N}}(x), \\ F_{\mathcal{N}}(1) &= F_{\mathcal{N}}(x \cdot x) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(x)\} = F_{\mathcal{N}}(x). \end{aligned}$$

□

Definition 3.4. A neutrosophic $\mathcal{N}$ -structure $X_{\mathcal{N}}$ over X is called a neutrosophic $\mathcal{N}$ -ideal of X if it satisfies (4) and

$$(\forall x, y \in X) \begin{pmatrix} T_{\mathcal{N}}(x \cdot y) \leq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}(x \cdot y) \geq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}(x \cdot y) \leq F_{\mathcal{N}}(y) \end{pmatrix}, \quad (5)$$

$$(\forall x, y_1, y_2 \in X) \begin{pmatrix} T_{\mathcal{N}}((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{T_{\mathcal{N}}(y_1), T_{\mathcal{N}}(y_2)\} \\ I_{\mathcal{N}}((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{I_{\mathcal{N}}(y_1), I_{\mathcal{N}}(y_2)\} \\ F_{\mathcal{N}}((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{F_{\mathcal{N}}(y_1), F_{\mathcal{N}}(y_2)\} \end{pmatrix}. \quad (6)$$

Example 3.5. From Example 3.2, $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -ideal of X .

Proposition 3.6. If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -ideal of X , then

$$(\forall x, y \in X) \begin{pmatrix} T_{\mathcal{N}}((y \cdot x) \cdot x) \leq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}((y \cdot x) \cdot x) \geq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}((y \cdot x) \cdot x) \leq F_{\mathcal{N}}(y) \end{pmatrix}. \quad (7)$$

Proof. Let $x, y \in X$. By (6), we have

$$\begin{aligned} T_{\mathcal{N}}((y \cdot x) \cdot x) &= T_{\mathcal{N}}((1 \cdot (y \cdot x)) \cdot x) \leq \max\{T_{\mathcal{N}}(1), T_{\mathcal{N}}(y)\} = T_{\mathcal{N}}(y), \\ I_{\mathcal{N}}((y \cdot x) \cdot x) &= I_{\mathcal{N}}((1 \cdot (y \cdot x)) \cdot x) \geq \min\{I_{\mathcal{N}}(1), I_{\mathcal{N}}(y)\} = I_{\mathcal{N}}(y), \\ F_{\mathcal{N}}((y \cdot x) \cdot x) &= F_{\mathcal{N}}((1 \cdot (y \cdot x)) \cdot x) \leq \max\{F_{\mathcal{N}}(1), F_{\mathcal{N}}(y)\} = F_{\mathcal{N}}(y). \end{aligned}$$

□

Lemma 3.7. If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -ideal of X , then

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \begin{cases} T_{\mathcal{N}}(x) \geq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}(x) \leq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}(x) \geq F_{\mathcal{N}}(y) \end{cases} \right). \quad (8)$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{aligned} T_{\mathcal{N}}(y) &= T_{\mathcal{N}}(1 \cdot y) \\ &= T_{\mathcal{N}}(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\leq \max\{T_{\mathcal{N}}(x \cdot y), T_{\mathcal{N}}(x)\} \\ &\leq \max\{T_{\mathcal{N}}(1), T_{\mathcal{N}}(x)\} \\ &= T_{\mathcal{N}}(x), \\ I_{\mathcal{N}}(y) &= I_{\mathcal{N}}(1 \cdot y) \\ &= I_{\mathcal{N}}(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\geq \min\{I_{\mathcal{N}}(x \cdot y), I_{\mathcal{N}}(x)\} \\ &\geq \min\{I_{\mathcal{N}}(1), I_{\mathcal{N}}(x)\} \\ &= I_{\mathcal{N}}(x), \end{aligned}$$

$$\begin{aligned}
 F_{\mathcal{N}}(y) &= F_{\mathcal{N}}(1 \cdot y) \\
 &= F_{\mathcal{N}}(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\
 &\leq \max\{F_{\mathcal{N}}(x \cdot y), F_{\mathcal{N}}(x)\} \\
 &\leq \max\{F_{\mathcal{N}}(1), F_{\mathcal{N}}(x)\} \\
 &= F_{\mathcal{N}}(x).
 \end{aligned}$$

□

Theorem 3.8. Every neutrosophic $\mathcal{N}$ -ideal of X is a neutrosophic $\mathcal{N}$ -subalgebra of X .

Proof. Let $X_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$ -ideal of X . By (5), we have

$$\begin{aligned}
 T_{\mathcal{N}}(x \cdot y) &\leq T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}, \\
 I_{\mathcal{N}}(x \cdot y) &\geq I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}, \\
 F_{\mathcal{N}}(x \cdot y) &\leq F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}.
 \end{aligned}$$

Hence, $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of X .

□

Proposition 3.9. If $\{X_{\mathcal{N}}^i \mid i \in \Delta\}$ is a family of neutrosophic $\mathcal{N}$ -subalgebras of X , then $\bigwedge_{i \in \Delta} X_{\mathcal{N}}^i$ is a neutrosophic $\mathcal{N}$ -subalgebra of X .

Proof. Let $\{X_{\mathcal{N}}^i \mid i \in \Delta\}$ be a family of neutrosophic $\mathcal{N}$ -subalgebras of X . Let $x, y \in X$. Then

$$\begin{aligned}
 (\bigwedge_{i \in \Delta} T_{\mathcal{N}_i})(x \cdot y) &= \sup_{i \in \Delta} \{T_{\mathcal{N}_i}(x \cdot y)\} \\
 &\leq \sup_{i \in \Delta} \{\max\{T_{\mathcal{N}_i}(x), T_{\mathcal{N}_i}(y)\}\} \\
 &\leq \max\{\sup_{i \in \Delta} \{T_{\mathcal{N}_i}(x)\}, \sup_{i \in \Delta} \{T_{\mathcal{N}_i}(y)\}\} \\
 &= \max\{(\bigwedge_{i \in \Delta} T_{\mathcal{N}_i})(x), (\bigwedge_{i \in \Delta} T_{\mathcal{N}_i})(y)\}, \\
 (\bigwedge_{i \in \Delta} I_{\mathcal{N}_i})(x \cdot y) &= \inf_{i \in \Delta} \{I_{\mathcal{N}_i}(x \cdot y)\} \\
 &\geq \inf_{i \in \Delta} \{\min\{I_{\mathcal{N}_i}(x), I_{\mathcal{N}_i}(y)\}\} \\
 &\geq \min\{\inf_{i \in \Delta} \{I_{\mathcal{N}_i}(x)\}, \inf_{i \in \Delta} \{I_{\mathcal{N}_i}(y)\}\} \\
 &= \min\{(\bigwedge_{i \in \Delta} I_{\mathcal{N}_i})(x), (\bigwedge_{i \in \Delta} I_{\mathcal{N}_i})(y)\}, \\
 (\bigwedge_{i \in \Delta} F_{\mathcal{N}_i})(x \cdot y) &= \sup_{i \in \Delta} \{F_{\mathcal{N}_i}(x \cdot y)\} \\
 &\leq \sup_{i \in \Delta} \{\max\{F_{\mathcal{N}_i}(x), F_{\mathcal{N}_i}(y)\}\} \\
 &\leq \max\{\sup_{i \in \Delta} \{F_{\mathcal{N}_i}(x)\}, \sup_{i \in \Delta} \{F_{\mathcal{N}_i}(y)\}\} \\
 &= \max\{(\bigwedge_{i \in \Delta} F_{\mathcal{N}_i})(x), (\bigwedge_{i \in \Delta} F_{\mathcal{N}_i})(y)\}.
 \end{aligned}$$

Hence, $\bigwedge_{i \in \Delta} X_{\mathcal{N}}^i$ is a neutrosophic $\mathcal{N}$ -subalgebra of X .

□

The following proposition can be proved similarly to Proposition 3.9.

Proposition 3.10. If $\{X_{\mathcal{N}}^i \mid i \in \Delta\}$ is a family of neutrosophic $\mathcal{N}$ -ideals of X , then $\bigwedge_{i \in \Delta} X_{\mathcal{N}}^i$ is a neutrosophic $\mathcal{N}$ -ideal of X .

Definition 3.11. Let $X_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$ -structure over a nonempty set X . The neutrosophic $\mathcal{N}$ -structures $\oplus X_{\mathcal{N}}$, $\otimes X_{\mathcal{N}}$, and $\odot X_{\mathcal{N}}$ are defined as $\oplus X_{\mathcal{N}} = (X, T_{\mathcal{N}}, \overline{T_{\mathcal{N}}}, F_{\mathcal{N}})$, $\otimes X_{\mathcal{N}} = (X, \overline{I_{\mathcal{N}}}, I_{\mathcal{N}}, F_{\mathcal{N}})$, and $\odot X_{\mathcal{N}} = (X, \overline{I_{\mathcal{N}}}, I_{\mathcal{N}}, \overline{F_{\mathcal{N}}})$.

Theorem 3.12. If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of X , then $\oplus X_{\mathcal{N}}$, $\otimes X_{\mathcal{N}}$, and $\odot X_{\mathcal{N}}$ are neutrosophic $\mathcal{N}$ -subalgebras of X .

Proof. Let $x, y \in X$. Then

$$\begin{aligned}\overline{T_N}(x \cdot y) &= -1 - T_N(x \cdot y) \\ &\geq -1 - \max\{T_N(x), T_N(y)\} \\ &= \min\{-1 - T_N(x), -1 - T_N(y)\} \\ &= \min\{\overline{T_N}(x), \overline{T_N}(y)\}, \\ \overline{I_N}(x \cdot y) &= -1 - I_N(x \cdot y) \\ &\leq -1 - \min\{I_N(x), I_N(y)\} \\ &= \max\{-1 - I_N(x), -1 - I_N(y)\} \\ &= \max\{\overline{I_N}(x), \overline{I_N}(y)\}.\end{aligned}$$

Hence, $\oplus X_N$, $\otimes X_N$, and $\odot X_N$ are neutrosophic $\mathcal{N}$ -subalgebras of X . $\square$

The following theorem can be proved similarly to Theorem 3.12.

Theorem 3.13. *If X_N is a neutrosophic $\mathcal{N}$ -ideal of X , then $\oplus X_N$, $\otimes X_N$, and $\odot$ are neutrosophic $\mathcal{N}$ -ideals of X .*

Theorem 3.14. *If X_N is a neutrosophic $\mathcal{N}$ -subalgebra of X , then the sets $X_{T_N} = \{x \in X \mid T_N(x) = T_N(1)\}$, $X_{I_N} = \{x \in X \mid I_N(x) = I_N(1)\}$, and $X_{F_N} = \{x \in X \mid F_N(x) = F_N(1)\}$ are subalgebras of X .*

Proof. Let $x, y \in X_{T_N}$. Then $T_N(x) = T_N(1) = T_N(y)$ and $T_N(x \cdot y) \leq \max\{T_N(x), T_N(y)\} = T_N(1)$. By (4), we have $T_N(x \cdot y) = T_N(1)$; hence $x \cdot y \in X_{T_N}$. Let $x, y \in X_{I_N}$. Then $I_N(x) = I_N(1) = I_N(y)$ and $I_N(x \cdot y) \geq \min\{I_N(x), I_N(y)\} = I_N(1)$. By (4), we have $I_N(x \cdot y) = I_N(1)$; hence $x \cdot y \in X_{I_N}$. Let $x, y \in X_{F_N}$. Then $F_N(x) = F_N(1) = F_N(y)$ and $F_N(x \cdot y) \leq \max\{F_N(x), F_N(y)\} = F_N(1)$. By (4), we have $F_N(x \cdot y) = F_N(1)$; hence $x \cdot y \in X_{F_N}$. Hence, the sets X_{T_N} , X_{I_N} , and X_{F_N} are subalgebras of X . $\square$

The following proposition can be proved similarly to Theorem 3.14.

Theorem 3.15. *If X_N is a neutrosophic $\mathcal{N}$ -ideal of X , then the sets X_{T_N} , X_{I_N} , and X_{F_N} are ideals of X .*

For any numbers $a^+, a^-, b^+, b^-, c^+, c^- \in [-1, 0]$ such that $a^+ > a^-, b^+ > b^-, c^+ > c^-$ and a nonempty subset G of X , define a neutrosophic $\mathcal{N}$ -structure

$$X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix} = \left(X, T_N^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix}, I_N^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix}, F_N^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} \right)$$

over X , where

$$\begin{aligned}T_N^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) &= \begin{cases} a^- & \text{if } x \in G \\ a^+ & \text{otherwise,} \end{cases} \\ I_N^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x) &= \begin{cases} b^+ & \text{if } x \in G \\ b^- & \text{otherwise,} \end{cases} \\ F_N^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x) &= \begin{cases} c^- & \text{if } x \in G \\ c^+ & \text{otherwise.} \end{cases}\end{aligned}$$

Lemma 3.16. *If the constant 1 of X is in a nonempty subset G of X , then the neutrosophic $\mathcal{N}$ -structure $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ over X satisfies (4).*

Proof. If $1 \in G$, then $T_N^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (1) = a^-$, $I_N^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (1) = b^+$, and $F_N^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (1) = c^-$. Thus,

$$(\forall x \in X) \begin{pmatrix} T_N^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (1) = a^- \leq T_N^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) \\ I_N^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (1) = b^+ \geq I_N^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x) \\ F_N^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (1) = c^- \leq F_N^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x) \end{pmatrix}. \quad (9)$$

Hence, $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ satisfies (4). $\square$

Lemma 3.17. *If the neutrosophic $\mathcal{N}$ -structure $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ over X satisfies (4), then the constant 1 of X is in a nonempty subset G of X .*

Proof. Assume that the neutrosophic $\mathcal{N}$ -structure $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ in X satisfies (4). Then $T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (1) \leq T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x)$ for all $x \in X$. Since G is nonempty, there exists $g \in G$. Thus, $T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (g) = a^-$ and so $T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (1) \geq a^- = T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (g) \geq T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (1)$, that is, $T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (1) = a^-$. Hence, $1 \in G$. $\square$

Theorem 3.18. *The neutrosophic $\mathcal{N}$ -structure $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ in X is a neutrosophic $\mathcal{N}$ -subalgebra of X if and only if a nonempty subset G of X is a subalgebra of X .*

Proof. Assume that $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ is a neutrosophic $\mathcal{N}$ -subalgebra of X . Let $x, y \in G$. Then $T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) = a^- = T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y)$. Thus,

$$\begin{aligned} T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x \cdot y) &\leq \max \left\{ T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x), T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) \right\} \\ &= \max \{ a^-, a^- \} \\ &= a^- \\ &\leq T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x \cdot y) \end{aligned}$$

and so $T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x \cdot y) = a^-$. Thus, $x \cdot y \in G$. Hence, G is a subalgebra of X .

Conversely, assume that G is a subalgebra of X . Let $x, y \in X$.

Case 1: Let $x, y \in G$. Then

$$\begin{aligned} T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) &= a^- = T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y), \\ I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x) &= b^+ = I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y), \\ F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x) &= c^- = F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y). \end{aligned}$$

Since G is a subalgebra of X , we have $x \cdot y \in G$ and so $T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x \cdot y) = a^-$, $I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x \cdot y) = b^+$, and $F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x \cdot y) = c^-$. Thus,

$$\begin{aligned} T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x \cdot y) &= a^- \leq a^- = \max \{ a^-, a^- \} = \max \left\{ T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x), T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) \right\}, \\ I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x \cdot y) &= b^+ \geq b^+ = \min \{ b^+, b^+ \} = \min \left\{ I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x), I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y) \right\}, \\ F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x \cdot y) &= c^- \leq c^- = \max \{ c^-, c^- \} = \max \left\{ F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x), F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y) \right\}. \end{aligned}$$

Case 2: Let $x \notin G$ or $y \notin G$. Then

$$\begin{aligned} T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) &= a^+ \text{ or } T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) = a^+, \\ I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x) &= b^- \text{ or } I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y) = b^-, \\ F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x) &= c^+ \text{ or } F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y) = c^+. \end{aligned}$$

Thus,

$$\begin{aligned} \max \left\{ T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x), T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) \right\} &= a^+, \\ \min \left\{ I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x), I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y) \right\} &= b^-, \\ \max \left\{ F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x), F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y) \right\} &= c^+. \end{aligned}$$

Therefore,

$$\begin{aligned} T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x \cdot y) &\leq a^+ = \max \left\{ T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x), T_{\mathcal{N}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) \right\} \\ I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x \cdot y) &\geq b^- = \min \left\{ I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x), I_{\mathcal{N}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y) \right\} \\ F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x \cdot y) &\leq c^+ = \max \left\{ F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x), F_{\mathcal{N}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y) \right\}. \end{aligned}$$

Hence, $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ is a neutrosophic $\mathcal{N}$ -subalgebra of X . □

The following theorem can be proved similarly to Theorem 3.18.

Theorem 3.19. The neutrosophic $\mathcal{N}$ -structure $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ over X is a neutrosophic $\mathcal{N}$ -ideal of X if and only if a nonempty subset G of X is an ideal of X .

Definition 3.20. Let f be an $\mathcal{N}$ -function on a nonempty set X . For any $t \in [-1, 0]$, the sets $U(f : t) = \{x \in X \mid f(x) \geq t\}$ is called an upper t -level subset of f , $L(f : t) = \{x \in X \mid f(x) \leq t\}$ is called a lower t -level subset of f , and $E(f : t) = \{x \in X \mid f(x) = t\}$ is called an equal t -level subset of f .

Theorem 3.21. A neutrosophic $\mathcal{N}$ -structure $X_{\mathcal{N}}$ over X is a neutrosophic $\mathcal{N}$ -subalgebra of X if and only if for all $a, b, c \in [-1, 0]$, the sets $L(T_{\mathcal{N}} : a)$, $U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are either empty or subalgebras of X .

Proof. Assume that $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of X . Let $a, b, c \in [-1, 0]$ be such that $L(T_{\mathcal{N}} : a)$, $U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are nonempty. Let $x, y \in L(T_{\mathcal{N}} : a)$. Then $T_{\mathcal{N}}(x) \leq a$ and $T_{\mathcal{N}}(y) \leq a$, so a is an upper bound of $\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$. By (3), we have $T_{\mathcal{N}}(x \cdot y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} \leq a$. Thus, $x \cdot y \in L(T_{\mathcal{N}} : a)$. Let $x, y \in U(I_{\mathcal{N}} : b)$. Then $I_{\mathcal{N}}(x) \geq b$ and $I_{\mathcal{N}}(y) \geq b$, so b is a lower bound of $\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$. By (3), we have $I_{\mathcal{N}}(x \cdot y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} \geq b$. Thus, $x \cdot y \in U(I_{\mathcal{N}} : b)$. Let $x, y \in L(F_{\mathcal{N}} : c)$. Then $F_{\mathcal{N}}(x) \leq c$ and $F_{\mathcal{N}}(y) \leq c$, so c is an upper bound of $\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$. By (3), we have $F_{\mathcal{N}}(x \cdot y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} \leq c$. Thus, $x \cdot y \in L(F_{\mathcal{N}} : c)$. Hence, $L(T_{\mathcal{N}} : a)$, $U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are subalgebras of X .

Conversely, assume that for all $a, b, c \in [-1, 0]$, the sets $L(T_{\mathcal{N}} : a)$, $U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are either empty or subalgebras of X . Let $x, y \in X$. Then $T_{\mathcal{N}}(x) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$ and $T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$. Thus, $x, y \in L(T_{\mathcal{N}} : \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}) \neq \emptyset$. By the assumption, we have $L(T_{\mathcal{N}} : \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\})$ is a subalgebra of X . Then $x \cdot y \in L(T_{\mathcal{N}} : \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\})$. Thus, $T_{\mathcal{N}}(x \cdot y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$. Let $x, y \in X$. Then $I_{\mathcal{N}}(x) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$ and $I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$. Thus, $x, y \in U(I_{\mathcal{N}} : \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}) \neq \emptyset$. By the assumption, we have $U(I_{\mathcal{N}} : \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\})$ is a subalgebra of X . Then $x \cdot y \in U(I_{\mathcal{N}} : \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\})$. Thus,

$I_{\mathcal{N}}(x \cdot y) \geq \max\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$. Let $x, y \in X$. Then $F_{\mathcal{N}}(x) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$ and $F_{\mathcal{N}}(x) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$. Thus, $x, y \in L(F_{\mathcal{N}} : \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}) \neq \emptyset$. By the assumption, we have $L(F_{\mathcal{N}} : \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\})$ is a subalgebra of X . Then $x \cdot y \in L(F_{\mathcal{N}} : \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\})$. Thus, $F_{\mathcal{N}}(x \cdot y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$. Hence, $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of X . $\square$

The following theorem can be proved similarly to Theorem 3.21.

Theorem 3.22. *A neutrosophic $\mathcal{N}$ -structure $X_{\mathcal{N}}$ over X is a neutrosophic $\mathcal{N}$ -ideal of X if and only if for all $a, b, c \in [-1, 0]$, the sets $L(T_{\mathcal{N}} : a)$, $U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are either empty or ideals of X .*

The following two corollaries are a straightforward result of Theorems 3.21 and 3.22.

Corollary 3.23. *A neutrosophic $\mathcal{N}$ -structure $X_{\mathcal{N}}$ over X is a neutrosophic $\mathcal{N}$ -subalgebra of X if and only if for all $a, b, c \in [-1, 0]$, the set $L(T_{\mathcal{N}} : a) \cap U(I_{\mathcal{N}} : b) \cap L(F_{\mathcal{N}} : c)$ is either empty or a subalgebra of X .*

Corollary 3.24. *A neutrosophic $\mathcal{N}$ -structure $X_{\mathcal{N}}$ over X is a neutrosophic $\mathcal{N}$ -ideal of X if and only if for all $a, b, c \in [-1, 0]$, the set $L(T_{\mathcal{N}} : a) \cap U(I_{\mathcal{N}} : b) \cap L(F_{\mathcal{N}} : c)$ is either empty or an ideal of X .*

Definition 3.25. Let $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ be neutrosophic $\mathcal{N}$ -structures of X and Y , respectively. The Cartesian product $X_{\mathcal{N}} \times Y_{\mathcal{N}} = (X \times Y, \Delta, \Theta, \Lambda)$ defined by

$$(\forall (x, y) \in X \times Y) \begin{pmatrix} \Delta(x, y) = \max\{T_{\mathcal{N}X}(x), T_{\mathcal{N}Y}(y)\} \\ \Theta(x, y) = \min\{I_{\mathcal{N}X}(x), I_{\mathcal{N}Y}(y)\} \\ \Lambda(x, y) = \max\{F_{\mathcal{N}X}(x), F_{\mathcal{N}Y}(y)\} \end{pmatrix}, \quad (10)$$

where Δ , Θ , and Λ are $\mathcal{N}$ -functions on $X \times Y$.

Remark 3.26. Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Then $(X \times Y, \diamond, (1_X, 1_Y))$ is a Hilbert algebra defined by $(x, y) \diamond (u, v) = (x \cdot u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.

Proposition 3.27. *If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$ -subalgebras of Hilbert algebras X and Y , respectively, then the Cartesian product $X_{\mathcal{N}} \times Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of $X \times Y$.*

Proof. Assume that $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$ -subalgebras of Hilbert algebras X and Y , respectively. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned} \Delta((x_1, y_1) \diamond (x_2, y_2)) &= \Delta(x_1 \cdot x_2, y_1 \star y_2) \\ &= \max\{T_{\mathcal{N}X}(x_1 \cdot x_2), T_{\mathcal{N}Y}(y_1 \star y_2)\} \\ &\leq \max\{\max\{T_{\mathcal{N}X}(x_1), T_{\mathcal{N}X}(x_2)\}, \max\{T_{\mathcal{N}Y}(y_1), T_{\mathcal{N}Y}(y_2)\}\} \\ &= \max\{\max\{T_{\mathcal{N}X}(x_1), T_{\mathcal{N}Y}(y_1)\}, \max\{T_{\mathcal{N}X}(x_2), T_{\mathcal{N}Y}(y_2)\}\} \\ &= \max\{\Delta(x_1, y_1), \Delta(x_2, y_2)\}, \\ \Theta((x_1, y_1) \diamond (x_2, y_2)) &= \Theta(x_1 \cdot x_2, y_1 \star y_2) \\ &= \min\{I_{\mathcal{N}X}(x_1 \cdot x_2), I_{\mathcal{N}Y}(y_1 \star y_2)\} \\ &\geq \min\{\min\{I_{\mathcal{N}X}(x_1), I_{\mathcal{N}X}(x_2)\}, \min\{I_{\mathcal{N}Y}(y_1), I_{\mathcal{N}Y}(y_2)\}\} \\ &= \min\{\min\{I_{\mathcal{N}X}(x_1), I_{\mathcal{N}Y}(y_1)\}, \min\{I_{\mathcal{N}X}(x_2), I_{\mathcal{N}Y}(y_2)\}\} \\ &= \min\{\Theta(x_1, y_1), \Theta(x_2, y_2)\}, \\ \Lambda((x_1, y_1) \diamond (x_2, y_2)) &= \Lambda(x_1 \cdot x_2, y_1 \star y_2) \\ &= \max\{F_{\mathcal{N}X}(x_1 \cdot x_2), F_{\mathcal{N}Y}(y_1 \star y_2)\} \\ &\leq \max\{\max\{F_{\mathcal{N}X}(x_1), F_{\mathcal{N}X}(x_2)\}, \max\{F_{\mathcal{N}Y}(y_1), F_{\mathcal{N}Y}(y_2)\}\} \\ &= \max\{\max\{F_{\mathcal{N}X}(x_1), F_{\mathcal{N}Y}(y_1)\}, \max\{F_{\mathcal{N}X}(x_2), F_{\mathcal{N}Y}(y_2)\}\} \\ &= \max\{\Lambda(x_1, y_1), \Lambda(x_2, y_2)\}. \end{aligned}$$

Hence, $X_{\mathcal{N}} \times Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of $X \times Y$. $\square$

The following proposition can be proved similarly to Proposition 3.27.

Proposition 3.28. If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$ -ideals of Hilbert algebras X and Y , respectively, then the Cartesian product $X_{\mathcal{N}} \times Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -ideal of $X \times Y$.

The following two theorems are a straightforward result of Propositions 3.27 and 3.28, and Theorems 3.12 and 3.13.

Theorem 3.29. If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$ -subalgebras of Hilbert algebras X and Y , respectively, then $\oplus(X_{\mathcal{N}} \times Y_{\mathcal{N}})$, $\otimes(X_{\mathcal{N}} \times Y_{\mathcal{N}})$, and $\odot(X_{\mathcal{N}} \times Y_{\mathcal{N}})$ are neutrosophic $\mathcal{N}$ -subalgebras of X .

Theorem 3.30. If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$ -ideals of Hilbert algebras X and Y , respectively, then $\oplus(X_{\mathcal{N}} \times Y_{\mathcal{N}})$, $\otimes(X_{\mathcal{N}} \times Y_{\mathcal{N}})$, and $\odot(X_{\mathcal{N}} \times Y_{\mathcal{N}})$ are neutrosophic $\mathcal{N}$ -ideals of X .

Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. A mapping $f : X \rightarrow Y$ of Hilbert algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \rightarrow Y$ is a homomorphism of Hilbert algebras, then $f(1_X) = 1_Y$. Let $f : X \rightarrow Y$ be a homomorphism of Hilbert algebras. For any neutrosophic $\mathcal{N}$ -structure $Y_{\mathcal{N}}$ over Y , we define a new neutrosophic $\mathcal{N}$ -structure $f^{-1}(Y_{\mathcal{N}}) = (X, T_{f^{-1}(Y_{\mathcal{N}})}, I_{f^{-1}(Y_{\mathcal{N}})}, F_{f^{-1}(Y_{\mathcal{N}})})$ over X by

$$(\forall x \in X) \begin{pmatrix} T_{f^{-1}(Y_{\mathcal{N}})}(x) = T_{\mathcal{N}Y}(f(x)) \\ I_{f^{-1}(Y_{\mathcal{N}})}(x) = I_{\mathcal{N}Y}(f(x)) \\ F_{f^{-1}(Y_{\mathcal{N}})}(x) = F_{\mathcal{N}Y}(f(x)) \end{pmatrix}.$$

Theorem 3.31. Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Let $f : X \rightarrow Y$ be a homomorphism and $Y_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$ -structure over Y . If $Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of Y , then $f^{-1}(Y_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$ -subalgebra of X .

Proof. Assume that $Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -subalgebra of Y . Let $x, y \in X$. Then

$$\begin{aligned} T_{f^{-1}(Y_{\mathcal{N}})}(x \cdot y) &= T_{\mathcal{N}Y}(f(x \cdot y)) \\ &= T_{\mathcal{N}Y}(f(x) \star f(y)) \\ &\leq \max\{T_{\mathcal{N}Y}(f(x)), T_{\mathcal{N}Y}(f(y))\} \\ &= \max\{T_{f^{-1}(Y_{\mathcal{N}})}(x), T_{f^{-1}(Y_{\mathcal{N}})}(y)\}, \\ I_{f^{-1}(Y_{\mathcal{N}})}(x \cdot y) &= I_{\mathcal{N}Y}(f(x \cdot y)) \\ &= I_{\mathcal{N}Y}(f(x) \star f(y)) \\ &\geq \min\{I_{\mathcal{N}Y}(f(x)), I_{\mathcal{N}Y}(f(y))\} \\ &= \min\{I_{f^{-1}(Y_{\mathcal{N}})}(x), I_{f^{-1}(Y_{\mathcal{N}})}(y)\}, \\ F_{f^{-1}(Y_{\mathcal{N}})}(x \cdot y) &= F_{\mathcal{N}Y}(f(x \cdot y)) \\ &= F_{\mathcal{N}Y}(f(x) \star f(y)) \\ &\leq \max\{F_{\mathcal{N}Y}(f(x)), F_{\mathcal{N}Y}(f(y))\} \\ &= \max\{F_{f^{-1}(Y_{\mathcal{N}})}(x), F_{f^{-1}(Y_{\mathcal{N}})}(y)\}. \end{aligned}$$

Hence, $f^{-1}(Y_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$ -subalgebra of X . □

The following theorem can be proved similarly to Theorem 3.31.

Theorem 3.32. Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Let $f : X \rightarrow Y$ be a homomorphism and $Y_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$ -structure over Y . If $Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$ -ideal of Y , then $f^{-1}(Y_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$ -ideal of X .

4 Conclusion

In this paper, we have introduced the notions of neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals of Hilbert algebras. Conditions for neutrosophic $\mathcal{N}$ -structures to be neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals of Hilbert algebras are provided. From our study, we found that the Cartesian product of neutrosophic $\mathcal{N}$ -subalgebras (ideals) of a Hilbert algebra is a neutrosophic $\mathcal{N}$ -subalgebra (ideal). Finally, we found that the homomorphic pre-image of a neutrosophic $\mathcal{N}$ -subalgebra (ideal) of a Hilbert algebra is a neutrosophic $\mathcal{N}$ -subalgebra (ideal).

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