A novel multivariate copula of Raftery type with multiple dependence parameters and its neutrosophic application in finance

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Abstract

This paper introduces an innovative multivariate exponential distribution, specifically of Raftery type, characterized by heterogeneous dependence parameters. Various properties of this distribution family are thoroughly investigated, with particular emphasis placed on the copula derived from this model. Notably, this copula is non-exchangeable and demonstrates multiple dependence parameters. Different properties associated with this novel copula, including the examination of estimation parameters, have been thoroughly investigated. The efficacy of the proposed copula is demonstrated through its successful application in modeling a real neutrosophic dataset associated with the New York and American Stock Exchanges.

Keywords: Raftery copula; multivariate copula; multiple dependence parameters ; mixed moment; moment method; neutrosophic theory; neutrosophic sets .

1 Introduction and motivation

Copula models allow for a more accurate representation of the dependence structure between financial variables, improving the assessment of joint risks. Creating a new copula model in finance is crucial for enhancing risk management and portfolio optimization. This innovation has the potential to enhance diversification strategies, refine hedging techniques, and foster overall stability in financial decision-making. Ultimately, it contributes to the development of more robust and reliable financial models. For a more in-depth exploration of copula applications in finance and risk management, see,[3] and,[2] and.[11]

Therefore, the primary objective of this article is to introduce a novel class of multivariate copulas, thereby enriching the existing set of copulas in the literature. This family of copulas is derived from a new multivariate exponential distribution involving various dependence parameters.

It is worth mentioning that multivariate exponential distributions are significant models with various applications in statistics and probability theory. They are used to describe multivariate data with dependent exponential marginal distributions. These kinds of models have received a lot of attention in the literature on the theory of distributions. For more details on the theoretical aspects and applications of different multivariate exponential distributions, see the comprehensive reference book by Johnson et al.[8] and the work by Basu.[1]

Raftery, in,[12] introduced an important class of multivariate exponential distribution with relevant interpretation in physics. This model provides a comprehensive framework for modeling correlation among the components

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of a multivariate exponential random vector, as it encompasses the multivariate Fréchet's upper bound distribution. Moreover, this distribution is continuous and devoid of singular component, facilitating the estimation of dependence parameter.

The multivariate Raftery exponential distribution, however, exhibits exchangeability, making it specifically suitable for modeling multivariate data characterized by identical exponential margins. This drawback arises due to the construction of the original multivariate Raftery exponential distribution using the common shock method, wherein dependence among the univariate components is induced through a single random variable. To address this issue, Saali et al.[13] have introduced a non-exchangeable multivariate Raftery exponential distribution utilizing the comonotonic shock method, as introduced in.[4] Furthermore, the multivariate copula derived from this model has been extensively studied in.[13] This copula offers the advantage of being straightforward to manage, given its exchangeability and the presence of only one easily estimated dependence parameter. However, this copula is constrained by the limitation of exclusively modeling random vectors wherein all pairs exhibit the same dependence structure. This may become impractical in certain real-world scenarios where the dependence levels among the pairs of random vectors to be described vary significantly.

Building upon the analysis above, the primary motivation of this study is to enhance the multivariate existing copula derived in[13] by introducing multiple dependence parameters. This extension offers increased flexibility in modeling the dependence structure among pairs of random vectors, accommodating diverse scenarios with distinct dependence parameters. To attain this objective, we first introduce a novel class of multivariate exponential random vectors. Notably, this model lacks exchangeability in both margins and dependence structure, coinciding with the model described in Saali et al.[13] when the dependence parameters are similar. Numerous properties of this novel model will be thoroughly examined, placing a specific emphasis on its associated copula. Dependence parameter estimation for the proposed copula will be discussed, along with its application to real-world neutrosophic data derived from financial context.[15]

The remainder of this paper is structured as follows. In Section 2, we delve into a detailed description of the proposed multivariate exponential distribution and elucidate the methodology for extracting its associated copula. Section 3 is dedicated to establishing key properties inherent in the proposed multivariate distribution. Section 4 derives the corresponding multivariate copula and scrutinize its various properties. In Section 5, we examine the estimation process for the dependence parameters of the proposed copula. Section 6 is dedicated to the practical application of the proposed copula in modeling real-world data related to neutrosophic datset associated with the New York and American Stock Exchanges.

2 Novel multivariate Raftery exponential distribution with multiple dependence parameters

2.1 Definition of the model

This section aims to define a family of multivariate exponential distributions characterized by various dependence parameters. The construction of marginal distributions is rooted in Raftery's work,[12] and the interdependence among them is governed by the comonotonic shock principle, as elucidated in.[4] Consider independent exponential random variables $\tilde{Y}_1, \ldots, \tilde{Y}_d, \tilde{Y}$ with parameters $\lambda_1, \ldots, \lambda_d, 1$, respectively. Saali et al.[13] propose a multivariate exponential random vector $(\tilde{X}_1, \ldots, \tilde{X}_d)$ of Raftery type, introducing a single dependence parameter:

$$\tilde{X}_j = (1-\theta)\tilde{Y}_j + J\lambda_j^{-1}\tilde{Y}, \quad j = 1, \ldots, d. \tag{1}$$

Here, J is a Bernoulli random variable with parameter $\theta \in (0,1)$ chosen independently of $\tilde{Y}_1, \ldots, \tilde{Y}_d, Z$. This model exhibits a homogeneous correlation structure, signifying that the Pearson correlation coefficient between any pair $(\tilde{X}_i, \tilde{X}_j)$, where $1 \leq i < j \leq d$, remains constant, see Proposition 2.3 in.[13] However, this uniformity might pose limitations in specific practical scenarios. To enhance the correlation structure, we propose extending the model described in (1) by introducing multiple dependent shocks, labeled as $J_1, \ldots, J_d$, in lieu of a singular shock J. To this end, suppose that $J_1 \ldots, J_d$ are comonotonic Bernoulli random variables with parameter $\Theta = (\theta_1, \ldots, \theta_d) \in (0,1)^d$. This implies that these variables are perfectly dependent, meaning there exists a uniformly distributed random variable V_0 over the interval $[0,1]$ such that

$$J_k = \mathbb{I}_{\{V_0 \leq \theta_k\}}, \quad k = 1, \ldots, d, \tag{2}$$

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where where $\mathbb{I}_A$ stands for the indicator function of the event A, and $\theta_1, \ldots, \theta_d$ form a decreasing sequence, denoted as $0 \leq \theta_d \leq \cdots \leq \theta_1 \leq 1$. Hence, the proposed multivariate exponential model with multiple dependence parameters is formally stated below. Let $Y_1, \ldots, Y_d, Y$ be independent exponential random variables with parameters $\lambda_1, \ldots, \lambda_d, 1$, respectively. Furthermore, assume that the uniform random variable V_0 involved in (2) is independent of the random vector $(Y_1, \ldots, Y_d, Y)$. A multivariate exponential random vector of Raftery type, characterized by multiple dependence parameters, is defined by the following equations,

$$\begin{aligned} X_1 &= (1-\theta_1)Y_1 + J_1\lambda_1^{-1}Y, \\ X_2 &= (1-\theta_2)Y_2 + J_2\lambda_2^{-1}Y, \\ &\vdots \\ X_d &= (1-\theta_d)Y_d + J_d\lambda_d^{-1}Y. \end{aligned} \tag{3}$$

Observe that the equations (3) give rise to a set of multivariate exponential distributions with fixed marginals. This is evident upon verification that the random variables $X_1, \ldots, X_d$ exhibit dependence and follow exponential distributions with parameters $\lambda_1, \ldots, \lambda_d$, respectively. Additionally, $\Theta = (\theta_1, \ldots, \theta_d)$ can be interpreted as the dependence parameter vector associated with this family, and importantly, it does not influence the marginal distributions. In the subsequent sections of the paper, we will denote the this family distributions as MEPD(Λ, Θ), where $\Lambda = (\lambda_1, \ldots, \lambda_d)$.

3 Properties of the proposed multivariate exponential distribution

3.1 High-order mixed moments of the proposed multivariate exponential distribution

In the following, we calculate the high-order mixed moments for the proposed multivariate exponential distribution, enabling the establishment of its correlation structure Let $(X_1, \ldots, X_d)$ be a multivariate exponential random vector defined in (3). Then for any, $1 \leq s \neq k \leq d$ and $n, m \geq 1$,

$$\begin{aligned} \mathrm{E}\left(X_s^n X_k^m\right) &= \frac{n!m!\min(\theta_k,\theta_s)}{\lambda_s^n\lambda_k^m}\sum_{i=1}^{n}\sum_{j=1}^{m}\binom{i+j}{i}(1-\theta_s)^{n-i}(1-\theta_k)^{m-j} \\ &+\frac{n!m!(1-\theta_k)^m\theta_s}{\lambda_s^n\lambda_k^m}\sum_{i=1}^{n}\frac{1}{i!}+\frac{n!m!(1-\theta_s)^n\theta_k}{\lambda_s^n\lambda_k^m}\sum_{j=1}^{m}\frac{1}{j!} \\ &+\frac{n!m!(1-\theta_s)^n(1-\theta_k)^m}{\lambda_s^n\lambda_k^m}. \end{aligned} \tag{4}$$

Proof. Given that the components of $(X_1, \ldots, X_d)$ are defined through the equations (3), it follows that for any pair $1 \leq s \neq k \leq d$,

$$X_s = (1-\theta)Z_s + J_s\lambda_s^{-1}Z \quad \text{and} \quad X_k = (1-\theta)Z_k + J_k\lambda_k^{-1}Z.$$

Since, (J_s, J_k), Z_s, Z_k and Z are independent, it follows from the binomial expansion formula,

$$\begin{aligned} \mathrm{E}\left(X_s^n X_k^m\right) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\binom{n}{i}\binom{m}{j}(1-\theta_s)^{n-i}(1-\theta_k)^{m-j}\lambda_s^{-i}\lambda_k^{-j}\mathrm{E}(J_s^iJ_k^j)\mathrm{E}(Y_s^{n-i})\mathrm{E}(Y_k^{m-j})\mathrm{E}(Y^{i+j}) \\ &+(1-\theta_k)^m\sum_{i=1}^{n}\binom{n}{i}(1-\theta_s)^{n-i}\lambda_s^{-i}\mathrm{E}(J_s^i)\mathrm{E}(Y_s^{n-i})\mathrm{E}(Y_k^m)\mathrm{E}(Y^j) \\ &+(1-\theta_s)^n\sum_{j=1}^{m}\binom{m}{j}(1-\theta_k)^{m-j}\lambda_k^{-j}\mathrm{E}(J_k^j)\mathrm{E}(Y_s^n)\mathrm{E}(Y_k^{m-j})\mathrm{E}(Y^j) \\ &+(1-\theta_s)^n(1-\theta_k)^m\mathrm{E}(Y_s^n)\mathrm{E}(Y_k^m). \end{aligned} \tag{5}$$

Using the well known expression of the high-order moments of the exponential distribution, we get,

$$\mathrm{E}(Y_s^{n-i}) = \frac{(n-i)!}{\lambda_s^{n-i}}, \quad \mathrm{E}(Y_k^{m-j}) = \frac{(m-j)!}{\lambda_k^{m-j}}, \quad \mathrm{E}(Y^{i+j}) = (i+j)!, \tag{6}$$

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and

$$\mathrm{E}(J_s^i J_k^j) = \mathrm{E}(J_s J_k) = \min(\theta_s, \theta_k). \tag{7}$$

Using standard binomial formulas, the expression (4) is then derived by inserting (6) and (7) in (5). which ends the proof. □

As a result, we obtain the Pearson correlation coefficients for pairs selected from the proposed exponential random vector. Let $(X_1, \dots, X_d)$ be a multivariate exponential random vector defined in (3). Then for any, $1 \leq s \neq k \leq d$,

$$\mathrm{cor}(X_s, X_k) = 2\min(\theta_s, \theta_k) - \theta_s\theta_k, \quad (\theta_s, \theta_k) \in [0,1]^2.$$

Proof. We observe from (4) that,

$$\begin{aligned} \mathrm{E}(X_s X_k) &= \lambda_s^{-1}\lambda_k^{-1}\left(\min(\theta_s,\theta_k) + (1-\theta_k)\theta_s + (1-\theta_s)\theta_k + (1-\theta_s)(1-\theta_s)\right) \\ &= \lambda_s^{-1}\lambda_k^{-1}\left(\min(\theta_s,\theta_k) + \theta_s\theta_k + 1\right). \end{aligned}$$

The deduction of the result follows immediately, utilizing the expressions, $\mathrm{E}(X_s)\mathrm{E}(X_k) = \lambda_s^{-1}\lambda_k^{-1}$ and $\mathrm{var}(X_s)\mathrm{var}(X_k) = \lambda_s^{-2}\lambda_k^{-2}$. □

Upon setting $\theta_s = \theta_k = \theta$, it follows that $\mathrm{cor}(X_s, X_k) = 2\theta - \theta^2$, aligning precisely with the correlation expression established in Proposition 2.3 of.[13]

3.2 Moment-generating function of the proposed multivariate exponential distribution

This section establishes the analytic expression of the moment-generating function for the family $\mathrm{MEPD}(\Lambda, \Theta)$. Let $\mathbf{X}^\top = (X_1, \dots, X_d)$ be the multivariate exponential random vector defined in (3). For any, $\mathbf{t}^\top = (t_1, \dots, t_d) \in \mathbb{R}^d$ such that $t_s < \lambda_s^{-1}$, $s = 1, \dots, d$, the moment-generating function of $\mathbf{X}$ is given by,

$$\mathrm{M}_{\mathbf{X}}(\mathbf{t}) = \left(1 - \theta_1 + \sum_{k=1}^{d}(\theta_k - \theta_{k+1})\prod_{s=1}^{k}\left(\frac{1}{1-t_s}\right)^{\lambda_s}\right)\prod_{s=1}^{d}\left(\frac{1}{1-t_s\lambda_s}\right)^{1-\theta_s}.$$

Proof.

$$\begin{aligned} \mathrm{M}_{\mathbf{X}}(\mathbf{t}) &= \mathrm{E}\left(\exp\left\{\mathbf{t}^\top\mathbf{X}\right\}\right) \\ &= \mathrm{E}\left(\exp\left\{\sum_{s=1}^{d} t_s X_s\right\}\right) \\ &= \mathrm{E}\left(\exp\left\{\sum_{s=1}^{d} t_s(1-\theta_s)Y_s + \left(\sum_{s=1}^{d} t_s J_s \lambda_s\right)Y\right\}\right) \\ &= \prod_{s=1}^{d} M_{Y_s}^{1-\theta_s}(t_s) \times \mathrm{E}\left(\exp\left\{\left(\sum_{s=1}^{d} t_s J_s\lambda_s\right)Y\right\}\right) \\ &= \prod_{s=1}^{d}\left(\frac{1}{1-t_s\lambda_s}\right)^{1-\theta_s} \times \mathrm{E}\left(\exp\left\{\left(\sum_{s=1}^{d} t_s J_s\lambda_s\right)Y\right\}\right). \end{aligned}$$

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The second term in the above expression can be simplified as follows,

$$\begin{aligned}
\mathrm{E}\left(\exp\left\{\left(\sum_{s=1}^{d} t_s J_s \lambda_s\right) Y\right\}\right) &= \sum_{k=0}^{d}(\theta_k-\theta_{k+1})\mathrm{E}\left(\exp\left\{\left(\sum_{s=1}^{d} t_s J_s \lambda_s\right) Y\right\} \big| \theta_{k+1}\le V_0\le\theta_k\right)\\
&= 1-\theta_1+\sum_{k=1}^{d}(\theta_k-\theta_{k+1})\mathrm{E}\left(\exp\left\{\left(\sum_{s=1}^{k} t_s\lambda_s\right) Y\right\}\right)\\
&= 1-\theta_1+\sum_{k=1}^{d}(\theta_k-\theta_{k+1})\prod_{s=1}^{k} M_Y(t_s)^{\lambda_s}\\
&= 1-\theta_1+\sum_{k=1}^{d}(\theta_k-\theta_{k+1})\prod_{s=1}^{k}\left(\frac{1}{1-t_s}\right)^{\lambda_s}.
\end{aligned}$$

□

3.3 Survival function linked to MEPD(Λ,Θ)

The survival function $\bar{H}\Theta$ of the random vector $(X_1,\ldots,X_d)$, as described by the equations (3), can be derived from the copula $C\Theta$ stated in Proposition 4. According to Sklar's theorem, we observe that:

$$\bar{H}_\Theta(\mathbf{x}) = C_\Theta\left\{\exp(-\lambda_1 x_1),\ldots,\exp(-\lambda_d x_d)\right\}. \tag{8}$$

Furthermore, define $\tilde{x}_i=\lambda_i x_i$ and $u_i=\exp(-\tilde{x}_i)$ for $i=1,\ldots,d$. Consequently, $u_{i:k}=\exp(-\tilde{x}_{k-i+1:k})$. By substituting $u_{i:k}$ for $\exp(-\tilde{x}_{k-i+1:k})$ and u_i for $\exp(-\tilde{x}_i)$ in (10), as per the result derived from (8), we directly obtain the multivariate survival function $\bar{H}_\Theta$.

4 Survival copula family corresponding to MEPD(Λ,Θ)

We are now focused on defining the survival copula derived from the multivariate family of distributions MEPD(Λ,Θ). To achieve this, let us first rewrite the equations (3) equivalently as follows,

$$\begin{aligned}
X_1 &= -\lambda_1^{-1}\big\{(1-\theta_1)\ln(V_1)+\mathbb{I}_{\{V_0\le\theta_1\}}\ln(V)\big\},\\
X_2 &= -\lambda_2^{-1}\big\{(1-\theta_2)\ln(V_2)+\mathbb{I}_{\{V_0\le\theta_2\}}\ln(V)\big\},\\
&\vdots\\
X_d &= -\lambda_d^{-1}\big\{(1-\theta_d)\ln(V_d)+\mathbb{I}_{\{V_0\le\theta_d\}}\ln(V)\big\},
\end{aligned}$$

where the random variables $V_0,V_1,\ldots,V_d$ and V are independent and uniformly distributed over [0,1]. Since the marginal random variable X_j follows an exponential distribution with parameter λ_j for $j=1,\ldots,d$, it is easy to verify that the survival copula C_Θ of the MEPD(Λ,Θ) family exactly corresponds to the distribution of the uniform random vector $(U_1,\ldots,U_d)$ defined by,

$$\begin{aligned}
U_1 &= \exp(-\lambda_1 X_1)=V_1^{1-\theta_k}V^{\mathbb{I}_{\{V_0\le\theta_1\}}},\\
U_2 &= \exp(-\lambda_2 X_2)=V_2^{1-\theta_2}V^{\mathbb{I}_{\{V_0\le\theta_2\}}},\\
&\vdots\\
U_d &= \exp(-\lambda_d X_d)=V_d^{1-\theta_d}V^{\mathbb{I}_{\{V_0\le\theta_d\}}}.
\end{aligned} \tag{9}$$

To obtain the survival copula, assume, without loss of generality, that $0\le\theta_d\le\cdots\le\theta_1\le 1$, and define $\theta_0=1$ and $\theta_{d+1}=0$. For $k=1,\ldots,d$, arrange the order statistics of the sample $u_1,\ldots,u_k$ as $u_{1:k},\ldots,u_{k:k}$, and set $u_{0:k}=0$ and $u_{k+1:k}=1$. For $i=1,\ldots,k$, denote the rank of u_i among $u_1,\ldots,u_k$ as $r_k(u_i)$.

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The survival copula for the random vector $(X_1, \ldots, X_d)$ is given for all $\mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d$ with $u_{d+1} = 1$ as follows:

$$C_\Theta(\mathbf{u}) = \sum_{k=0}^{d} (\theta_k - \theta_{k+1}) J_k \prod_{s=k+1}^{d+1} u_s^{\frac{1}{1-\theta_s}}, \tag{10}$$

where $J_0 = 1$ and for $k = 1, \ldots, d$,

$$J_k = u_{1:k} + \sum_{i=1}^{k} \frac{1}{1-\Theta_{i,k}} \prod_{s=1}^{i} u_{s:k}^{\frac{1}{1-\theta_{r_k(u_s)}}} \times \left(u_{i+1:k}^{1-\Theta_{i,k}} - u_{i:k}^{1-\Theta_{i,k}} \right),$$

and

$$\Theta_{i,k} = \sum_{s=1}^{i} \frac{1}{1-\theta_{r_k(u_s)}}, \quad i = 1, \ldots, k.$$

Proof. For all $\mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d$, one has from the equations (9),

$$\begin{aligned}
C_\Theta(\mathbf{u}) &= \mathrm{P}\left(U_1 \le u_1, \ldots, U_d \le u_d\right) \\
&= \sum_{k=0}^{d} (\theta_k - \theta_{k+1}) \mathrm{P}\left(U_1 \le u_1, \ldots, U_d \le u_d | \theta_{k+1} \le V_0 \le \theta_k\right) \\
&= (1-\theta_1) \prod_{s=1}^{d} u_s^{\frac{1}{1-\theta_s}} + \theta_d \mathrm{P}\left(V_1^{1-\theta_1} V \le u_1, \ldots, V_d^{1-\theta_d} V \le u_d\right) \\
&\quad + \sum_{k=1}^{d-1} (\theta_k - \theta_{k+1}) \mathrm{P}\left(V_1^{1-\theta_1} V \le u_1, \ldots, V_k^{1-\theta_k} V \le u_k, V_{k+1}^{1-\theta_{k+1}} \le u_{k+1}, \ldots V_d^{1-\theta_d} \le u_d\right) \\
&= (1-\theta_1) \prod_{s=1}^{d} u_s^{\frac{1}{1-\theta_s}} + \theta_d J_d + \sum_{k=1}^{d-1} (\theta_k - \theta_{k+1}) \prod_{s=k+1}^{d} u_s^{\frac{1}{1-\theta_s}} J_k, \qquad (11)
\end{aligned}$$

where for $k = 1, \ldots, d$,

$$J_k = \int_0^1 F\left\{ \left(\frac{u_1}{t}\right)^{\frac{1}{1-\theta_1}} \right\} \times \cdots \times F\left\{ \left(\frac{u_k}{t}\right)^{\frac{1}{1-\theta_k}} \right\} dt,$$

and F represents the uniform distribution over $[0,1]$. As $\theta_0 = 1$, $\theta_{d+1} = 0$ and $u_{d+1} = 1$, the expression in (11) can be significantly as follows,

$$C_\Theta(\mathbf{u}) = \sum_{k=0}^{d} (\theta_k - \theta_{k+1}) J_k \prod_{s=k+1}^{d+1} u_s^{\frac{1}{1-\theta_s}}.$$

Furthermore, J_k can be rewritten in terms of the order statistics $u_{1:k}, \ldots, u_{k:k}$ along with the rank function $r_k(u_i)$, $i = 1, \ldots, k$, as follows,

$$\begin{aligned}
J_k &= \int_0^1 F\left\{ \left(\frac{u_{1:k}}{t}\right)^{\frac{1}{1-\theta_{r_k(u_1)}}} \right\} \times \cdots \times F\left\{ \left(\frac{u_{k:k}}{t}\right)^{\frac{1}{1-\theta_{r_k(u_k)}}} \right\} dt \\
&= u_{1:k} + \sum_{i=1}^{k} \int_{u_{i:k}}^{u_{i+1:k}} F\left\{ \left(\frac{u_{1:k}}{t}\right)^{\frac{1}{1-\theta_{r_k(u_1)}}} \right\} \times \cdots \times F\left\{ \left(\frac{u_{k:k}}{t}\right)^{\frac{1}{1-\theta_{r_k(u_k)}}} \right\} dt \\
&= u_{1:k} + \sum_{i=1}^{k} \int_{u_{i:k}}^{u_{i+1:k}} \left(\frac{u_{1:k}}{t}\right)^{\frac{1}{1-\theta_{r_k(u_1)}}} \times \left(\frac{u_{i:k}}{t}\right)^{\frac{1}{1-\theta_{r_k(u_k)}}} dt \\
&= u_{1:k} + \sum_{i=1}^{k} \frac{1}{1-\Theta_i} \prod_{s=1}^{i} u_{s:k}^{\frac{1}{1-\theta_{r_k(u_s)}}} \times \left(u_{i+1:k}^{1-\Theta_i} - u_{i:k}^{1-\Theta_i} \right),
\end{aligned}$$

where

$$\Theta_i = \sum_{s=1}^{i} \frac{1}{1-\theta_{r_k(u_s)}}.$$

□

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Observing that the survival copula C_θ, as established in Proposition 1 in Saali et al (2023), is a special case derived from C_Θ when $\theta_1 = \cdots = \theta_d = \theta$, it is noteworthy to mention that this connection illuminates the relationship between these two copulas. Furthermore, in the case of $d = 2$, the copula C_Θ is explicitly defined for any $(u_1, u_2) \in [0,1]^2$ as

$$\begin{aligned} C_\Theta(u_1,u_2) &= \frac{\theta_2}{\theta_{r(u_1)}} u_{1:2} + \frac{\theta_1-\theta_2}{\theta_1} u_1 u_2^{\frac{1}{1-\theta_2}} \\ &\quad + \frac{\theta_2(1-\theta_{r(u_1)})^2}{\theta_1(1-\theta_1\theta_2)} u_1^{\frac{1}{1-\theta_1}} u_2^{\frac{1}{1-\theta_2}} \left(1 - \frac{\theta_1}{\theta_{r(u_1)}} u_{2:2}^{-\frac{1-\theta_1\theta_2}{(1-\theta_1)(1-\theta_2)}}\right), \end{aligned} \tag{12}$$

where $r(u_i) = \text{rank}(u_i)$, $i = 1, 2$, $u_{1:2} = \min(u_1, u_2)$, $u_{2:2} = \max(u_1, u_2)$ and $0 \le \theta_2 \le \theta_1 \le 1$. This copula is novel and coincide with Raftery copula when $\theta_1 = \theta_2 = \theta$, namely,

$$C_\theta(u_1,u_2) = u_{1:2} + \frac{1-\theta}{1+\theta} u_1^{\frac{1}{1-\theta}} u_2^{\frac{1}{1-\theta}} \left(1 - u_{2:2}^{-\frac{1+\theta}{1-\theta}}\right). \tag{13}$$

The density function the copula described in (12) is given by,

$$\begin{aligned} c_\Theta(u_1,u_2) &= \frac{\theta_1-\theta_2}{\theta_1(1-\theta_2)} u_2^{\frac{\theta_2}{1-\theta_2}} + \frac{\theta_2(1-\theta_{r(u_1)})}{\theta_1(1-\theta_1\theta_2)(1-\theta_{r(u_2)})} u_1^{\frac{\theta_1}{1-\theta_1}} u_2^{\frac{\theta_2}{1-\theta_2}} \\ &\quad + \frac{\theta_2}{1-\theta_1\theta_1} u_{1:2}^{\frac{\theta_{r(u_1)}}{1-\theta_{r(u_1)}}} u_{2:2}^{-\frac{1}{1-\theta_{r(u_1)}}}. \end{aligned} \tag{14}$$

Observe that the copula featuring two dependence parameters, as illustrated in (12), possesses the capability to effectively model an extensive spectrum of dependence patterns. This encompasses asymmetric relationship which elude adequate capture by a sole dependence parameter, as seen in the conventional copula defined in (4.5). The exploration of asymmetric dependence in financial modeling, as underscored in the work by Patton,[11] adds a compelling dimension to the financial context.

5 Properties of the proposed copula

5.1 Algorithm of simulation

It is straightforward to simulate from the proposed copula C_Θ using the following algorithm:

1. Assume that $0 \le \theta_d \le \theta_{d-1} \le \cdots \le \theta_1 \le 1$.
2. Generate independent values $v_0, v_1, \dots, v_d$ and v from the uniform distribution on $[0,1]$.
3. Set $u_1 = v_1^{1-\theta_1} v^{\mathbb{I}v_0 \le \theta_1}, \dots, u_d = v_d^{1-\theta_d} v^{\mathbb{I}v_0 \le \theta_d}$.
4. The desired vector is $(u_1, \dots, u_d)$.

Hereafter several illustrations demonstrating the implementation of the preceding algorithm to simulate data from the bivariate copula specified in equation (12).

5.2 High-order mixed moments of the proposed copula

The mixed moments play a crucial role in estimating the copula parameters by using the moment method. This section derive explicit form of the high-order mixed moments of the proposed copula. Let $(U_1, \dots, U_d)$ be a uniform vector distributed according to the copula C_Θ. Then, for any $1 \le s < k \le d$ and $n, m \ge 1$,

$$\mathrm{E}(U_s^n U_k^m) = \frac{(n+m+1)(n+1-n\theta_s) - \theta_k}{[n(1-\theta_s)+1][m(1-\theta_k)+1](n+m+1)(n+1)}. \tag{15}$$

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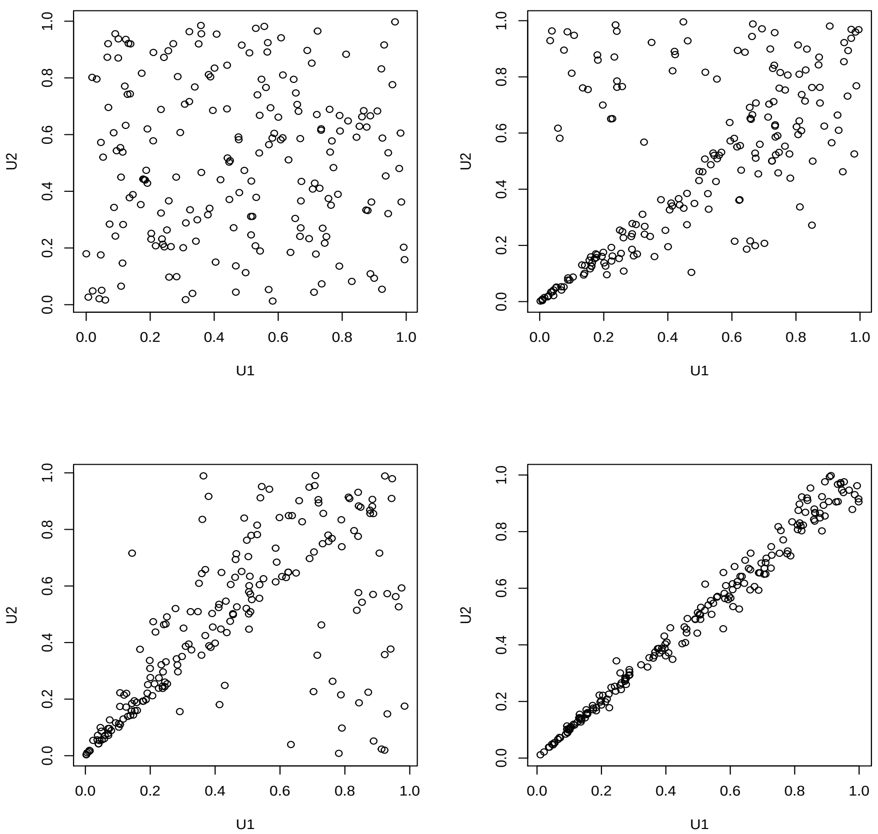

Figure 1: *Simulations from the copula C_Θ expressed in* (12)*, for* Θ *equals* (*0.3, 0.1*)*,* (*0.9, 0.7*)*,* (*0.7, 0.9*)*,* (*0.95, 0.95*)*, respectively.*

Proof. Given that $(U_1, \ldots, U_d)$ follows the distribution C_Θ, it can be inferred from (9) that, for any s and k within the range 1 to d,

$$U_s = V_s^{1-\theta_s} V^{\mathbb{I}_{\{V_0 \leq \theta_s\}}} \quad \text{and} \quad U_k = V_k^{1-\theta_k} V^{\mathbb{I}_{\{V_0 \leq \theta_k\}}}$$

where $V_s, V_k, V,$ and V_0 are independent standard uniform random variables. For the sake of simplicity, suppose without loss of generality that $1 \leq s < k \leq d$, which implies that $\theta_k \leq \theta_s$. Using the fact that $0 \leq \theta_d \leq \cdots \leq \theta_1 \leq 1$, it follows that,

$$\begin{aligned}
\mathrm{E}(U_s^n U_k^m) &= \mathrm{E}\left(V_s^{n(1-\theta_s)} V_k^{m(1-\theta_k)} V^{n\mathbb{I}_{\{V_0 \leq \theta_s\}} + m\mathbb{I}_{\{V_0 \leq \theta_k\}}}\right) \\
&= \mathrm{E}\left(V_s^{n(1-\theta_s)} V_k^{m(1-\theta_k)} V^{n+m} | V_0 \leq \theta_k\right) \theta_k \\
&\quad + \mathrm{E}\left(V_s^{n(1-\theta_s)} V_k^{m(1-\theta_k)} V^{n} | \theta_k \leq V_0 \leq \theta_s\right) (\theta_s - \theta_k) \\
&\quad + \mathrm{E}\left(V_s^{n(1-\theta_s)} V_k^{m(1-\theta_k)} | V_0 \geq \theta_s\right) (1-\theta_s),
\end{aligned}$$

which can be rewritten as,

$$\begin{aligned}
&\theta_k \mathrm{E}\left(V_s^{n(1-\theta_s)}\right) \mathrm{E}\left(V_k^{m(1-\theta_k)}\right) \mathrm{E}\left(V^{n+m}\right) \\
&\quad + (\theta_s - \theta_k) \mathrm{E}\left(V_s^{n(1-\theta_s)}\right) \mathrm{E}\left(V_k^{m(1-\theta_k)}\right) \mathrm{E}\left(V^{n}\right) \\
&\quad + (1-\theta_s) \mathrm{E}\left(V_s^{n(1-\theta_s)}\right) \mathrm{E}\left(V_k^{m(1-\theta_k)}\right).
\end{aligned}$$

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The above expression, in turn, is reduced to

$$
\begin{aligned}
& \frac{\theta_k}{(n(1-\theta_s)+1)(m(1-\theta_k)+1)(n+m+1)} \\
& + \frac{\theta_s-\theta_k}{(n(1-\theta_s)+1)(m(1-\theta_k)+1)(n+1)} \\
& + \frac{1-\theta_s}{(n(1-\theta_s)+1)(m(1-\theta_k)+1)} \\
= \quad & \frac{(n+m+1)(n+1-n\theta_s)-\theta_k}{(n(1-\theta_s)+1)(m(1-\theta_k)+1)(n+m+1)(n+1)},
\end{aligned}
$$

which ends the proof.

□

5.3 Spearman's rho

Consider a uniform vector $(U_1,\ldots,U_d)$ distributed according to a copula C_Θ. For any distinct indices s and k within the range 1 to d, the Spearman's rho of the copula associated with (U_s,U_k) is given by,

$$\rho_{sk} = \frac{\min(\theta_k,\theta_s)\,(4-3\max(\theta_k,\theta_s))}{(2-\theta_s)(2-\theta_k)}. \tag{16}$$

Proof. Spearman's rho for the copula of the uniform random vector (U_s,U_k) is expressed as,

$$\rho_{sk} = 12\mathrm{E}(U_sU_k) - 3.$$

Suppose without loss of generality that $\theta_k \le \theta_s$, then from (15), one gets

$$\mathrm{E}(U_sU_k) = \frac{3(2-\theta_s)-\theta_k}{6(2-\theta_s)(2-\theta_k)},$$

which implies that

$$\rho_{sk} \quad = \quad \frac{2\left\{3(2-\theta_s)-\theta_k\right\}}{(2-\theta_s)(2-\theta_k)} - 3 = \frac{\theta_k(4-\theta_s)}{(2-\theta_s)(2-\theta_k)} = \frac{\min(\theta_k,\theta_s)\,(4-3\max(\theta_k,\theta_s))}{(2-\theta_s)(2-\theta_k)}.$$

□

6 Estimation procedure and real data application

6.1 Estimation method

In this section, we examine the process of estimating the dependence parameter vector denoted as $\Theta = (\theta_1,\ldots,\theta_d)$ for the proposed copula C_Θ. Our approach focuses on the moment method, utilizing the bivariate Spearman coefficients as established in Proposition 5.3. This method provides a robust framework for parameter estimation, offering a reliable estimation of the interdependence inherent in the proposed copula structure.

The initial step involves generating estimates, $\hat{\rho}_{s,k}$, for the population Spearman's rho $\rho_{s,k}$, for any $(s,k) \in \{1,\ldots,d\}^2$. Therefore, the second step involves calculating the mean of $\rho_{s,k}$ for any fixed $s \in \{1,\ldots,d\}$, where $k \in \{1,\ldots,d\} \setminus \{s\}$, denoted as ρ_s. This is achieved through the formula:

$$\rho_s = \frac{1}{d-1}\sum_{k\neq s}\rho_{s,k},$$

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Similarly, the corresponding mean for $\hat{\rho}_{s,k}$ is computed as:

$$\hat{\rho}_s = \frac{1}{d-1} \sum_{k \neq s} \hat{\rho}_{s,k}.$$

Subsequently, the point estimates of the dependence parameter vector $\Theta = (\theta_1, \ldots, \theta_d)$, denoted $\hat{\Theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_d)$ is derived by solving the following minimization problem:

$$\min_{\Theta \in [0,1]^d} \sum_{s=1}^{d} \left(\rho_s - \hat{\rho}_s\right)^2,$$

that is,

$$\hat{\Theta} =_{\Theta \in [0,1]} \sum_{s=1}^{d} \left(\rho_s - \hat{\rho}_s\right)^2. \tag{17}$$

6.2 Real data application

To assess the efficacy of the current methodology, we employ a neutrosophic financial asset real dataset, focusing on the analysis of the Acme data. This dataset is accessible through both the datasets R package and the source.[14] The neutrosophic data comprises 60 rows and 3 columns, and the information was collected over a span of five years. The dataset encompasses excess returns for the Acme Cleveland Corporation, as well as those for all equities listed on the New York and American Stock Exchanges. These excess returns are calculated in relation to the return on a risk-free investment, such as US Treasury notes. The key variables contributing to the dataset are denoted as X_1 and X_2, representing the market's overall excess return and that of the Acme Cleveland Corporation, respectively.[6]

Our objective is to utilize the suggested bivariate copula C_Θ, as defined in (12), to accurately model a bivariate distribution that characterizes the neutrosophic real dataset represented by the bivariate random pair (X_1, X_2). This can be achieved through a two-phase approach. During the initial phase, we select the marginal distributions for both X_1 and X_2. In the subsequent phase, we validate that the copula defined in (12) apply captures the dependence structure between X_1 and X_2. This involves employing goodness-of-fit techniques, ensuring a comprehensive and robust modeling process.

First step. Tables 1 reveal that the Market and Acme variables conform to Generalized Extreme Value distributions, determined through the bootstrap technique employing the Kolmogorov-Smirnov (KS) Test. This particular distribution stands out as the most suitable choice among various distribution families, adeptly capturing the characteristics of the marginal distributions of Market and Acme. For further insights into the modeling, Tables 2 and 3 present the Maximum Likelihood Estimates (MLEs) of the model parameters.

Table 1: Test statistics and p-value tests

	Test statistic	p-value
Market	0.06482	0.94845
Acme	0.08498	0.74689

Table 2: MLEs of the model parameters for Market

	Market
Gen.Extreme Value	shape=-0.45224, scale=0.05506, location=-0.06509

Second step. Let us conduct a comprehensive evaluation of the Goodness-Of-Fit (GOF) test for the copula C_Θ, employing Cramér-von Mises statistics via the bootstrap algorithm pioneered by.[5] The estimation for the dependence parameter vector of the proposed copula is denoted as $\hat{\Theta} = (\hat{\theta}_1, \hat{\theta}_2) = (0.652, 0.507)$. These

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Table 3: MLEs of the model parameters for Acme

	Acme
Gen.Extreme Value	shape=0.03371, scale=0.0842, location=-0.12047

parameter values are derived using the moment method detailed in the preceding section, wherein they serve as the optimal solution to the minimization problem presented in (17). The accuracy and reliability of our copula model hinge on the calculated of the p-value of the Goodness-Of-Fit (GOF) introduced by.[5] The resulting p-value of this GOF test applied to the current data set using the copula C_Θ is 0.6761905, surpassing the significance level of 0.05. This outcome signifies that the proposed copula C_Θ effectively captures the interdependence between the variables Market and Acme. The following table succinctly summarizes these significant findings, highlighting the robust performance of the copula model in characterizing the underlying dependence structure.

Table 4: GOF test for $\hat{C}_\Theta$

	Cramér-von Mises
Test statistic	$S_n = 0.1086406$
p-value	0.6761905

Building upon the earlier analysis, we can deduce the bivariate distribution H that characterizes the interplay between the random variables Market and Acme. This derivation is facilitated by the application of Sklar's theorem (see[10]), which enables the representation of the distribution H through the copula C_Θ and the marginal distributions F_1 and F_2 for the Market and Acme, respectively. Consequently, the expression of the distribution H is given by:

$$H(x, y) = C_{\hat{\Theta}}(F_1(x), F_2(y)).$$

Here, the marginal distributions F_1 and F_2 have been identified in Table 6.2 and Table 6.3, providing a comprehensive understanding of their respective characteristics.

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