$\mu - L -$Closed Subsets of Noetherian Generalized Topological Spaces

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Abstract

In the final years of the 20th century, the notion of generalized topological spaces was introduced, marking a significant shift in the field of topology. This paper focuses on a subset of $\wp(X)$ on a non-empty set X that is closed under arbitrary unions, defining a generalized topology and subsequently a generalized topological space (GTS) denoted by (X, μ). Within this framework, we explore the concept of Noetherian generalized topological spaces and delve into the properties of $\mu - L -$closed subsets within the Noetherian GTS. The investigation reveals that subspaces of a $\mu -$Noetherian GTS X, with the induced topology, inherit the $\mu -$Noetherian property and exhibit finitely many non-empty $\mu -$irreducible components. Furthermore, the study extends to the analysis of hereditary properties, regular $\mu - G_\delta$, $\mu - d_\delta$, $\mu -$irreducible $L -$closed subsets, and the product properties of $\mu - L -$closed subsets under $(\mu, \mu') -$continuous functions. We also establish the closure property of finite unions in $\mu -$Noetherian GTS and clarify the homeomorphic nature of $\mu -$Noetherian GTS (X, μ) to itself.

Keywords: GTS, $\mu -$Noetherian, $(\mu, \mu') -$continuous function; fuzzy topology; neutrosophic topology.

1. Introduction

The concept of generalized topological spaces, initially introduced by Ákos Császár [1], has significantly influenced the development of topology. This paper builds upon the foundational work of Császár, Almuhur et al. [2], and Hdeib and Pareek [3], who respectively contributed to the understanding of generalized topology, $\mu - L -$closed subsets, and $L -$closed spaces. These concepts have laid the groundwork for the exploration of Noetherian generalized topological spaces.

The generalized topological space (X, μ), as presented in the study, behaves modulo small sets wrapped in an ideal, aligning closely with the essence of generalized topology. The topological separation axioms, originally designed to distinguish non-homeomorphic topological spaces, are fundamental to the discussion. These axioms are pivotal in asserting that two topological spaces, X and Y, are non-homeomorphic if one satisfies a separation axiom that the other does not.

Recent years have witnessed extensive studies on the structure and characteristics of various aspects of generalized topological spaces [4]. Research by Sarsak [5] on axioms $\mu - D_0, \mu - D_1, \mu - D_2, \mu - T_0, \mu - T_1, \mu - R_0\ and\ \mu - R_1$ of Separation has further enriched the discourse, encompassing enlarged versions of the final five axioms as delineated by Á Császár.

Moreover, the isomorphism between generalized topologies and strongly generalized interior operators has been recognized as a natural isomorphism from the perspective of category theory. The contributions of Noorie and Bala [6] have demonstrated that the concepts of generalized topology, strongly generalized interior operator, and monotonic functions extend to the category of sets [7-8].

Definition 1.1: A subset E of (X, μ) is called $D_\mu -$open if there exists U_1 a $\mu -$open proper subset of X and a $\mu -$open subset U_2 such that $E = U_1 - U_2$.

Lemma 1.2: If U is a $\mu -$open proper subset of (X, μ), then U is $D_\mu -$open.

Remark 1.3: In the GTS (X, μ) and $E \subset X$, we have [3]:

(i) $i_\mu(E) = \cup \{U: U\ is\ \mu - open, U \subset E\}$ denotes the interior of E and it is the largest $\mu -$open set contained in E.

(ii) $C_\mu(E) = \cap \{F: F$ is $\mu -$closed, $E \subset F\}$ denotes the closure of E and it is the smallest $\mu -$closed set containing E.

DOI: https://doi.org/10.54216/IJNS.230313
Received: August 19, 2023 Revised: November 25, 2023 Accepted: January 26, 2024

(iii) $i_\mu(E)$ and $C_\mu(E)$ are monotonic and idempotent.
(iv) $M_\mu(E)$=∪$\{U{:}U$ is μ −open, $U \subset X\}$ [1].
(v) $C_\mu(C_\mu\ (E))$= $C_\mu\ (E)$.
(vi) $i_\mu((i_\mu(E))$= $i_\mu(E)$.
(vii) E is μ −open iff E=$i_\mu(E)$.
(viii) E is μ −closed iff $E = C_\mu\ (E)$.

Definition 1.4: For the GTS (X,μ) and $A \subset X$, then:
(i) If $X = M_\mu$, then X is called strong.
(ii) X is $\mu - D_0$ if for all $x \neq y \in X$, there exists μ −open set containing x but not y or containing y but not x.
(iii) X is called a quasi topological space [10] if it is closed under finite intersection.
(iv) E is called generalized-closed or briefly g −closed [11] if whenever $E \cap M_\mu \subset U \in \mu$, $C_\mu(E) \cap M_\mu(E) \subset U$.
(v) X is μT_0 if for all $x \neq y \in M_\mu(E)$, there exist U a μ −open subset : $x \in U$ but not y or $y\ \in\ U$ but not x.
(vi) X is μT_1 if for all $x \neq y \in M_\mu(E)$, there exists U_1, U_2 two μ −open disjoint subsets of X such that $x \in U_1, y \notin U_1$ and $y \in U_2, x \notin U_2$.
(vii) X is μT_2 if for all $x \neq y \in M_\mu(E)$, there exists U_1, U_2 two μ −open disjoint subsets of X such that $x \in U_1$ and $y \in U_2$.
(ix) X is μ −regular if for all F a μ −closed subset of X and $x \notin F$,there exists U_1, U_2 two μ −open subsets: $x \in U_1$ and $F \cap M_\mu(E) \subset U_2$.
(x) X is μ −normal [12] if for all F_1, F_2 two disjoint μ −closed subsets, there exists U_1, U_2 two disjoint μ −open subsets: $F_1 \cap M_\mu(E) \subset U_1$ and $F_2 \cap M_\mu(E) \subset U_2$.

Definition1.5: For the GTS (X,μ), if $\varphi \neq M \subset X$, then [13]:
(i) The generalized topological subspace on M is $\{W \cap M{:}\ W \in \mu\}$ and denoted by μ_M.
(ii) A subset E of the strong GTS $M_\mu(E)$ is a μG_σ set if $E = \bigcap_{\alpha\in\Lambda}\{W : W \in \mu\}$.
(iii) A subset E of the strong GTS $M_\mu(E)$ is a μF_δ set if $E = \bigcup_{\alpha\in\Lambda}\{F{:}\,F \in X - \mu\}$.
(vi) A subset W of the strong GTS $M_\mu(E)$ is a μd_δ −open if $\exists F_\alpha$ a regular μF_δ such that $W = \bigcup_{\alpha\in\Lambda}\{F_\alpha \cap M_\mu(E)\}$
(v) A subset N of the strong GTS $M_\mu(N)$ is a μd_δ −closed if $\exists F_\alpha$ a regular μF_δ such that $N = \bigcup_{\alpha\in\Lambda}\{F_\alpha \cap (X - M_\mu(N)\}$.

2. Preliminaries

Typically, intersection of μd_δ −closed subsets of the GTS X is closed. The concepts of generalized topological spaces and neighborhood systems were first presented by Á Császár. Additionally, he added generalized neighborhood system (short by GNS) and generalized topological spaces to the notion of continuous functions and the accompanying closure and interior operators. By utilizing a closure operator constructed on GNS, he specifically looked into characterizations of (ψ,ψ') −continuous function (generalized continuous function).

Strong generalized neighborhood space has been generated by a strong generalized neighborhood system or SGNS for short. Additionally, the SGNS induces and generates a structure that is a topological generalization (the collection of all sg-open sets on SGNS). The quasi-topology that is regarded as a generalized topology [9] was also established by Á Császár.

The space $(X,\psi(x))$ is strong generalized neighborhood space and W is a strong generalized neighborhood of x in X. The GTS (X,μ) is shorted by X that is assumed to be non-empty throughout the paper unless it is stated that it is empty.

Definition 2.1: For set X with the power set e^X. If $\psi{:}\,X \to e^{e^X}$ is a continuous function, then ψ is called a generalized neighborhood system (short bt GNS) whenever $x \in W$ and $W \in \psi(x)$ [14].

Definition 2.2: For the set X with the power set e^X. If $\psi{:}\,X \to e^{e^X}$ is a continuous function satisfying the following conditions, then it is termed a SGNS on X [15] if:
(i) $x \in W$ whenever $W \in \psi(x)$.
(ii) For all $W_1, W_2 \in \psi(x),\ W_1 \cap W_2 \in \psi(x)$.

Remark 2.3: A SGNS is a generalized neighborhood system.

Definition 2.4: If ψ is a GNS, $W \subset X$ and for some $U \subset W$ [2]:
(i) $i_\psi(W) = \{x \in W{:}\,U \in \psi(x), U \subset W\}$ is the interior of W on ψ.
(ii) $\gamma_\psi(W) = \{x \in W{:}\,U \in \psi(x), U \cap W \neq \varphi\}$ is the closure of W on ψ.
(iii) $I(W) = \{x \in W{:}\,W \in \psi(x)\}$ is a weak interior of W on X.
(iv) $C(W) = \{x \in W{:}\,X - W \notin \psi(x)\}$ is a weak interior of W on X.

Lemma 2.5: If ψ is a GNS and $V, W \subset X$, then:
(i) $i_\psi(V) \subset V \subset \gamma_\psi(V)$.

DOI: https://doi.org/10.54216/IJNS.230313
Received: August 19, 2023 Revised: November 25, 2023 Accepted: January 26, 2024

(ii) i_{ψ} $(V\cap W)=$ $i_{\psi}(V)\cap i_{\psi}$ (W).

Lemma 2.6: If ψ is a generalized neighborhood system and $V,W\subset 2^{X}$, then [11]:

(i) $\gamma_{\psi}(V)$=$X-i_{\psi}(X-V)$.

(ii) $i_{\psi}(V)=X-\gamma\_\{\psi\}(X-V)$

(iii) γ_{ψ} $(V\cup W)=$ $\gamma_{\psi}(V))\cup$ $\gamma_{\psi}(W)$.

Definition 2.7: For the set X, the subfamily q_{X} of e^{X} is called quasi-topology if is closed under finite intersection and arbitrary union. The space (X,q_{X}) is called q −topological space [11].

Remark 2.8: Each space is a q −space on a non-empty set X, but the converse is not true. The subset W of a q −topological space (X,q_{X}) is q −open and $X-W$ is q −closed.

Definition 2.9: If (X,q_{X}) is a q −topological space and $E,F\subset q_{X}$, then [11]:

(i) $QO(X)=\{W\colon W\ is\ q_{X}-open\}$

(ii) $QC(X)=\{F\colon F\ is\ q_{X}-closed\}$.

(iii) $qInt(E)$ $=\cup\{W\subset E\colon W\in q\_\{X\}\}$ is the q −interior of E.

(iv) $qCl(A)$ $=\cap\{E\subset F\colon X-F\in q_{X}\}$ is the q −closure of E.

Remark 2.10: If (X,q_{X}) is a q −topological space and $E\subset q_{X}$, then:

(i) $qInt(E)$ contains $x\in X$ iff there exists $W\in q_{X}$ such that $W\subset E$

(ii) $qCl(E)$ contains $x\in X$ iff there exists $W\in q_{X}$ such that $E\cap W$ is non-empty.

Definition 2.11: For the collection H of subsets of the non-empty set X, if $\cap H$ is non-empty, then H is said to be an m −family on X [11].

Remark 2.12: If ψ is a generalized neighborhood system on X, then whenever the m −family H converges to x in X, then H is finer than $\psi(x)$ [11].

3. $\mu-L$ −closed Subsets of Noetherian Generalized Topological Spaces

In 1986, Hdieb and Pareek [6] presented the concept of L −closed spaces in which each Lindelöf subset is closed. A GTS X is $\mu-L$ −closed if every μ −Lindelöf subset of X is μ −closed [5].

Lemma 3.1: If (X,μ) is a GTS, Y is a μ −closed subspace of X and $E\subset Y$ is μ −semi-closed, then E is μ −closed subset of X [5].

Lemma 3.2: If (X,μ) is a GTS, Y is a μ −closed subspace of X and $E\subset Y$ is μ −semi-closed, then $E=\cap\{W:\ W$ is μ −open subset of $Y\}$ [5].

Definition 3.3: For the GTS (X,μ), if every subspace is μ −Lindelöf, then X is hereditarily μ −Lindelöf [2].

Theorem 3.4: If the GTS (X,μ) is hereditarily μ −Lindelöf, then it is $\mu-L$ −closed whenever it is countable and discrete.

Theorem 3.5: If M is a subset of the $\mu-L$ −closed GTS (X,μ), then it is μ −Lindelöf iff for all $F=\{F_{\alpha}\colon\alpha\in\Lambda\}$ such that F_{α} is μ −closed subsets of X for all $\alpha\in\Lambda$ and $(\bigcap_{\alpha\in\Lambda}F_{\alpha})\cap M\neq\emptyset$, then $\cap F\neq\emptyset$.

Lemma 3.6: For the $\mu-L$ −closed GTS (X,μ), if $E\subset X$, then μ_{E} is a $\mu-L$ −closed generalized topological subspace. Furthermore, if $B\subset E$,then there exists F a $\mu-L$ −closed subset of X : $B=F\cap E$ [13].

Theorem 3.7: For the $\mu-L$ −closed GTS (X,μ), if $E\subset X$ and $B\subset E$, then B is μ −Lindelöf iff B is μ_{E} −Lindelöf.
Proof: Let $E=\{W_{\alpha}\colon W_{\alpha}\in\mu_{E},\alpha\in\Lambda\}$ be a cover of E, where $W_{\alpha}=\{U_{\alpha}\cap E\colon W_{\alpha}\in\mu,\ \alpha\in\Lambda\}$ and $U=\{U_{\alpha}\colon U_{\alpha}\in\mu,\alpha\in\Lambda\}$ is a countable μ −open cover of X. But B is a μ −Lindelöf subset of X, $\exists\{\alpha_{1},\alpha_{2},\ldots\}$ a countable subset of Λ such that $B\subset\bigcup_{\alpha\in\Lambda}\{U_{\alpha_{i}}\colon\ i\in\mathbb{N}\}$. Thus, E is μ_{B} −Lindelöf. Conversely, we claim that B is μ_{B} −Lindelöf. Considering the cover W of B consisting of μ −open subsets U_{α}. Now, $B=\{W_{\alpha}\cap B\colon\alpha\in\Lambda\}$ is a μ_{E} −open and since B is a μ_{E} −Lindelöf, there exists $\{\alpha_{1},\alpha_{2},\ldots\}$ a countable subset of Λ such that $\subset\bigcup_{\alpha\in\Lambda}\{U_{\alpha_{i}}\colon i\in\mathbb{N}\}$. Therefore, B is μ −Lindelöf.

Theorem 3.8: For the $\mu-L$ −closed GTS (X,μ), if M is a regular μG_{δ} subset of X, then $M\cup(X-M_{\mu})$ is μd_{δ} −closed.

Corollary 3.9: If X is a $\mu-L$ −closed GTS and M is μd_{σ} −closed subset of X, then M is μ −closed.

Theorem 3.10: If the GTS (X,μ) is finer than (X,μ') and (X,μ') is $\mu-L$ −closed GTS, then is (X,μ) is $\mu-L$ −closed [13].

Corollary 3.11: If X is $\mu-L$ −closed and M is a regular μG_{δ} subset of X, then M is μG_{δ} set.

Definition 3.12: (i) A GTS X is μ −Noetherian if the descending chain condition holds for its $\mu-L$ −closed subsets.

(ii) A GTS X is locally μ −Noetherian if the neighbourhood of each point $x\in X$ is μ −Noetherian.

Lemma 3.13: If X is a $\mu-L$ −closed μ −Noetherian GTS, then:

(i) Each subspaces of X is $\mu-L$ −closed;

(ii) Each subset of X with the induced GTS is μ −Noetherian.

Theorem 3.14: If X is a $\mu-L$ −closed μ −Noetherian GTS, then each Hausdorff μ −Noetherian subspace of X is

DOI: https://doi.org/10.54216/IJNS.230313
Received: August 19, 2023 Revised: November 25, 2023 Accepted: January 26, 2024

finite, discrete and $\mu - L$ −closed.

Definition 3.15: (i) If X is a non-empty μ −Noetherian GTS, then X is μ −irreducible if whenever $X = M_1 \cup M_2$ with M_i is μ −closed for all $i \in \mathbb{N}$, we have $X = M_1$ or $X = M_2$.

(ii) $M \subset X$ is a μ −irreducible component of X if M is a maximal μ −irreducible subset of X.

Theorem 3.16: If X is a $\mu - L$ −closed μ −Noetherian GTS, then:

(i) Each subspace of X with the induced topology is μ −Noetherian.

(ii) X has finitely many non-empty μ −irreducible components.

Proof: (i) Suppose that M is a L −closed subset of X such that $\ldots \subset M_3 \subset M_2 \subset M_1$. Consider M_i=$M \cap N_i$ for some μ −closed subsets N_i of $\forall i \in \mathbb{N}$. Now, if $\ldots \subset N_1 \cap N_2 \cap N_3 \subset N_1 \cap N_2 \subset N_1$, then M is $\mu - L$ −closed.

(ii) Suppose that F is a μ −closed subsets of X that does not have finitely many μ −irreducible components, then F is partially ordered and because of the descending chain condition, there exists $M \in F$ such that M is a smallest element and not a finite union of μ −irreducible components. Thus M is not μ-irreducible. Now, if M_1 and $M_2$are strictly smaller μ −closed subsets of X, $M = M_1 \cup M_2 = \bigcup_{i\in\mathbb{N}} {M_1}_i \cup (\bigcup_{j\in\mathbb{N}} {M_2}_j)$ which is not a decomposition of M into its μ −irreducible components, a contradiction.

4. Product Properties of μ-L-Closed Sets in Noetherian Generalized Topological Spaces

Definition 4.1: If (X,μ) and (Y,μ') are two μ −Noetherian GTS and the function $f:(X,\mu) \to (Y,\mu')$ is (μ,μ') −continuous, bijective and f^{-1} is (μ',μ) −continouos, then f is (μ,μ') −homeomorphism.

Definition 4.2: Á Császár generalized product GTS on $\prod_{\alpha\in\Lambda} X_\alpha$ is generalized by the basis $\{W_n: W_n \in \mu_n, W_n = M_{\mu_n}$ except for finite number of indecies$\}$ where $X = \{(X_\alpha, \mu_\alpha): \alpha \in \Lambda\}$ and Λ is a non-empty countable set. (X,μ) is the Cs´asz´ar generalized product GTS.

Lemma 4.3: If (X,μ) and (Y,μ') are two Noetherian GTS and the function $f:(X,\mu) \to (Y,\mu')$ is (μ,μ') −continuous, then there exists M,N⊆X such that, the following are equivalent:

(i) $f(C_\mu(M)) \subset C_{\mu'}(f(M))$

(ii) $C_{\mu'}(f^{-1}(M)) \subset f^{-1}(C_\mu(M))$.

Theorem 4.4: Let (X,μ) and (Y,μ') be two μ −Noetherian GTS, X is μ −Lindelöf and Y is μ −regular. If $f:(X,\mu) \to (Y,\mu')$ is (μ,μ') −continuous surjective, then Y is μ −Lindelöf.

Proof: Let $W = \{W_\alpha: \alpha \in \Lambda\}$ be a family of μ −open subsets of Y that forms a μ −open cover of it, then $f^{-1}(W_\alpha)$ is a family of μ' −open subset of X because of the (μ,μ') −continuity of f.

If $Y = \bigcup_{\alpha\in\Lambda}\{W_\alpha : \alpha \in \Lambda\}$ and since f is surjective, we have $f^{-1}(Y) = f^{-1}(\bigcup_{\alpha\in\Lambda}\{W_\alpha : \alpha \in \Lambda\}) = \bigcup_{\alpha\in\Lambda}\{f^{-1}(W_\alpha) : \alpha \in \Lambda\} = X$. Now, X is μ −Lindelöf , hence there exists $\{\alpha_1, \alpha_2, \ldots\}$ a countable subset of Λ such that $X = \bigcup_{i\in\mathbb{N}}\{f^{-1}(W_{\alpha_i}): \alpha \in \Lambda\}$ and since Y is bijective, $Y = f(X) = \bigcup_{i\in\mathbb{N}}\{f(f^{-1}(W_{\alpha_i})): \alpha \in \Lambda\} \subset \bigcup_{i\in\mathbb{N}}\{W_{\alpha_i} : \alpha \in \Lambda$, thus Y is μ −Lindelöf.

Theorem 4.5: Being a $\mu - L$ −closed μ −Noetherian GTS is a topological property.

Proof: If X is $\mu - L$ −closed μ −Noetherian GTS. if $f:(X,\mu) \to (Y,\mu')$ is (μ,μ') −homeomorphism and E is a μ −Lindelöf subset of X, then $f(E)$ is μ −Lindelöf since f is (μ,μ') −continuous. But X is $\mu - L$ −closed μ −Noetherian, hence E is μ −closed. Therefore, $f(E)$ is μ −closed, and Y is $\mu - L$ −closed.

Lemma 4.6:If (X,μ) and (Y,μ') be are two μ −Noetherian GTS such that each μ-open cover of X has a countable refinement of μ −open subsets.

Lemma 4.7: If (X,μ) and (Y,μ') be are two μ −Noetherian GTS and $f:(X,\mu) \to (Y,\mu')$ is a (μ,μ') −continuous surjective function, then:

(i) Y is μ −Lindelöf.

(ii) If Y is $\mu - L$ −Lindelöf, then X is μ −L-Lindelöf.

Proof: (ii) If $f:(X,\mu) \to (Y,\mu')$ is a (μ,μ') −continuous surjective function and Y is $\mu - L$ −Lindelöf, let E be a $\mu - L$ −Lindelöf subset of X, then $f(E)$ is $\mu - L$ −Lindelöf because f is (μ,μ') −continuous. Since Y is $\mu - L$ −closed, hence $f(E)$ is $\mu - L$ −closed subset of Y and $A = f^{-1}(f(A))$ is a $\mu - L$ −closed subset of X because f is injective. Therefore, X is $\mu - L$ −Lindelöf.

Remark 4.8: A μ −Noetherian GTS (X,μ) is homeomorphic to itself.

Theorem 4.9: If (X,μ) and (Y,μ') be are two μ −Noetherian GTS, X is μ −Lindelöf, Y is $\mu - L$ −closed and $f:(X,\mu) \to (Y,\mu')$ is a (μ,μ') −continuous surjective function, then f is μ −homeomorphism.

Proof: Suppose that $f:(X,\mu) \to (Y,\mu')$ is (μ,μ') −closed $W = \{W_\alpha: \alpha \in \Lambda\}$ is a family of $\mu - L$ −closed subsets of X, and $E \in W$ is a proper subset of X. Since X is μ −Noetherian μ −Lindelöf and f is a (μ,μ') −continuous surjective function, E is a μ −Lindelöf. Therefore, $f(E)$ is μ −Lindelöf. But Y is $\mu - L$ −closed, hence $f(E)$ is a $\mu - L$ −closed subset of Y. Hence, f is μ −closed μ −homeomorphic.

Corollary 4.10: If $\mathrm{f}:(X,\mu) \to (Y,\mu')$ is a (μ,μ') −continuous from a μ −Noetherian μ −Lindelöf GTS to μ −

DOI: https://doi.org/10.54216/IJNS.230313
Received: August 19, 2023 Revised: November 25, 2023 Accepted: January 26, 2024

L −closed space is μ −closed, then every (μ,μ') −continuous bijective function is μ −homeomorphism.

Theorem 4.11: If (X,μ) and (Y,μ') are two μ −Noetherian GTS, $f:(X,\mu)\rightarrow(Y,\mu')$ is a (μ,μ') −continuous, X is $\mu-L$ −closed and $\{(x,f(x)):x\in X\}$ is a (μ,μ') −Lindelöf subset of $X\times Y$, then f is (μ,μ') −continuous.

Proof: Let π_X and π_Y be two projection functions, $\pi_{X'}=\pi_X|_f$ and $\pi_{Y'}=\pi_Y$ are (μ,μ') −continuous bijective functions. Assume that $E=\{(x,f(x)):x\in X\}$, then E is a μ −Lindelöf subset of f and each μ −closed subset of f is μ −Lindelöf. Now, $E\subset f$ is μ −closed, thus it is μ −Lindelöf because X is $\mu-L$ −closed. But $\pi_{X'}$ is (μ,μ') −closed projection, hence for all $\subset f$, we have $\pi_{X'}(W)$ is μ −open subset of X. Therefore, $f=\pi_Y\circ(\pi_{X'})^{-1}$ is (μ,μ') −continuous.

Theorem 4.12: If $X=\{(X_\alpha,\mu_\alpha):\alpha\in\Lambda\}$, Λ is a non-empty countable set is a family of μ −Noetherian GTS and $\pi_\alpha:\prod_{\alpha\in\Lambda}X_\alpha\rightarrow X_\alpha$ is the projection function, then $f:(X,\mu)\rightarrow\prod_{\alpha\in\Lambda}(X_\alpha,\mu_\alpha)$ is $(\mu_{\alpha_1},\mu_{\alpha_2},\dots)$ −continuous iff $f\circ\pi_\alpha$ is $(\mu_{\alpha_1},\mu_{\alpha_2},\dots)$ −continuous.

Theorem 4.13: If $f:(X,\mu)\rightarrow(Y,\mu')$ is a (μ,μ') −continuous, then $f(X)$ is μ' −Noetherian whenever X is μ −Noetherian.

Proof: Assume that $M_1\supset M_2\supset\dots$ is a descending chain of $\mu-L$ −closed subsets of $f(X)$, so $f^{-1}(M_1)\supset f^{-1}(M_2)\supset\dots$ is a descending chain of $\mu-L$ −closed subsets of X, thus such chain stabilizes. But $f(f^{-1}(M_n))=M_n\ \forall n\in\mathbb{N}$. Therefore, $M_1\supset M_2\supset\dots$ stabilizes.

Lemma 4.14: Finite union of μ-Noetherian GTS is μ-Noetherian GTS.

Proof: Suppose that $X\supset X_i\}\ \forall i=1,2,\dots,n$. Let $F=\{F_\alpha:\alpha\in\Lambda\}$ be a family of decreasing sequence of $\mu-L$ −closed subsets of $X=\bigcup_{i=1}^{n}X_i$. If $F_i=M_i\cap X$ for some decreasing sequence of $\mu-L$ −closed subsets M_i, there exists $i_\circ\in\mathbb{N}$ such that for all $i_\circ\le i$, $M_{i_\circ}\cap X_i=M_i\cap X$.

If $i_m=max\{i_\circ,i_1,\dots,i_n\}$, then $F_i\cap M_i\cap X=F_i=M_i\cap\bigcup_{i=1}^{n}X_i=\bigcup_{i=1}^{n}(F_i\cap X_i)$

which stabilizes for all $i_\circ\le i$.

5. Conclusion

The μ −open subsets of Noetherian generalized topological spaces play an important role in the formation of this space. Subspaces of μ −Noetherian GTS X are μ −Noetherian and having finitely many non-empty μ −irreducible components. Under (μ,μ') −continuous functions, some hereditary properties and sets are satisfied such as regular μG_δ, μd_δ, μ −irreducible L −closed subsets and product properties of $\mu-L$ −closed subsets. If $f:(X,\mu)\rightarrow(Y,\mu')$ is a (μ,μ') −continuous, then in μ −Noetherian spaces, $f(X)$ is μ' −Noetherian. In the future, we hope that our results will be extended to fuzzy topological spaces, and neutrosophic topological spaces.

Conflicts of interest: The authors declare that there are no conflicts of interest.

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Received: August 19, 2023 Revised: November 25, 2023 Accepted: January 26, 2024

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DOI: https://doi.org/10.54216/IJNS.230313
Received: August 19, 2023 Revised: November 25, 2023 Accepted: January 26, 2024