A Specific Category of Harmonic Functions Characterized By A Generalized Komatu Operator in Conjunction With The (R-K) Integral Operator and Applications to Neutrosophic Complex Field

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Abstract
In our work, we introduced a distinct subclass of univalent harmonic functions referred to as a subclass of chiral functions. These functions are defined by combining the generalized Komatu operator with the integral operator (R − K), which has positive coefficients within the unit disc A. Also, we generalize the same subclass into neutrosophic complex numbers. Throughout our investigation, we establish several properties associated with these functions, including coefficient estimates, the convex formula, the integral operator, and the Hadamard product. On the other hand, we present the Neutrosophic convex formula and the neutrosophic integral operator.

Keyword. Spiral-like functions generalized integral operator; sufficient coefficient, convex combination; neutrosophic complex numbers; neutrosophic convex formula.

1. Introduction and Preliminaries

Many authors have studied many of these classes of univalent and multivalent functions, as well as meromorphic functions that include many integral operators, and they obtained many results and important theorems that were used later in the classical analysis. These functions have many applications in many different mathematical fields such as mathematical physics, complex analysis, and analytical number theory. Also, some important and useful theorems and properties were presented for neutrosophic analysis [10, 14], neutrosophic functions [11, 13], and spaces [12,15].

Let AR represent the set of functions defined in the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \, , k \in \mathbb{N}, a_k \geq 0, z \in \mathfrak{A} \tag{1.1}$$

In open unit disk $\mathfrak{A} = \{z \in \mathbb{C}: |z| < 1\}$.
$f(z)$ is an analytic and univalent function. Where $|G| \leq \frac{\Pi}{2}$, the equation (1.1) is said in the class $S_{A,R}$
If and only if
$\text{Re}\left\{e^{iG} \frac{zN'(z)}{N(z)}\right\} > 0$, the class $S_{A,R}$ is a class of all G-spiral-like functions.
Let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \, , k \in \mathbb{N}, b_k \geq 0, z \in \mathfrak{A} \tag{1.2}$$

The convolution is defined as

DOI: https://doi.org/10.54216/IJNS.230304
Received: August 12, 2023 Revised: November 12, 2023 Accepted: January 22, 2024

i) $G = 0$ then $S_{A,R}(0) = S$ is a renowned class of functions known for their starlikeness with respect to the origin.
ii) $G \neq 0$ then $S_{A,R}(G) = \emptyset$.
[1], [2] and **[3]** show that the radius of starlike ness of $S_{A,R}(G) = \frac{1}{\cos G + |\sin G|}$
Buti **[4]** define the generalized integral operator of the function $f(z) \in S_{A,R}$ is form

$$P_{\lambda,\alpha,\theta,k}^{\mu,\beta,l}\big(f(z)\big) = \frac{\theta k(\lambda-\beta+2)^{\mu-\alpha+1}}{l^{\mu-\alpha+1}\Gamma(\lambda-\alpha+1)}\int_0^1 \left[\log\frac{1}{t^l}\right]^{\mu-\alpha} f\left(\frac{zt}{\theta k}\right)dt \tag{1.3}$$

Such that $l, t, \theta, k,$ approximate to 0 and $\lambda - \alpha < 1$ and from (1.3) we have

$$P_{\lambda,\alpha,\theta,k}^{\mu,\beta,l}\big(f(z)\big) = z + \sum_{k=2}^{\infty}\left[\frac{\lambda-\beta+2}{\lambda-\beta+n+1}\right]^{\mu-\alpha+1} a_k z^k \tag{1.4}$$

If $\beta, \alpha, l, \theta, k,$ approximate to one, we get Komatu operator **[5]**
Remark. The (R-K) utilizing $e^{iA} = (cosA + isinA)$ is defined by:

$$[\mathrm{R}-\mathrm{K}]_q^{A,r,c}\big(f(z)\big) = \frac{r}{q}e^{-iA}\int_0^1 t^{r+1+c}\, g\left(\frac{qze^{iA}t^c}{t^c}\right)dt \tag{1.5}$$

$$= z + \sum_{k=2}^{\infty}\psi(\mathcal{A}, n, c, r)a_k\, z^k,$$

And $\psi(\mathcal{A}, n, c, r) = \frac{q^{n-1}e^{iA(n-1)}}{r(r-c(n-1))}, r > c(n-1), c > 0.$
A function f in the class $S_{A,R}$ refers to a complex-valued continuous harmonic function is univalent and normalized by $f(0) = 0,$ and $f'(z) - 1 = 0.$ Additionally, f in the form $f = \mathfrak{h} + \overline{g},$ where $\mathfrak{h}$ and $\overline{g}$ *are* both analytic functions in AR. We refer to $\mathfrak{h}$ as the analytic part and $\overline{g}$ as the co-analytic part of f.
We can express it as follow

$$\mathfrak{h}(z) = z + \textstyle\sum_{k=2}^{\infty} a_k z^k, \qquad \overline{g}(z) = \sum_{k=1}^{\infty} b_k z^k \quad (0 \le b_1 < 1) \tag{1.6}$$

A necessary and sufficient condition for f to be locally univalent and sense-preserving in AR is that $|\mathfrak{h}'(z)| < |g'(z)|$ in $\mathfrak{A}$.(see **[6]**), as stated in **[7]**. Hence, for $f = \mathfrak{h} + \overline{g} \in S_{A,R},$
In this paper, we general the integral operator$\big(f(z)\big)$in (1.5) of harmonic $f = \mathfrak{h} + \overline{g}$ is as

$$[\mathrm{R}-\mathrm{K}]_q^{A,r,c}\big(f(z)\big) = [\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c}\mathfrak{h}(z) + \overline{[\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c} g(z)}\ \ \forall z \in \mathfrak{A},$$

Where

$$[\mathrm{R}-\mathrm{K}]_q^{A,r,c}\mathfrak{h}(z) = z + \sum_{k=2}^{\infty}\psi(\mathcal{A}, n, c, r)a_k\, z^k$$

$$\overline{[\mathrm{R}-\mathrm{K}]_q^{A,r,c} g(z)} = \sum_{k=2}^{\infty}\psi(\mathcal{A}, n, c, r)b_k\, z^k$$

Using the integral operator $[\mathrm{R}-\mathrm{K}]_q^{A,r,c}\big(f(z)\big)$, we introduce the class of harmonic univalent functions as illustrated below.
Definition 1.1 for $0 \le v < 1$ the function $f = \mathfrak{h} + \overline{g}$ is in the class $MR_p(v)$ If satisfy the inequality.

$$\boldsymbol{Re}\left\{\frac{\Big([\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c}\big(f(z)\big)\Big)'\, \boldsymbol{z}}{[\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c}\big(f(z)\big)}\right\} \ge \boldsymbol{v} \qquad |\boldsymbol{z}| = \boldsymbol{r} < \mathbf{1}. \tag{1.7}$$

Let $MR_{\overline{p}}(v) \subseteq MR_p(v)$ of the form

$$\mathfrak{h}(z) = z - \sum_{k=2}^{\infty}|a_k|\, z^k, \quad g(z) = \sum_{k=2}^{\infty}|a_k|\, z^k, \quad |b_1| < 1.$$

2. Sufficient coefficient condition

Theorem (2.1). The function $f = \mathfrak{h} + \overline{g}$ such that $\mathfrak{h}(z)$ and $g(z)$ of the form (1.4)

$$\sum_{k=2}^{\infty}\big(k\,\psi' + \psi(1-\vartheta)\big)(\mathcal{A}, n, c, r)|a_k|\,|z^k| + \sum_{k=1}^{\infty}\big(k\,\psi' - \psi(1+\vartheta)\big)(\mathcal{A}, n, c, r)|b_k|\,|z^k| \le (2-\vartheta) \tag{2.1}$$

DOI: https://doi.org/10.54216/IJNS.230304
Received: August 12, 2023 Revised: November 12, 2023 Accepted: January 22, 2024

Where $a_1 = 1$, $\vartheta \in [0,1)$ and $\psi(\mathcal{A}, n, c, r)$ form

$$\psi(\mathcal{A}, n, c, r) = \frac{q^{n-1}e^{iA(n-1)}}{r(r - c(n-1))}, r > c(n-1), c > 0. \tag{2.2}$$

Such that f is sense –preserving, univalent in $\mathfrak{A}$, where $f \in MR_p(v)$

Proof. If $|z_1| \neq |z_2| < q$,then

$$\left|\frac{f(z_1) - f(z_2)}{\mathfrak{h}(z_1) - \mathfrak{h}(z_2)}\right| \geq 1 - \left|\frac{g(z_1) - g(z_2)}{\mathfrak{h}(z_1) - \mathfrak{h}(z_2)}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right|$$

Attended

$$> 1 - \frac{\sum_{k=1}^{\infty}[k]_q|b_k|}{1 - \sum_{k=2}^{\infty}[k]_q|a_k|} \geq 1$$
$$- \frac{\sum_{k=1}^{\infty}[(k\,\psi' + \psi)(\mathcal{A}, n, c, r)/(1-\vartheta)]|b_k|}{1 - \sum_{k=2}^{\infty}\big[[(k\,\psi' - \psi)(\mathcal{A}, n, c, r)/(1-\vartheta)]\big]|a_k|} \geq 0$$

Which proves the multivalent. Observe that, f is sense-preserving in$\mathfrak{A}$, because

$$|\mathfrak{h}(z)'| \geq \left(1 - \sum_{k=2}^{\infty} \psi'(\mathcal{A}, n, c, r)|a_k||z|^{k-1}\right) > \left(1 - \sum_{k=2}^{\infty} \frac{(k\,\psi' + \psi)(A, n, c, r)}{2 - \vartheta}|a_k|\right)$$
$$\geq \left(\sum_{k=1}^{\infty} \frac{(k\,\psi' + \psi)(\mathcal{A}, n, c, r)}{2 - \vartheta}|b_k|\right) > \left(\sum_{k=1}^{\infty} \frac{(k\,\psi' + \psi)(\mathcal{A}, n, c, r)}{2 - \vartheta}|b_k||z|^{k-1}\right)$$
$$\geq \sum_{n=1}^{\infty}[k]_q\,|b_k||z|^{k-1} \geq |g(z)'|.$$

We obtain $\lim_{q\to 1}\big[|D_q\mathfrak{h}(z)| \geq |D_q g(z)|\big] = [|\mathfrak{h}(z)'| \geq |g(z)'|]$.

Now, we show that $\in MR_p(v)$. From (1.7), we can write

$$Re\left\{\frac{\left([\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c}(f(z))\right)'}{[\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c}(f(z))}\right\} = Re\left\{\frac{C(z)}{E(z)}\right\},$$

where

$$C(z) = \left([\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c}\mathfrak{h}(z) + \overline{[\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c} g(z)}\right)' z$$
$$= z + \sum_{k=2}^{\infty} k\,\psi'(\mathcal{A}, n, c, r)a_k\,z^k + \sum_{k=2}^{\infty} k\,\psi'(\mathcal{A}, n, c, r)\overline{b_k}\,\overline{z^k}$$
$$E(z) = z + \sum_{k=2}^{\infty} \psi(\mathcal{A}, n, c, r)a_k\,z^k + \sum_{k=1}^{\infty} \psi(\mathcal{A}, n, c, r)\overline{b_k}\,\overline{z^k}$$

From the fact that $Re(w) \geq \vartheta$ if and only if $|1 - \vartheta + w| \geq |1 + \vartheta - w|$, it suffices to prove

$$|C(z) + (1-\vartheta)E(z)| - |C(z) - (1+\vartheta)E(z)| \geq 0 \tag{2.3}$$

$$\left|z + \sum_{k=2}^{\infty} k\,\psi'(\mathcal{A}, n, c, r)a_k\,z^k + \sum_{k=1}^{\infty} k\,\psi'(\mathcal{A}, n, c, r)\overline{b_k}\,\overline{z^k}\right.$$
$$\left. + (1-\vartheta)\left(z + \sum_{k=2}^{\infty} \psi(\mathcal{A}, n, c, r)a_k\,z^k + \sum_{k=1}^{\infty} \psi(\mathcal{A}, n, c, r)\overline{b_k}\,\overline{z^k}\right)\right|$$
$$- \left|z + \sum_{k=2}^{\infty} k\,\psi'(\mathcal{A}, n, c, r)a_k\,z^k + \sum_{k=1}^{\infty} k\,\psi'(\mathcal{A}, n, c, r)\overline{b_k}\,\overline{z^k}\right.$$
$$\left. - (1+\vartheta)\left(z + \sum_{k=2}^{\infty} \psi(\mathcal{A}, n, c, r)a_k\,z^k + \sum_{k=2}^{\infty} \psi(\mathcal{A}, n, c, r)\overline{b_k}\,\overline{z^k}\right)\right|$$
$$= \left|(2-v)z + \sum_{k=2}^{\infty}\big(k\,\psi' + \psi(1-\vartheta)\big)(\mathcal{A}, n, c, r)a_k\,z^k + (2-\vartheta)\sum_{k=1}^{\infty}\big(k\,\psi' + \psi(1-\vartheta)\big)(\mathcal{A}, n, c, r)\overline{b_k}\,\overline{z^k}\right|$$
$$- \left|-\vartheta z + \sum_{k=2}^{\infty}\big(k\,\psi' + \psi(1+\vartheta)\big)(\mathcal{A}, n, c, r)a_k\,z^k - \sum_{k=2}^{\infty}\big(k\,\psi' + \psi(1+\vartheta)\big)(\mathcal{A}, n, c, r)\overline{b_k}\,\overline{z^k}\right|$$

DOI: https://doi.org/10.54216/IJNS.230304
Received: August 12, 2023 Revised: November 12, 2023 Accepted: January 22, 2024

$$\geq (2-\vartheta)|z| - \sum_{k=2}^{\infty}\big(k\,\psi' + \psi(1-\vartheta)\big)(\mathcal{A},n,c,r)|a_k|\,|z^k| - \sum_{k=1}^{\infty}\big(k\,\psi' + \psi(1-\vartheta)\big)(\mathcal{A},n,c,r)|b_k|\,|z^k| - \vartheta|z|$$

$$-\sum_{k=2}^{\infty}\big(k\,\psi' - \psi(1+\vartheta)\big)(\mathcal{A},n,c,r)|a_k|\,|z^k| - \sum_{k=2}^{\infty}\big(k\,\psi' - \psi(1+\vartheta)\big)(\mathcal{A},n,c,r)|b_k|\,|z^k|$$

$$\geq 2(1-\vartheta)|z|\left\{1 - \frac{\sum_{k=2}^{\infty}(k\,\psi' + \psi)(\mathcal{A},n,c,r)}{1-\vartheta}|a_k||z|^{k-1} - \frac{\sum_{k=1}^{\infty}\big(k\,\psi' - \psi(1+\vartheta)\big)(\mathcal{A},n,c,r)}{1-\vartheta}\,|b_k||a_k||z|^{k-1}\right\}$$

$$\sum_{k=2}^{\infty}\big(k\,\psi' + \psi(1-\vartheta)\big)(\mathcal{A},n,c,r)|a_k|\,|z^k| + \sum_{k=1}^{\infty}\big(k\,\psi' - \psi(1+\vartheta)\big)(\mathcal{A},n,c,r)|b_k|\,|z^k| \leq (2-\vartheta)$$

Theorem (2.2). Let $f = \mathfrak{h} + \overline{g}$ then $f \in MR_{\bar{p}}\,(v)$if and only if

$$\textstyle\sum_{k=2}^{\infty}\big(k\,\psi' + \psi(1-\vartheta)\big)(\mathcal{A},n,c,r)|a_k|\,|z^k| + \sum_{k=1}^{\infty}\big(k\,\psi' - \psi(1+\vartheta)\big)(\mathcal{A},n,c,r)|b_k|\,|z^k| \leq (2-\vartheta) \qquad (2.4)$$

such that $\vartheta \in [0,1)$, $a_1 = 1$ and $\psi(\mathcal{A},n,c,r)$ given by (2.2)

Proof. Since

$MR_{\bar{p}}(v) \subseteq MR_p(v)$To demonstrate the only if part of the theorem, we will focus on functions. $f \in MR_{\bar{p}}(v)$.

We notice that is equivalent to (1.7)

$$Re\left\{\frac{\left([\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c}(f(z))\right)'\boldsymbol{z}}{[\mathrm{R}-\mathrm{K}]_q^{\mathcal{A},r,c}(f(z))} - \vartheta\right\} \geq 0. \qquad (2.4)$$

$$Re\left\{\frac{z + z\sum_{k=2}^{\infty}k\,\psi'(\mathcal{A},n,c,r)a_k{}^k + \sum_{k=1}^{\infty}k\,\psi'(\mathcal{A},n,c,r)\overline{b_k}\,\overline{z^k}}{z + \sum_{k=2}^{\infty}\psi(\mathcal{A},n,c,r)a_k\,z^k + \sum_{k=1}^{\infty}\psi(\mathcal{A},n,c,r)\overline{b_k}\,\overline{z^k}} - \vartheta\right\} \geq 0 \qquad (2.5)$$

$$Re\left[\frac{(1-\vartheta)z - \sum_{k=2}^{\infty}(k\,\psi' + \psi v)(\mathcal{A},n,c,r)a_k\,z^k + \sum_{k=1}^{\infty}(k\,\psi' - \psi v)(\mathcal{A},n,c,r)\overline{b_k}\,\overline{z^k}}{z + \sum_{k=2}^{\infty}\psi(\mathcal{A},n,c,r)a_k\,z^k + \sum_{k=1}^{\infty}\psi(\mathcal{A},n,c,r)\overline{b_k}\,\overline{z^k}}\right] \geq 0 \qquad (2.6)$$

The values of z in $\mathfrak{A}$ is hold. Its positive real axis such that $0 \leq z = r < 1$, we must get

$$\frac{(1-\vartheta) - (k\,\psi' - \psi v)b_1 - \sum_{k=2}^{\infty}(k\,\psi' + \psi v)(\mathcal{A},n,c,r)|a_k|\,z^k + \sum_{k=1}^{\infty}(k\,\psi' - \psi v)(\mathcal{A},n,c,r)\left|\overline{b_k}\right|\overline{z^k}r^{n-1}}{1 + |b_1| + \sum_{k=2}^{\infty}(|a_k| + |b_k|)\psi(\mathcal{A},n,c,r)r^{n-1}} \geq 0$$

Since equation (2.4) fails to hold, the equation in (2.6) becomes negative when r is sufficiently close to 1. Consequently, there exists a value $z_0 = r_0$ in the range (0,1) where the quotient in (2.6) is less than zero. Therefore, we can conclude that f belongs to the closure of the function space, $f \in MR_{\bar{p}}(v)$

3. Convex combination

Theorem (2.3). Let $f_{n,i}$(z) in $MR_{\bar{p}}(v)$ where $i = 1,2,\ldots m, by$

Therefore $c_i(z)$ is given by

$$c_i(z) = \sum_{i=1}^{\infty} t_i f_{n,i}\,(z).\,0 \leq t_i \leq 1$$

And in $MR_{\bar{p}}(v)$ wherever$\sum_{i=1}^{\infty} t_i = 1$

Proof. By definition of, $c_i(z)$ we get

$$c_i(z) = z + \sum_{w=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i\left|a_{w,i}\right|\right)z^k + \sum_{w=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i\left|b_{w,i}\right|\right)\bar{z}^w$$

Furthermore, because $f_{n,i}$(z) in $MR_{\bar{p}}$, per $i = 1,2,\ldots,m$, then by theorem (2.1) we get

$$\sum_{n=2}^{\infty}\Omega_k\,(w,l)\left(\sum_{i=1}^{\infty} t_i\left|a_{k,i}\right|\right) + \sum_{n=1}^{\infty}\Omega_k\,(w,l)\left(\sum_{i=1}^{\infty} t_i\left|b_{w,i}\right|\right)$$

$$\sum_{i=1}^{\infty} t_i\left\{\sum_{n=2}^{\infty}\psi(\mathcal{A},n,c,r)\left|a_{w,i}\right| + \sum_{n=1}^{\infty}\psi(\mathcal{A},n,c,r)\left|b_{w,i}\right|\right\} \leq \sum_{i=1}^{\infty} t_i = 1$$

4. Hadamard product.

We must show that the class $MR_{\bar{p}}(v)$ is closed under the hadamard product

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DOI: https://doi.org/10.54216/IJNS.230304
Received: August 12, 2023 Revised: November 12, 2023 Accepted: January 22, 2024

$$f_k(z) = z - \sum_{w=2}^{\infty} |a_w|\, z^w + \sum_{w=1}^{\infty} |b_w|\, \bar{z}^w$$

And

$$Q_w(w) = z - \sum_{w=2}^{\infty} |L_w|\, z^w + \sum_{w=1}^{\infty} |A_w|\, \bar{z}^w$$

Is given as

$$(f_w * Q_w)(z) = f_w(z) * Q_w(z) = z - \sum_{w=2}^{\infty} |a_w L_w| z^w + \sum_{w=1}^{\infty} |b_w A_w|\, \bar{z}^w$$

4. Integral Operator

The next theorem we test the closure quality of the class$MR_{\bar{p}}(v)$ Bernardi-Libera-Livingston integral **[8], [9]** we get

$$T_u\big(f(z)\big) = \frac{u+1}{z^u} \int_0^2 t^{u-1} f(t)dt, \qquad u > -1 \tag{4.1}$$

Theorem (4.1). Suppose that $f_n \in MR_{\bar{p}}(v)$. Therefore $T_u(f_n(z)) \in MR_{\bar{p}}(v)$

Proof. By the definition of $T_u(f_n(z))$ defined by (4.1)

$$T_u\big(f_n(z)\big) = \frac{u+1}{z^u} \int_0^2 t^{u-1} \left(t - \sum_{w=2}^{\infty} |a_w| t^w + \sum_{w=1}^{\infty} |b_w|\, \bar{t}^w \right) dt,$$

$$= z - \sum_{w=2}^{\infty} \frac{u+1}{u+w} |a_w| z^w + \sum_{w=1}^{\infty} \frac{u+1}{u+w} |b_w|\, \bar{z}^w$$

$$= z - \sum_{k=2}^{\infty} d_w z^w + \sum_{w=1}^{\infty} l_w\, \bar{z}^w$$

$d_w = \frac{u+1}{u+w} |a_w|$ And $l_w = \frac{u+1}{u+w}\, |b_w|$

Therefore

$$\sum_{n=2}^{\infty} MR_p(v) \frac{u+1}{u+w} |a_w| + \sum_{n=1}^{\infty} MR_p(v) \frac{u+1}{u+w} |b_w| \le 1$$

From theorem (2.1)

$$T_u\big(f_n(z)\big) \in MR_{\bar{p}}(v)$$

6. Neutrosophic Convex combination

Theorem 5.1. Let $f_{n,i}(z_1 + z_2 I)$ in $MR_{\bar{p}}(v_1 + v_2 I)$ where $i = 1,2, \dots m, by$

Therefore $c_i(z_1 + z_2 I)$ is given by

$$c_i(z_1 + z_2 I) = \sum_{i=1}^{\infty} t_i f_{n,i}\,(z_1 + z_2 I).\, 0 \le t_i \le 1$$

And in $MR_{\bar{p}}(v_1 + v_2 I)$ wherever$\sum_{i=1}^{\infty} t_i = 1$

Proof. By definition of, $c_i(z_1 + z_2 I)$ we get

$$c_i(z_1 + z_2 I) = z_1 + z_2 I + \sum_{w=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{w,i}| \right) (z_1 + z_2 I)^k + \sum_{w=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{w,i}| \right) \overline{(z_1 + z_2 I)}^w$$

Furthermore, because $f_{n,i}(z_1 + z_2 I)$ in $MR_{\bar{p}}$, per $i = 1,2, \dots, m$, then by theorem (2.1) we get

$$\sum_{n=2}^{\infty} \Omega_k\,(w,l) \left(\sum_{i=1}^{\infty} t_i |a_{k,i}| \right) + \sum_{n=1}^{\infty} \Omega_k\,(w,l) \left(\sum_{i=1}^{\infty} t_i |b_{w,i}| \right)$$

$$\sum_{i=1}^{\infty} t_i \{ \sum_{n=2}^{\infty} \psi(\mathcal{A}, n, c, r)\, |a_{w,i}| + \sum_{n=1}^{\infty} \psi(\mathcal{A}, n, c, r)\, |b_{w,i}| \} \le \sum_{i=1}^{\infty} t_i = 1$$

7. Neutrosophic Hadamard product.

We must show that the class $MR_{\bar{p}}(v_1 + v_2 I)$ is closed under the hadamard product

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DOI: https://doi.org/10.54216/IJNS.230304
Received: August 12, 2023 Revised: November 12, 2023 Accepted: January 22, 2024

$$f_k(z_1 + z_2 I) = z_1 + z_2 I - \sum_{w=2}^{\infty} |a_w| (z_1 + z_2 I)^w + \sum_{w=1}^{\infty} |b_w| \overline{(z_1 + z_2 I)}^w$$

And

$$Q_w(w) = z_1 + z_2 I - \sum_{w=2}^{\infty} |L_w| (z_1 + z_2 I)^w + \sum_{w=1}^{\infty} |A_w| \overline{(z_1 + z_2 I)}^w$$

Is given as

$$(f_w * Q_w)(z_1 + z_2 I) = f_w(z_1 + z_2 I) * Q_w(z_1 + z_2 I)$$
$$= z_1 + z_2 I - \sum_{w=2}^{\infty} |a_w L_w| (z_1 + z_2 I)^w + \sum_{w=1}^{\infty} |b_w A_w| \overline{(z_1 + z_2 I)}^w$$

8.Neutrosophic Integral Operator

$$T_u\big(f(z_1 + z_2 I)\big) = \frac{u+1}{(z_1 + z_2 I)^u} \int_0^2 t^{u-1} f(t)dt, \qquad u > -1 \qquad (7.1)$$

Theorem (7.1). Suppose that $f_n \in MR_{\bar{p}}(v_1 + v_2 I)$. Therefore $T_u(f_n(z_1 + z_2 I)) \in MR_{\bar{p}}(v)$

Proof. By the definition of $T_u(f_n(z_1 + z_2 I))$ defined by (7.1)

$$T_u\big(f_n(z_1 + z_2 I)\big) = \frac{u+1}{(z_1 + z_2 I)^u} \int_0^2 t^{u-1} \left(t - \sum_{w=2}^{\infty} |a_w| t^w + \sum_{w=1}^{\infty} |b_w| \bar{t}^w \right) dt,$$
$$= z_1 + z_2 I - \sum_{w=2}^{\infty} \frac{u+1}{u+w} |a_w| (z_1 + z_2 I)^w + \sum_{w=1}^{\infty} \frac{u+1}{u+w} |b_w| \overline{(z_1 + z_2 I)}^w$$
$$= z_1 + z_2 I - \sum_{k=2}^{\infty} d_w (z_1 + z_2 I)^w + \sum_{w=1}^{\infty} l_w \overline{(z_1 + z_2 I)}^w$$

$d_w = \frac{u+1}{u+w} |a_w|$ And $l_w = \frac{u+1}{u+w} |b_w|$

Therefore

$$\sum_{n=2}^{\infty} MR_p(v_1 + v_2 I) \frac{u+1}{u+w} |a_w| + \sum_{n=1}^{\infty} MR_p(v_1 + v_2 I) \frac{u+1}{u+w} |b_w| \le 1$$

From theorem (2.1)

$$T_u\big(f_n(z_1 + z_2 I)\big) \in MR_{\bar{p}}(v_1 + v_2 I)$$

9. Conclusion

In this paper, we studied and presented properties of univalent harmonic functions, where we obtained some theorems and properties associated with a class defined by an integral operator. Also, we have presented neutrosophic Hadamard product, neutrosophic convexity approach, and neutrosophic integral operator. In the future, we aim to generalize our results to refined neutrosophic function theory, and plithogenic theory.

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DOI: https://doi.org/10.54216/IJNS.230304
Received: August 12, 2023 Revised: November 12, 2023 Accepted: January 22, 2024

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Received: August 12, 2023 Revised: November 12, 2023 Accepted: January 22, 2024