(σ, τ)-Derivations on Prime Inverse Gamma Semi-Ring

Sundus taha kathem[1,*], Abdulrahman Hameed Majeed [2]

Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
Emails: Sundus.phd.math@gmail.com; Abdulrahman.Hameed2021@gmail.com

Abstract

The concept of inverse Γ-semiring M is a generalization of inverse semiring. This paper investigates the concept (σ, τ)- derivation on inverse Γ-semiring and extend a few results of this map on prime inverse Γ- semiring that acts as a homomorphism or as an anti- homomorphism, where σ, τ are automorphisms on M.

Keywords: (σ, τ)- derivation; Inverse Γ-semiring; Jordan ideal; Left ideal; Prime inverse Γ-semiring

1. Introduction

The notion of the Γ - ring was first proposed by (Nobusawa, 1964) as an expansion of the definition of the classical ring. In the concept of Γ-ring in the sense of Nobusawa, (Barnes, 1966) weakened the conditions slightly. The definition of Γ semirings, a generalization of Γ-ring, ring and semiring, was introduced by (Rao, 1995, 1997) in 1995. Many mathematicians achieved interesting results on Γ-semirings after the paper (Rao, 1995, 1997) was written.

Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. M is said to be a Γ-semiring if there exists a map $M \times \Gamma \times M \rightarrow M$ that send the triples (x, α, y) to $x\alpha y$ such that

(i) $m\alpha(n + r) = m\alpha n + m\alpha r$,

(ii) $(m + n)\alpha r = m\alpha r + n\alpha r$,

(iii) $m(\alpha + \beta)n = m\alpha n + m\beta n$,

(iv) $(m\alpha n)\beta r = m\alpha(n\beta r)$,

for all $m, n, r \in M$ and $\alpha, \beta \in \Gamma$.

A Γ-semiring M is additively inverse if for every element $m \in$ M there exists a unique element $m' \in$ M such that $m + m' + m = m$ and $m' + m + m' = m'$. According to (Rao, 1997), for all $m, n \in M$, we have $(\text{man})' = m'\alpha n = m\alpha n', (m + n)' = m' + n', m'n' = mn, (m')' = m$.The center of a Γ-semiring M is defined as $Z(M) = \{m\epsilon M: m\alpha n = n\alpha m, \forall n \in M, \alpha\epsilon\Gamma\}$.An additively inverse Γ-semiring M is said to be a Γ-MA-semiring if $(m + m')\epsilon Z(M)$ for all $m \in$ M. M is said to be right (left) multiplicatively cancellable (Halder, 2022) if xγy = zγy ; (resp.xγy = xγz) for all x, y, z ∈ M and for all γ ∈ Γ implies that x = z (resp. y = z) (Salih & Karim, 2015). The commutator of any elements $m, n \in M$ can be defined as $[m, n]_{\alpha} = m\alpha n + n'\alpha m$, and $[m, n]_{\alpha} = 0$ implies $m\alpha n = n\alpha m$ and $(mon)_{\gamma} = m\gamma n + n\gamma m$ for any $m, n \in M$ and $\gamma \in \Gamma$ (Zaghir & Majeed, 2021). A non-empty subset I of M is said to be a left ideal ofM if for all a, b ∈ I then a + b ∈ I and If a ∈ I, s ∈ M and γ ∈ Γ then sγa ∈ I (Mandal, 2014). An additive subsemigroup U of inverse Γ -semiring M, is said to be a Jordan ideal of M if for all $r \in M, u \in U$, $u\alpha r + r\alpha u$, is in U ,then any ideal I of M is a Jordan ideal. Recall that M is called a prime inverse Γ-semiring if $x\Gamma M\Gamma y = 0$ implies $x = 0$ or $y = 0$, for $x, y \in M$, M a semiprime is named if $x\Gamma M\Gamma x = 0$ implies $x = 0$, for $x \in M$. Obviously, all prime inverseΓ-semiring is semiprime inverse Γ-semiring. In addition,
M is called a commutative if $x\alpha y = y\alpha x$ for all $x, y \in$ M and $\alpha \in \Gamma$, and M is called a 2-torsion free if $2x = 0$

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implies $x = 0$, for $x \in$ M . We know the identities of the basic commutator:$[x\beta y, z]_\alpha = [x, z]_\alpha \beta y + x[\beta, \alpha]_C y + x\beta[y, z]_\alpha$ and $[x, y\beta z]_\alpha = [x, y]_\alpha \beta z + y[\alpha, \beta]_X z + y\beta[x, z]_\alpha$ for all x, $y, z \in M$ and α, β $\in\Gamma$. We take an assumption: $a\alpha b\beta c = a\beta b\alpha c$, for all a, b, $c \in M$ and $\alpha, \in \Gamma(*)$.The above two identities are reduced, according to the hypothesis (*), to:$[x\beta y, z]_\alpha = [x, z]_\alpha \beta y + x\beta[y, z]_\alpha$, $[x, y\beta z]_\alpha = [x, y]_\alpha \beta z$ +
$y\beta[x, z]_\alpha$ and $(ao(b\beta c))_\alpha = (aob)_\alpha \beta c - b\beta[a, c]_\alpha = b\beta(a \circ c)_\alpha + [a, b]_\alpha \beta c$, $((a\beta b)oc)_\alpha = a\beta(boc)_\alpha - [a, c]_\alpha \beta c = (a \circ c)_\alpha b\beta + a\beta[b, c]_\alpha$ for all x, y, $z \in M$ and α, $\beta \in \Gamma$ (Hamil, 2021). In (Argaç et al., 1987) the concept of (σ,τ)-derivations in rings defined as follow an additive mapping d: M $\longrightarrow$ M is called (σ,τ)- derivation if $d(x\alpha y) = d(x)\alpha\ \sigma(y) + \tau(x)\alpha d(y)$ and Jordan (σ,τ)-derivation if $d(x\alpha x) = d(x)\alpha\ \sigma(x) + \tau(x)\alpha d(x)$
holds for all x, y $\in$ M and $\alpha \in \Gamma$ where σ, τ are endomorphisms of M. Majeed and Zaghir introduced $(\alpha, \beta) -$ Derivations on Prime Inverse Semirings where α, β are automorphisms on M (Zaghir & Majeed, 2021). Several authors have also addressed this topic (see, e.g., (Ibraheem & Majeed, 2019; Rasheed et al., 2020; Rasheed & Majeed, 2019) .An additive mapping d : M → M is called a derivation if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all a, b $\in$ M and $\alpha \in \Gamma$. An additive mapping ϕ : M → M is said to be a homomorphism if $\phi(a\alpha b) = \phi(a)\alpha\phi(b)$ for all a, b $\in$ M and $\alpha \in \Gamma$. And, an additive mapping ψ : M → M is called an anti-homomorphism if $\psi(a\alpha b) = \psi(b)\alpha\psi(a)$ for all a, b $\in$ M and $\alpha \in \Gamma$. A derivation d of M is said to act as a homomorphism [resp. as an anti-homomorphism] on a subset S of M if $d(a\alpha b) = d(a)\alpha d(b)$ [resp. $d(a\alpha b) = d(b)\alpha d(a)$] for all a, b $\in$ S and $\alpha \in \Gamma$ (Paul & Chakraborty, 2015). In this paper, we expanded some findings on (σ, τ)- derivation on inverse Γ-semiring that acts as a homomorphism or as an anti- homomorphism, where σ, τ are automorphisms on M. The results of this research serve researchers who work in this field to advance their work.

Research Method

To prove the main Theorems in this paper, we need the following Lemmas.

Definition 2.1

Let M be a $\Gamma-$semiring and U be a nonzero ideal of M, then the set Z(U)= {a $\in$U : a α b = b α a , $\forall$ b $\in$U ,$\alpha \in \Gamma$ } called the center of U.

Lemma 2.2

Let M be a $\Gamma-$semiring and U be a nonzero ideal of M, if U is commutative as Γ- semiring , then Z(U)= U.

Proof:

It is clear.

Lemma 2.3

Let M be a prime inverse Γ- semiring and U be a nonzero left (right) ideal of M, then Z (U)$\subseteq$Z(M).

Proof

Let 0$\neq$ a $\in$ Z(U), since Z(U) $\subseteq$U, then a $\in$U

Let x $\in$M, then $x\beta a$, $x \in U$ (By definition of left ideal).

Since a $\in$Z(U), we have

$[x\beta a, a]_\alpha = x\beta[a, a]_\alpha + [x, a]_\alpha \beta a = 0$

then,

$[x, a]_\alpha \beta a = 0$ for all x$\in$ M, $\alpha, \beta \in \Gamma$ (1)

Replace x by xγy in (1), where y $\in$M, $\gamma \in \Gamma$, we get

$[x\gamma y, a]_\alpha \beta a = 0$ for all x$\in$ M, $\alpha, \beta, \gamma \in \Gamma$

x γ $[x, a]_\alpha \beta$ a+ $[x, a]_\alpha \gamma y \beta a$ =
0.By using (1) we get

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$[x\ ,a]_\alpha\ \gamma y\beta\ a\ = 0$ for all x, y∈ M ,a∈ U and for all $\alpha, \beta, \gamma\ \in \Gamma$.

Then

$[x\ ,a]_\alpha$ ΓMΓ $a\ = 0$.

By primness and since U is nonzero ideal of M then

$[x\ ,a]_\alpha = 0$ for all x ∈M and a ∈ Z(U).
Then Z(U)⊆Z(M).

Lemma 2.4

Let M be an inverse Γ- semiring and U be a nonzero ideal of M, if U commutative of M , then U⊆Z(M). If M is prime then M is commutative.

Proof

Since U is commutative of M ,then
by (Lemma2.2)we have
U= Z(U).

By (lemma 2.3) we have
Z(U) ⊆Z(M),
then U ⊆ Z(M)

Now, If M is prime, Let x, y∈ M and a∈ U
Then $\alpha\beta x\ \in$ Z(M)that is
$[\alpha\beta x, y]_\alpha = 0$ for all y ∈M.

$\alpha\beta\ [\ x, y]_\alpha + [\alpha\ , y]_\alpha\beta x = \alpha\beta\ [\ x, y]_\alpha\ = 0$ for all a ∈ U.Then
UΓ$[\ x, y]_\alpha = 0$
UΓMΓ$[\ x, y]_\alpha = 0$
By primness of M and since U is nonzero ideal ,then

$[\ x, y]_\alpha\ = 0$ for all x, y∈ M.

Then M is commutative.

Lemma 2.5 (Halder, 2022)

Let M be an additively inverse Γ -semiring, for all $m, n \in M$, if $m + n = 0$ then $n\ = m'$ and $m + m'\ = 0$, forall $m, n\ \in\ M$ and $\alpha \epsilon \Gamma$.

Lemma 2.6

Let M be a cancellative prime inverse Γ-semiring. If $r\ = s$, where $r, s\ \in\ M$, then $r + s' = 0$.

Proof.

If $r\ = s$, where $r, s\ \in\ M$, then by adding $(s + s')$ to the two sides, we get:

$r + (s' + s) = s + (s' + s)$.

Since M is an inverse Γ-semiring, we get

$r + s' + s\ = s\ (r$

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$+ s') + s = s.$

Since M is a cancellative inverse Γ-semiring,”
we get r + s' =0.
Lemma 2.7

Let I be a non- zero left ideal on M, which is a semiprime inverse Γ-semiring. If $I\alpha u = 0$ ($u\alpha I = 0$) for all $u \in M, \alpha \in \Gamma$, then $u = 0$.

Proof

If $I\alpha u = 0$ for all $u \in M, \alpha \in \Gamma$,then we have:
$u\Gamma M\Gamma I = 0$. Since M is a prime inverse Γ-semiring and I is a nonzero left ideal
Therefore$u = 0$
Now, we want to show that if $I\alpha u = 0$, then $u = 0$. Suppose that $u \neq 0$. Define K by

$K = \{r \in M \mid I\alpha r = 0\}$

Since $0 \neq u \in K$, it is clear that K is a nonzero right ideal of M, such that $I\Gamma K = \{0\}$. On the other side, $K \cap I$ is a right ideal of I and

$$(K \cap I)\,\Gamma\mathrm{I}\Gamma(K \cap I) \subset I\Gamma K = \{0\}$$

Since I is a semiprime inverse Γ- semiring, then we get:

$(K \cap I) = \{0\}$. Then, we have $\subset K \cap I = \{0\}$.

Since M is a prime inverse Γ - semiring and I is a nonzero left ideal of M, we obtain $K = \{0\}$. Thus, we get $u = 0$.

Lemma 2.8

Let I be a non- zero left ideal on M, which is a semiprime as an inverse Γ - semiring. If $d(I) = 0$, then d =0 on M.

Proof:

By the assumption, $d(I) = 0$, then for all $x \in I$ and $s \in M$:

d(sαx) = d(s)α $\sigma(x) + r(s)$ xαd(x), for all s ∈ M, α ∈ Γ.

Since σ is an automorphism on M, we get $\sigma^{-1}(d(s))\Gamma I = 0$ for all $s \in M$.By
Lemma (2.7), we get $\sigma^{-1}(d(s)) = 0$, for all $s \in M$.
Again, since σ is an automorphism on M, we get $d(s) = 0$ for all $s \in M$, therefore $d = 0$ on M.

Lemma 2.9

If M be a prime inverse Γ - semiring and J be a nonzero Jordan ideal of M, then $2(r\alpha s + s'\alpha r)\Gamma J \subseteq J$ and $2J\Gamma(r\alpha s + s'\alpha r) \subseteq J$.

Proof

Let $r, s, s' \in$ M and $x \in J$. Then by definition of Jordan ideal we have,

$$x\alpha(r\alpha s + s'\alpha r) + (r\alpha s + s'\alpha r)\alpha x + (x\alpha r + r\alpha x)'\alpha s + s\alpha(x\alpha r + r\alpha x)' + (x\alpha s + s\alpha x)\alpha r \\ + r\alpha(x\alpha s + s\alpha x) \in J$$

As M is a Γ-MA semiring ($(i.e)(m + m')\epsilon Z(M)$ for all $m \in M$and $Z(M) = \{m\epsilon M : m\alpha n = n\alpha m, \forall n \in M, \alpha\epsilon\Gamma\}$).

This implies that,

$$x\alpha r\alpha s + x\alpha s\alpha r' + r\alpha s\alpha x + s\alpha r'\alpha x + x\alpha r\alpha s' + r\alpha x\alpha s' + s\alpha x\alpha r' + s\alpha r'\alpha x + x\alpha s\alpha r + s\alpha x\alpha r + r\alpha x\alpha s + r\alpha s\alpha x \in J$$

$$2r\alpha s\alpha x + 2s\alpha r'\alpha x + x\alpha(r\alpha s + r'\alpha s) + x\alpha(s\alpha r + s'\alpha r) + (r\alpha x + r'\alpha x)\alpha s + (s\alpha x + s'\alpha x)\alpha r \in J$$

Since, M be a Γ-MA semiring ($((i.e)(m + m')\epsilon Z(M)$ for all $m \in M$and $Z(M) = \{m\epsilon M : m\alpha n = n\alpha m, \forall n \in M, \alpha\epsilon\Gamma\}$),we get

$$2r\alpha s\alpha x + 2s\alpha r'\alpha x + (r\alpha s + r'\alpha s)\alpha x + (s\alpha r + s'\alpha r)\alpha x + s\alpha(r\alpha x + r'\alpha x) + r\alpha(s\alpha x + s'\alpha x) \in J$$

Therefore,

$$2r\alpha s\alpha x + 2r'\alpha s\alpha x + 2r\alpha s\alpha x + 2s\alpha r'\alpha x + 2s\alpha r\alpha x + 2s\alpha r'\alpha x \in J$$

Since M be an additively inverse$\Gamma -$ semiring, we get
$2r\alpha s\alpha x + 2s\alpha r'\alpha x \in J$ for all $x \in J$, $\alpha\epsilon\Gamma$ and $r, s \in M$.
And hence, we get
$2(r\alpha s + s'\alpha r)\alpha x \in J$ for all $x \in J, \alpha\epsilon\Gamma$ and $r, s \in M$.

That is, $2(r\alpha s + s'\alpha r)\Gamma J \subseteq J$ for all $r, s \in M$.

Similarly, to show that $2J\Gamma(r\alpha s + s'\alpha r) \subseteq J$,for all $r, s \in M$.

Lemma 2.10

Let J be a nonzero Jordan ideal of M. $If\ u \in M\ and\ u\alpha J = 0\ or\ (J\alpha u = 0\ ), then\ u = 0$.

Proof

Let $u\alpha J = 0$. Since J is a nonzero Jordan ideal of M, we have $(x \circ s)_\alpha \in J$ for all $x \in J$ and $s \in M$.

$$0 = u\alpha(x \circ s)_\alpha = u\alpha(x\alpha s + s\alpha x) = u\alpha x\alpha s + u\alpha s\alpha x$$

By assumption, we get:

$u\alpha s\alpha x = 0$ for all $x \in J$, $\alpha \in \Gamma$ and $s \in M$.
That is, $u\Gamma S\Gamma x = 0$ for all $x \in J$.
Since M is a prime inverse Γ- semiring and J is a nonzero Jordan ideal of M,
we get $u = 0$.
Using the similar way, we can show that if $J\alpha u = 0$ then $u = 0$.

Lemma 2.11

Let M be 2-tortion free prime inverse Γ- semiring and J is a nonzero Jordan ideal of M. If $u, v \in M$ such that $u\alpha J\alpha v = 0$, then either $u = 0$ or $v = 0$.

Proof:

By the assumption, we have $u\alpha J\alpha v = 0$.

By Lemma (2.9), we get $2(r\alpha s + s'\alpha r)\alpha J \subseteq J$ for all $r, s \in M and\ \alpha \in \Gamma$.

Then we have $2u\alpha(r\alpha s + s'\alpha r)\alpha x\alpha v = 0$, for all $x \in J$, $\alpha \in \Gamma$ and $r, s \in M$.This implies that, since M is 2-tortion free, we get:

$$u\alpha(r\alpha s + s'\alpha r)\alpha x\alpha v = 0, for\ all\ x \in J\ , \alpha \in \Gamma and\ r, s \in M.$$

By replacing s by sβu in the above equation, we get:

$$0 = u\alpha(r\alpha(s\beta u) + (s\beta u)'\alpha r)\alpha x\alpha v = u\alpha r\alpha s\beta u\alpha x\alpha v + u\alpha s\beta u'\alpha r\alpha x\alpha v$$

Since M is an inverse Γ- semiring, we get:

$$\begin{aligned} 0 &= u\alpha r\alpha(s + s' + s)\beta u\alpha x\alpha v + u\alpha s\beta u'\alpha(r + r' + r)\alpha x\alpha v \\ &= u\alpha r\alpha s\beta u\alpha x\alpha v + u\alpha r\alpha(s + s')\beta u\alpha x\alpha v + u\alpha s\beta u'\alpha r\alpha x\alpha v + u\alpha s\beta u'\alpha(r + r')\alpha x\alpha v \end{aligned}$$

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Since M is an additively inverse Γ- semiring, this gives

$$0 = u\alpha r\alpha s\beta u\alpha x\alpha v + u\alpha r\alpha(s + s')\beta u\alpha x\alpha v + u\alpha s\beta u'\alpha r\alpha x\alpha v + u\alpha s\beta(r + r')\alpha u'\alpha x\alpha v$$

$= u\alpha r\alpha s\beta u\alpha x\alpha v + u\alpha r\alpha s\beta u\alpha x\alpha v + u\alpha r\alpha s'\beta u\alpha x\alpha v + u\alpha s\beta u'\alpha r\alpha x\alpha v + u\alpha s\beta r\alpha u'\alpha x\alpha v +$ $u\alpha s\beta r\alpha u\alpha x\alpha v$

$$= u\alpha s\beta(r\alpha u + u'\alpha r)\alpha x\alpha v + u\alpha r\alpha s\beta u\alpha x\alpha v + u\alpha r s'\beta u\alpha x\alpha v + u\alpha r\alpha s\beta u\alpha x\alpha v + u\alpha s'\beta r\alpha u\alpha x\alpha v$$

$$= u\alpha s\beta(r\alpha u + u'\alpha r)\alpha x\alpha v + u\alpha r\alpha s\beta u\alpha x\alpha v + u\alpha s'\alpha r\beta u\alpha x\alpha v$$

$= u\alpha s\beta(r\alpha u + u'\alpha r)\alpha x\alpha v \ + u\alpha(r\alpha s + s'\alpha r)\beta u\alpha x\alpha v \ \ for\ all\ x \in J, \alpha, \beta \in \Gamma\ and\ r, s \in M$

By using $u\alpha J\alpha v = 0$ we get:

$u\alpha s\beta(r\alpha u + u'\alpha r)\alpha x\alpha v = 0\ for\ all\ x \in J,\ \alpha,\ \beta \in \Gamma\ and\ r,\ s \in M.$
And hence, we get, $u\Gamma M\Gamma(r\alpha u + u'\alpha r)\alpha x\alpha v = 0$.
Since M is a prime inverse Γ- semiring, we have :

either $u = 0$ or $(r\alpha u + u'\alpha r)\alpha x\alpha v = 0,\ for\ all\ x \in J, \alpha, \beta \in \Gamma\ and\ r, s \in M.$
If $(r\alpha u + u'\alpha r)\alpha x\alpha v = 0$, for all $x \in J$ and $r \in M$, then we have:
$r\alpha u\alpha x\alpha v + u'\alpha r\alpha x\alpha v = 0$, for all $x \in J$, $\alpha \in \Gamma$ and $r \in M$.

By the hypothesis we get : $u'\alpha r\alpha x\alpha v = 0$ for all $x \in J$, $\alpha \in \Gamma$ and $r \in M$,
that is, $u\Gamma M\Gamma x\alpha v = 0$, for all $x \in J$.
Again, since M is a prime inverse Γ- semiring, we get :

either $u = 0$ or $x\alpha v = 0$, for all $x \in J$. If$x\alpha v = 0$, for all $x \in J$, that is $J\alpha v = 0$, then by (Lemma 2.10), we get $v = 0$.

Lemma 2.12

Let M be 2-tortion free and J is a nonzero Jordan Ideal of M, If J is commutative, then $J \subseteq$ Z(M).

Proof

By Lemma (2.9), we have: $2(r\alpha s + s'\alpha r)\alpha J \subseteq J$ for all $r, s \in M\ and\ \alpha \in \Gamma$.
And since J is a commutative Jordan ideal of M, we have:
$2(r\alpha s + s'\alpha r)\alpha x\alpha y + 2y'\alpha(r\alpha s + s'\alpha r)\alpha x = 0$, for all $x, y \in J, \alpha \in \Gamma$

and $r, s \in M$.

Since M is 2- tortion free, we get:

$(r\alpha s + s'\alpha r)\alpha x\alpha y + y'\alpha(r\alpha s + s'\alpha r)\alpha x = 0$, for all $x, y \in J, \alpha \in \Gamma$ and$r, s \in M$. (1)
By Lemma(2.5), we get:
$(r\alpha s + s'\alpha r)\alpha x\alpha y = y'\alpha(r\alpha s + s'\alpha r)\alpha x$, for all $x, y \in J, \alpha \in \Gamma$

and $r, s \in M$.

Hence, by equation (1) we obtain:

$y\alpha(r\alpha s + s'\alpha r)\alpha x + y'\alpha(r\alpha s + s'\alpha r)\alpha x = 0$, for all $x, y \in J, \alpha \in \Gamma$

and $r, s \in M$.
Then we have:
$((r\alpha s + s'\alpha r)\alpha y + y'\alpha(r\alpha s + s'\alpha r))\alpha x = 0$, for all $x, y \in J, \alpha \in \Gamma$

and $r, s \in M$.
This gives:
$((r\alpha s + s'\alpha r)\alpha y + y'\alpha(r\alpha s + s'\alpha r))\alpha J = 0$, for all $y \in J, \alpha \in \Gamma$

and $r, s \in M$.

By (Lemma 2.10), we get:

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$(r\alpha s + s'\alpha r)\alpha y + y'\alpha(r\alpha s + s'\alpha r) = 0$, for all $y \in J, \alpha \in \Gamma$

and $r, s \in M$.

By replacing s by $r\beta s$ in the above equation, we get:

$$\begin{aligned} 0 &= (r\alpha\, r\beta s + \ r\beta s'\alpha r)\alpha y + y'\alpha(r\alpha\, r\beta s + \ r\beta s'\alpha r) \\ &= r\alpha(r\beta s + s'\beta r)\alpha y + y'\alpha r\alpha(r\beta s + s'\beta r) \end{aligned}$$

Since J is commutative, we get :

$r\alpha y\alpha(r\beta s + s'\beta r) + y'\alpha r\alpha(r\beta s + s'\beta r) = 0$ for all $y \in J$ and $r, s \in M$and $\alpha, \beta \in \Gamma$.
Hence we get,
$(\ r\alpha y + y'\alpha r)\alpha(r\beta s + s'\beta r) = 0$, for all $y \in J$ and $r, s \in M$ and $\alpha, \beta \in \Gamma$. (2)
By further replacing s by $s\gamma y$ in equation (2), we have:
$(\ r\alpha y + y'\alpha r)\alpha(r\beta s\gamma y + s\gamma y'\beta r) = 0$, for all $y \in J$ and $r, s \in$ M and $\gamma, \alpha, \beta \in \Gamma$.Since M is an inverse Γ- semiring, we have:

$$0 = (\ r\alpha y + y'\alpha r)\alpha(r\beta(s + s' + s)\gamma y + s\gamma y'\beta(r + r' + r))$$

$$= (\ r\alpha y + y'\alpha r)\alpha(r\beta s\gamma y + r\beta(s + s')\gamma y + s\gamma y'\beta r + s\gamma y'\beta(r + r'))$$

Since M is additively inverse Γ- semiring, we get:

$$\begin{aligned} 0 &= (\ r\alpha y + y'\alpha r)\alpha(r\beta s\gamma y + (s + s')\gamma r\beta y + s\gamma y'\beta r + s\gamma(r + r')\beta y') \\ &= (\ r\alpha y + y'\alpha r)\alpha(r\beta s\gamma y + s'\gamma r\beta y + s\gamma r\beta y + s\gamma y'\beta r + s\gamma r\beta y' + s'\gamma r\beta y') \\ &= (\ r\alpha y + y'\alpha r)\alpha(s\gamma(r\beta y + y'\beta r) + r\beta s\gamma y + s'\beta r\gamma y) \\ &= (\ r\alpha y + y'\alpha r)\alpha(s\gamma(r\beta y + y'\beta r) + (r\beta s + s'\beta r)\gamma y) \\ &= (\ r\alpha y + y'\alpha r)\alpha s\gamma(r\beta y + y'\beta r) + (\ r\alpha y + y'\alpha r)\alpha(r\beta s + s'\beta r)\gamma y \end{aligned}$$

By using equation (2) we get, $(\ r\alpha y + y'\alpha r)\alpha s\gamma(r\beta y + y'\beta r) = 0$, for all $y \in J, \alpha, \beta, \gamma \in \Gamma$. and $r, s \in M$.

And hence we get: $(\ r\alpha y + y'\alpha r)\Gamma M\Gamma(r\beta y + y'\beta r) = 0$, for all $y \in J, \alpha, \beta, \gamma \in \Gamma$. and $r, s \in M$.
Since M is a prime inverse Γ- semiring, we get:

$r\alpha y + y'\alpha r = 0$, for all $y \in J$, $\alpha \in \Gamma$ and $r \in M$.
Hence, we get: J ⊆ Z(M).

Proposition 2.13

Let M be 2-tortion free and J is a nonzero Jordan ideal and an inverse sub-Γ-semiring on M. If $d(J) = 0$, then $d = 0$ or $J \subseteq Z(M)$

Proof

By the assumption,

$d(J) = 0$ (3)

This yields:

$0 = d(x\alpha s + s\alpha x)$ for all $x \in J, \alpha \in \Gamma$, $s \in M$

=d(x)α $\sigma(s)$ + $r(x)$ αd(s)+ d(s)α $\sigma(x)$ + $r(s)$ αd(x).

By using equation (3), we get

τ(x) αd(s)+ d(s)α σ(x)=0, for all $x \in J, s \in M$. (4)
(By Lemma 2.5), we get:
τ(x) αd(s) = d(s)α σ(x)$'$ for all $x \in J, s \in M$. (5)
By replacing s by $r\beta s$, where $r \in M$ *and* $\beta \in \Gamma$, in equation (4), we get:

$$0 = \tau(x)\ \alpha d(\ r\beta s) + d(\ r\beta s)\alpha\ \sigma(x)$$

= τ(x) α$d(r)$β σ(s) + τ(x) α τ(r)β$d(s)$ + $d(r)$β σ(s)α σ(x) + τ(r)β$d(s)$α σ(x)By

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using equation (5), we get
$= d(r)\alpha\ \sigma(x)'\beta\sigma(s) + \tau(x)\ \alpha\ \tau(r)\beta d(s) + d(r)\beta\sigma(s)\alpha\ \sigma(x) + \tau(r)'\beta\tau(x)\ \alpha d(s)$Hence,
$d(r)\alpha(\ \sigma(x)'\beta\sigma(s) +\ \sigma(s)\beta\ \sigma(x)\ ) + (\tau(x)\ \beta\tau(r) + \tau(r)'\beta\tau(x))\alpha d(s) = 0$ (6)
By replacing s by $s\gamma y, s \in M$ in equation (6), we get:
$0 = d(r)\alpha(\ \sigma(x)'\beta\sigma(s\gamma y) +\ \sigma(s\gamma y)\beta\ \sigma(x)\ ) + (\tau(x)\ \beta\tau(r) + \tau(r)'\beta\tau(x))\alpha d(s\gamma y)$

$0 = d(r)\alpha(\ \sigma(x)'\beta\sigma(s)\gamma\sigma(y)) + \sigma(s)\gamma\sigma(y)\beta\ \sigma(x)\ ) + (\tau(x)\ \beta\tau(r) + \tau(r)'\beta\tau(x))\alpha d(s)\gamma\sigma(y) + \tau(s)\gamma d(y)$By using equations (3) and (5), we get:
$0 = d(r)\alpha(\ \sigma(x)'\beta\sigma(s)\gamma\sigma(y)) + \sigma(s)\gamma\sigma(y)\beta\ \sigma(x)\ ) + (\tau(x)\ \beta\tau(r) + \tau(r)'\beta\tau(x))\alpha d(s)\gamma\sigma(y)$Since M is an inverse Γ- semiring, then we have:
$d(r)\alpha\sigma(s)\gamma\sigma(y + y' + y)\beta\ \sigma(x) + d(r)\alpha$

$\sigma(x)'\beta\sigma(s + s' + s)\gamma\sigma(y) + (\tau(x)\ \beta\tau(r) + \tau(r)'\beta\tau(x))\alpha d(s)\gamma\sigma(y) = 0$That is,
$d(r)\alpha\ \sigma(s)\gamma\sigma(y)\ \beta\ \sigma(x) + d(r)\alpha\ \sigma(s)\gamma\sigma(y' + y)\beta\ \sigma(x) + d(r)\alpha\ \sigma(x)'\beta\sigma(s)\gamma\sigma(y) + d(r)\alpha\ \sigma(x)'\beta\sigma(s' + s)\gamma\sigma(y)$
$+ (\tau(x)\ \beta\tau(r) + \tau(r)'\beta\tau(x))\alpha d(s)\gamma\sigma(y) = 0$

By adding:

$d(r)\alpha\ \sigma(s)\beta\sigma(x)'\gamma\sigma(y) + d(r)\alpha\ \sigma(s)\beta\ \sigma(x)\gamma\sigma(y)$ to the above equation, on the two sides, we get:

$d(r)\alpha\ \sigma(s)\beta\ \sigma(y)\gamma\sigma(x) + d(r)\alpha\ \sigma(s)\beta\sigma(x)'\gamma\sigma(y) + d(r)\alpha\ \sigma(s)\gamma\sigma(y' + y)\beta\ \sigma(x) + d(r)\alpha\ \sigma(s)\gamma\sigma(y)\ \beta\ \sigma(x) + d(r)\alpha\ \sigma(x)'\beta\sigma(s)\gamma\sigma(y) + d(r)\alpha\ \sigma(x)'\beta\sigma(s' + s)\gamma\sigma(y) + (\tau(x)\ \beta\tau(r) + \tau(r)'\beta\tau(x))\alpha d(s)\gamma\sigma(y) = d(r)\alpha\ \sigma(s)\beta\sigma(x)'\gamma\sigma(y) + d(r)\alpha\ \sigma(s)\beta\ \sigma(x)\gamma\sigma(y)$

Since M is an additively inverse Γ- semiring, we get:

$d(r)\alpha\ \sigma(s)\beta\ (\sigma(y)\gamma\sigma(x) + \sigma(x)'\gamma\sigma(y)) + d(r)\alpha\ (\sigma(s)\ \beta\ \sigma(x) + \sigma(x)'\beta\sigma(s))\gamma\sigma(y) + d(r)\alpha\ \sigma(x)'\beta\sigma(s' + s)\gamma\sigma(y) + d(r)\alpha\ \sigma(s)\gamma\sigma(y' + y)\beta\ \sigma(x) + (\tau(x)\ \beta\tau(r) + \tau(r)'\beta\tau(x))\alpha d(s)\gamma\sigma(y) = d(r)\alpha\ \sigma(s)\beta\sigma(x)'\gamma\sigma(y) + d(r)\alpha\ \sigma(s)\beta\ \sigma(x)\gamma\sigma(y)$ for all $x, y \in J$ and $r, s \in M$

By using equation (6), we get

$$d(r)\alpha\ \sigma(s)\beta\ \sigma(y)\gamma\sigma(x) + d(r)\alpha\ \sigma(s)\beta\sigma(x)'\gamma\sigma(y) + d(r)\alpha\sigma(s)\beta\ \sigma(x)\gamma\sigma(y) + d(r)\alpha\sigma(s)\beta\sigma(x)'\gamma\sigma(y) + d(r)\alpha\sigma(s)\beta\ \sigma(x)\gamma\sigma(y) + d(r)\alpha\sigma(s)\beta\sigma(x)'\ \gamma\sigma(y) = d(r)\alpha\sigma(s)\beta\sigma(x)'\gamma\sigma(y) + d(r)\alpha\ \sigma(s)\beta\ \sigma(x)\gamma\sigma(y)$$

Since M is an inverse Γ- semiring, and using the cancellative low, we get:

$d(r)\alpha\ \sigma(s)\beta\ (\sigma(y)\gamma\sigma(x) + \sigma(x)'\gamma\sigma(y)) = 0$ for all $x, y \in J$ and $r, s \in M$.
And hence,
$\sigma^{-1}(d(r))\Gamma M\Gamma(y\gamma x + x'\gamma y) = 0$ for all $x, y \in J, \gamma, \alpha \in \Gamma$ and $r \in M$

Since M is a prime inverse Γ- semiring and σ is an automorphism on M, then we have

either $d(r) = 0$ or $y\gamma x + x'\gamma y = 0$ for all $x, y \in J$ and $r \in M$. If $y\gamma x + x'\gamma y = 0$ for all $x, y \in J$, then it follows that J is commutative.

By using (Lemma 2.12), we get: $J \subseteq Z(M)$.

Lemma 2.14

Let M be a prime inverse Γ- semiring. Suppose that σ and τ are two automorphisms of M and d: M→M is (σ, τ)-derivation such that for all x ∈ Mwe have $a\alpha$ d (x) =0 or d (x) α a= 0, where a ∈ M, then either a = 0 or d = 0.

Proof

Let aα d (x) = 0.

Since d is (σ, τ)- derivation of M, then:

d (xβy) = $d(x)\beta\sigma(y) + \tau(x)\beta d(y)$ for all x, y ∈ M.

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Then $a\alpha\ (d(x)\beta\sigma(y) + \tau(x)\beta d(y)) = 0$
$a\alpha d(x)\beta\sigma(y) + a\alpha\tau(x)\beta d(y) = 0$
$a\alpha\tau(x)\beta d(y) = 0.$

And hence,

$\tau^{-1}\ (a\Gamma M\Gamma d(y)) = 0$ for all , $y \in M, \beta, \alpha \in \Gamma$.

Since M is a prime inverse Γ- semiring and τ is an automorphism on M, then we have :
either a = 0 or d (y) = 0 for all y ∈ M.
i.e. either a = 0 or d = 0.
Similarly for d (x) α a = 0.

2.Results and Discussions

Theorem 3.1.1

Let I be a non- zero left ideal on M, which is a semiprime as an inverse Γ- semiring. If $d(I)\alpha u = 0$, $(u\alpha d(I) = 0)$, then $u = 0$, for all $u \in M$.

Proof

$d(x)\alpha u = 0$, for all $x \in I, u \in M$. (7)

By replacing x by $s\beta x$, $s \in M$, $\alpha \in \Gamma$ in equation (7), we have:

$$0 = d(s\beta x)\alpha u = (d(s)\beta\sigma(x) + \tau(s)\beta d(x))\alpha u$$
$$= d(s)\beta\sigma(x)\alpha u + \tau(s)\beta d(x)\alpha u$$

By using equation (7), we get:

$d(s)\beta\sigma(x)\alpha u = 0$, for all $x \in I, u, s \in M$ (8)
By replacing x by $r\delta x$, $r \in M$ in equation (8), we get
$0 = d(s)\beta\sigma(r\delta x)\alpha u$
$= d(s)\beta\sigma(r)\delta\sigma(x)\alpha u$

Therefore

$d(s)\Gamma M\Gamma\sigma(x)\alpha u = 0$ for all $x \in I, u, s \in M$.
Since M is a prime inverse Γ- semiring, we get :
either $d(s) = 0$ or $\sigma(x)\alpha u = 0$ for all $x \in I, u, s \in M$.

Since d is a nonzero (σ, r)- derivation on M, then we have: $\sigma(x)\alpha u = 0$ for all $x \in I$. Therefore, I $\Gamma\sigma^{-1}(u) = 0$,for all $u \in M$.

By (Lemma 2.7) and since σ is an automorphism on$M$, we get: $u = 0$.If
$u\alpha d(I) = 0$ (9)

$$0 = u\alpha d(x\beta y) = u\alpha(d(x)\beta\sigma(y) + r(x)\beta d(y))$$
$$= u\alpha d(x)\beta\sigma(y) + u\alpha r(x)\beta d(y)$$

By using equation (9), we get: $u\alpha r(x)\beta d(y) = 0$, for all $x, y \in I, u \in M$.
Therefore, $r^{-1}(u)\Gamma I\Gamma r^{-1}(d(y)) = 0$, for all $y \in I, u \in M$.
Since I is a left ideal of M, we get $r^{-1}(u)\Gamma SI\ \Gamma r^{-1}(d(y)) = 0$, for all $y \in I, u \in M$.Since M is a prime inverse Γ- semiring, we get :
either $r^{-1}(u) = 0$ or $\Gamma I\Gamma r^{-1}(d(y)) = 0$, for all $y \in I, u \in M$.

If $r^{-1}(u) = 0$, for all $u \in M$, then since r is an automorphism on M, then we have: $u = 0$.

If $\Gamma I\Gamma r^{-1}(d(y)) = 0$, for all $y \in I$, then by (Lemma 2.1.3) and since r is an automorphism on M, we get: $d(I) = 0$.

And then by (Lemma2.8), we have: $d = 0$ *on* M.

But this is a contradiction, since d *is a nonzero* (σ, r)*- derivation on* M*. This yields that* $u = 0$.

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Theorem 3.1.2

Let M be a Cancellative prime inverse Γ- semiring, I be a non- zero left Ideal on M which is semiprime as an inverse Γ- semiring and let $d: M \rightarrow M$ is (σ, r) −derivation on M whereσ, r are two automorphism on M.. If $d$acts as a homomorphism on I, then $d = 0$ on M.

Proof

Since d *acts as a homomorphism on* I, *then we have:*

$d(x\alpha y) = d(x)\alpha d(y)$ *for all* $x, y \in I.$ *(10)*
And since d *is a* (σ, τ)- *derivation on* M, *we get:*
$d(x\alpha y) = d(x)\alpha\sigma(y) + \tau(x)\alpha d(y)$ *for all* $x, y \in I.$ *(11)*
By replacing y *by* $y\beta z$, $z \in I$ *in equation (10), we get:*
$d(x\alpha\ y\beta z) = d(x\alpha\ y)\beta d(z)$

$= (d(x)\alpha\sigma(y) + \tau(x)\alpha d(y)\ )\beta d(z)$

$= d(x)\alpha\sigma(y)\beta d(z) + \tau(x)\alpha d(y)\ \beta d(z)$ *for all* $x, y, z \in I.$ *(12)*
Again, by replacing y *by* $y\beta z$, $z \in I$ *in equation (11), we get:*
$d(x\alpha(y\beta z)) = d(x)\alpha\sigma(y\beta z) + \tau(x)\alpha d(y\beta z)$

$= d(x)\alpha\sigma(y)\beta\sigma(z) + \tau(x)\alpha d(y)\beta d(z)$ *for all* $x, y, z \in I.$ *(13)*
By the equivalence between equations (12) and (13), we get
$d(x)\alpha\sigma(y)\beta d(z) = d(x)\alpha\sigma(y)\beta\sigma(z)$ *for all* $x, y, z \in I$.*By (Lemma 2.6), we get:*
$d(x)\alpha\sigma(y)\beta d(z) + d(x)\alpha\sigma(y)\beta\sigma(z)' = 0$ *for all* $x, y, z \in I.$

Therefore:
$d(x)\alpha\sigma(y)\beta(d(z) + \sigma(z)') = 0$ *for all* $x, y, z \in I$.*And hence:*
$\sigma^{-1}(d(x))\Gamma I\Gamma\sigma^{-1}(d(z) + \sigma(z)') = 0,$ *for all* $x, z \in I.$
Since I *is left ideal on* M, *we get:*
$\sigma^{-1}(d(x))\Gamma MI\Gamma\sigma^{-1}(d(z) + \sigma(z)') = 0,$ *for all* $x, z \in I$

And since M *is a prime inverse* Γ- *semiring, we get :*
either $\sigma^{-1}(d(x)) = 0$ *or* $I\Gamma\sigma^{-1}(d(z) + \sigma(z)') = 0,$ *for all* $x, z \in I$.*If* $\sigma^{-1}(d(x)) = 0,$ *for all* $x \in I$. *Since* σ *is an automorphism on* M, *then by (Lemma 2.8), we get:* $d = 0$ *on* M.
If $I\Gamma\sigma^{-1}(d(z) + \sigma(z)') = 0,$ *for all* $z \in I$, *then by (Lemma 2.1.3) ,we obtain* $\sigma^{-1}(d(z) + \sigma(z)') = 0,$ *for all* $z \in I.$
Since σ *is an automorphism on* M, *we have:* $d(z) + \sigma(z)' = 0,$ *for all* $z \in I$.*By (Lemma 2.6), we get*
$d(z) = \sigma(z)$ *,for all* $z \in I$ *(14)*

By replacing z *by* $z\delta y$ *in equation (14), we get:*

$d(z\delta y) = \sigma(z\delta y)$

$d(z)\delta\sigma(y) + \tau(z)\delta d(y) = \sigma(z)\delta\sigma(y),$ *for all* $y, z \in I.$

By using equation (14) with the Cancellative low, we get: $\tau(z)\delta d(y) = 0,$ *for all* $y, z \in I$. *Therefore,* $\tau^{-1}((d(y)) = 0,$ *for all* $y \in I$. *Since* τ *is an automorphism on* M *and by (Lemma 2.7), we get:* $d(y) = 0,$ *for all* $y \in I.$

By (Lemma 2.8), we get: $d = 0$ *on* M.

Theorem 3.1.3

Let M be a Cancellative prime inverse Γ- semiring , I be a nonzero left ideal of M which is a semiprime as inverse

Γ- semiring and d : M $\rightarrow M$ is a (σ, τ)- derivation on M, where σ, τ are automorphisms on M. If d acts as an antihomomorphism on I, then $d = 0$ on M.

Proof

Since d acts as an anti- homomorphism on I, then we have:

$d(x\alpha y) = d(x)\alpha d(y)$ for all $x, y \in I$ (15)

And since d is a (σ, τ)- derivation on M, we get:

$d(x\alpha y) = d(x)\alpha\sigma(y) + \tau(x)\alpha d(y)$, for all $x, y \in I$. (16)

By replacing y by $x\beta y$ in equation (15), we get:

$d(x\alpha\ (x\beta y)) = d(x\beta y)\alpha d(x)$

$= (d(x)\beta\sigma(y) + \tau(x)\beta d(y)\ )\alpha d(x)$

$= d(x)\beta\sigma(y)\alpha d(x) + \tau(x)\beta d(y)\ \alpha d(x)$ for all $x, y, z \in I$. (17)

And by replacing y by $x\beta y$ in equation (16), we get:

$d(x\alpha\ (x\beta y)) = d(x) \quad \alpha\sigma(x\beta y) + \tau(x)\alpha d(x\beta y)$

$= d(x)\alpha\sigma(x)\beta\sigma(y) + \tau(x)\alpha d(x)\beta d(y)$, for all $x, y \in I$ (18)

By the equivalence of equations (17) and (18), we get

$d(x)\beta\sigma(y)\alpha d(x) = d(x)\alpha\sigma(x)\beta\sigma(y)$, for all $x, y \in I$ (19)

By replacing y by $y\delta z$, $z \in I$ in equation (19), we get:

$d(x)\beta\sigma(\ y)\delta\sigma(z)\alpha d(x) = d(x)\alpha\sigma(x)\beta\sigma(\ y)\delta\sigma(z)$, for all $x, y, z \in I$.

By using equation (19), we get:

$d(x)\beta\sigma(\ y)\delta\sigma(z)\alpha d(x) = \mathrm{d}(x)\beta\sigma(y)\alpha d(x)\delta\sigma(z)$, for all $x, y, z \in I$.

By using (Lemma 2.6), we get:

$d(x)\beta\sigma(\ y)\delta\sigma(z)\alpha d(x) + \mathrm{d}(x)\beta\sigma(y)\delta d(x)'\alpha\sigma(z) = 0$, for all $x, y, z \in I$.

$d(x)\beta\sigma(\ y)\delta(\sigma(z)\alpha d(x) + d(x)'\alpha\sigma(z)) = 0$, for all $x, y, z \in I$. $\sigma^{-1}(d(x))\Gamma I\Gamma\sigma^{-1}(\sigma(z)\alpha d(x) + d(x)'\alpha\sigma(z)) = 0$, for all $x, z \in I$

Since I is left ideal on M and M is a prime inverse Γ- semiring, we get :

either $d(x) = 0$ or $I(\sigma(z)\alpha d(x) + d(x)'\alpha\sigma(z)) = 0$, for all $x, z \in I$.If $d(x) = 0$, for all $x \in I$, it gives $d(I) = 0$.

By Lemma (2.8), we get: $d = 0$ on M.

If $\sigma(z)\alpha d(x) + d(x)'\alpha\sigma(z) = 0$, for all $x, z \in I$ (20)
By Lemma (2.5), we get: $\sigma(z)\alpha d(x) = d(x)'\alpha\sigma(z)$for all $x, z \in I$.
By replacing z by $r\delta z, r \in M$ in equation (20), we have:

$$\begin{aligned} 0 &= \sigma(r\delta z)\alpha d(x) + d(x)'\alpha\sigma(r\delta z) \\ &= \sigma(r)\delta\sigma(z)\alpha d(x) + d(x)'\alpha\sigma(r)\delta\sigma(z) \\ &= \sigma(r)\alpha d(x)\delta\sigma(z) + d(x)'\alpha\sigma(r)\delta\sigma(z) \\ &= (\sigma(r)\alpha d(x) + d(x)'\alpha\sigma(r))\delta\sigma(z) \end{aligned}$$

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$= (\sigma(r)\alpha d(x) + d(x)'\alpha\sigma(r))\delta\sigma(I)$, for all $x \in I, r \in M$.

Therefore

$\sigma^{-1}[(\sigma(r)\alpha d(x) + d(x)'\alpha\sigma(r))]\Gamma I = 0$, for all $x \in I, \delta \in \Gamma, r \in M$.

Since I is a nonzero left ideal on M and M is a prime inverse Γ- semiring, we get which forces d to be a homomorphism of M.

It follows that d = 0 on M, by (Theorem 3.1.2).

Theorem 3.1.4

Let M be 2-tortion free, J is a nonzero Jordan ideal and a sub inverse Γ- semiring of M, and $d: M \to M$ is a (σ, τ)-derivation on M, where σ, τare automorphisms on M. If d acts as a homomorphism on J, then either $d = 0$ or $J \subseteq Z(M)$.

Proof

Assume that $J \nsubseteq Z(M)$.

If d acts as a homomorphism on J, then we have:

$d(x\alpha y) = d(x)\alpha d(y)$ for all $x, y \in J$. (21)

And since d is a (σ, τ)- derivation on M, we get:

$d(x\alpha y) = d(x)\alpha\sigma(y) + \tau(x)\alpha d(y)$ for all $x, y \in J$. (22)
By the equivalence of equations (21), (22), we get:
$d(x)\alpha d(y) = d(x)\alpha\sigma(y) + \tau(x)\alpha d(y)$for all $x, y \in J$ (23)
Now, by replacing y by $y\beta b, b \in J, \beta \in \Gamma$ in equation (23), we get:
$d(x)\alpha d(y\beta b) = d(x)\alpha\sigma(y\beta b) + \tau(x)\alpha d(y\beta b)$

$d(x)\alpha d(y)\beta\sigma(b) + d(x)\alpha\tau(y)\beta d(b) = d(x)\alpha\sigma(y)\beta\sigma(b) + \tau(x)\alpha d(y)\beta\sigma(b) + \tau(x)\alpha\tau(y)\beta d(b)$

$= (d(x)\alpha\sigma(y) + d(x)\alpha\tau(x)\alpha d(y))\beta\sigma(b) + \tau(x)\alpha\tau(y)\beta d(b)$

By using equation (23) and the cancellative low, we get:

$(d(x)\alpha\sigma(y) + \tau(x)\alpha d(y))\beta\sigma(b) + d(x)\alpha\tau(y)\beta d(b) = (d(x)\alpha\sigma(y) + \tau(x)\alpha d(y))\beta\sigma(b) + \tau(x)\alpha\tau(y)\beta d(b)$

$d(x)\alpha\tau(y)\beta d(b) = \tau(x)\alpha\tau(y)\beta d(b)$for all $x, y, b \in J$.

By(Lemma 2.6), we get:

$0 = d(x)\alpha\tau(y)\beta d(b) + \tau(x)'\alpha\tau(y)\beta d(b)$

$= (d(x)\alpha\tau(y) + \tau(x)'\alpha\tau(y))\beta d(b)$ for all $x, y, b \in J$

$= (d(x) + \tau(x)')\alpha\tau(y)\beta d(b)$ for all $x, y, b \in J$

And hence, $\tau^{-1}(d(x) + \tau(x)')\, \Gamma J\, \Gamma\tau^{-1}(d(b)) = 0$ for all $x, b \in J$.By (Lemma 2.11), we get:

either $d(b) = 0$ or $d(x) + \tau(x)' = 0$ for all $x, b \in J$.

If $d(b) = 0$ for all $b \in J$, then by (Proposition 2.13), we get:

$d = 0$ on M.

If $d(x) + \tau(x)' = 0$ for all $x \in J$, then by Lemma(2.5), we get:

$d(x) = \tau(x)$ for all $x \in J$ (24)

Using equation (23) in equation (24), we obtain:

$d(x)\alpha\sigma(y) = 0$ for all $x, y \in J$. (25)

Now, by replacing y by $y\delta b$, we get:

$d(x)\alpha\sigma(y\delta b) = 0$ for all $x, y, b \in J$.

$d(x)\alpha\sigma(y)\delta\sigma(b) = 0$

That is, $\sigma^{-1}(d(x))\Gamma J\Gamma b = 0$ for all $x, b \in J$. By (Lemma2.11), we get:

either $d(x) = 0$ or $b = 0$ for all $x, b \in J$.

But J is a nonzero Jordan ideal on M, hence we get: $d(x) = 0$ for all $x \in J$.
By (Proposition 2.13) , we get: $d = 0$ on M.

3. Discussions

Let M be $a$prime inverse Γ-semiring and d a derivation of M. If $d(x\alpha y) = d(x)\alpha d(y)$ (resp., $(x\alpha y) = d(y)\alpha d(x)$) holds, for all $x, y \in R$and $\alpha \in \Gamma$ then we say that d acts as a homomorphism (resp., anti-homomorphism) on M.

Bell and Kappe [Bell, H. E. and Kappe, L. C.(1989)] proved that if d is a derivation of a prime ring M which acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal I of M, then $d = 0$ on M.

Further, this result was extended by M. Ashraf for (σ, r)-derivation in [Ashraf, M., Rehman, N. and Quadri.(1999).] as follows:

Let M be a prime ring, I is a nonzero right ideal of R. Suppose that σ, r are automorphisms of M and $d: M \longrightarrow M$ is a (σ, r)-derivation of M.

(i) If d acts as a homomorphism on I, then $d = 0$ on M.

(ii) If d acts as an anti homomorphism on I, then $d = 0$ on M.

Recently Khatam AD. Zaghir, Abdulrahman Hameed Majeed extended the result for (σ, r)-derivation in prime and semiprime inverse semiring .

The purpose of this section is to extend the above study to a(σ, r)-derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero left ideal of M which is a semiprime as inverse Γ- semiring and d : M → M is a (σ, τ)- derivation on M, where σ, τ are automorphisms on M.

4. Conclusion

Γ-semiring is one of the most important and modern branches of algebra in mathematics now a days. In this paper we tried to characterize some special properties of prime inverse Γ-semiring on a nonzero Jordan ideal anda nonzero ideal with (σ, τ)- derivation M that acts as a homomorphism or as an anti- homomorphism, where σ, τ are automorphisms on M which may help future researchers to proceed further. In this paper, we expanded some findings on (σ, τ)- derivation on inverse Γ-semiring that acts as a homomorphism or as an anti- homomorphism, where σ, τ are automorphisms on M. suggestions for future research to improve on these limitations. He studied these results of σ-Prime inverse gamma semiring.

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