New Notion for Neutrosophic Soft Normed Linear Spaces

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Abstract

An idea about neutrosophic soft normed space with linear tends to every soft points set throughout the field with scalar $\mathfrak{K}$ that can be examined using a different approach in this study. Additionally, the terms Cauchy and Convergence are defined. A few theorems are proved that is related to these ideas.

Keywords: Neutrosophic set; Neutrosophic soft normed spaces;Soft set; Soft normed spaces.

1. Introduction

In 1999, soft set theory was first developed by Molodostov [10]. In 2002, Maji [9] studied a new idea called fuzzy soft set. The sets with crisp soft have been further developed to the sets with fuzzy soft by those exact authors. In 2012, saw the introduction of the linear spaces with soft by Das, Majumdar along withSamanta [3] and also the linear spaces with soft norms.

Zadeh [16] first established the idea called Fuzzy Set (FS). Atanassov [1] first developed the term and concept that is an intuitionistic *FS* on 1986. In 1984, Katsaras [7], developed that the concept on normed space with fuzzy. Fuzzy Banach space research was done in 2005 bySaadati along with Vaezapour [12]. The idea was first presented in 2006 by Saadati along with Park as the Intuitionistic Fuzzy Normed Linear Space (IFNLS).

Fuzzy soft was first used in 2013 by Zahedi, Kilicman and RazakSalleh [8] a link involving norm with fuzzy soft as well as norm with fuzzy throughout the set was developed. The idea of fuzzy soft with convergence and Cauchy were first suggested by Beaula and Merlin [2] in 2015 within fuzzy soft normed linear space.

Smarandache [13] defines a subclass of the crisp set called the Neutrosophic Set (NS). Neutrosophy is a theoretical framework that made its way into print in 1998. Parameterizing the universal set, we get the soft set, a collection of subsets. Any collection of phrases, natural integers, etc., may be used as the parameter set. Therefore, soft sets theory has appealing uses in a wide variety of contexts. Additionally, in neutrosophic soft metric spaces, we present various topological structures of this

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newly discovered space, which gives the open ball with soft and closed ball with soft. Neutrosophic soft normed linear space has been studied for its many topological and structural characteristics.

2. Preliminaries

Definition (2.1): Let Ξ be a set that is non-empty as well as I is a closed interval of real numbers with the value $I = [0, 1]$. A function from Ξ into $I = [0, 1]$ is a $FS\dot{\mathfrak{I}}$ in Ξ (or a subset with fuzzy from Ξ). "If $\dot{\mathfrak{I}}$ is a FS in Ξ after this $\dot{\mathfrak{I}}$ can be defined as characteristic function which connects all $\breve{\mathfrak{v}} \in \Xi$ to real number $\dot{\mathfrak{I}}(\breve{\mathfrak{v}})$ within the interval I. The grade of membership towards $\breve{\mathfrak{v}}$ in $\dot{\mathfrak{I}}$ is represented by the function $\dot{\mathfrak{I}}(\breve{\mathfrak{v}})$. $\dot{\mathfrak{I}}$ can be described completely as":

$$\dot{\mathfrak{I}} = \left\{\left(\breve{\mathfrak{v}}, \dot{\mathfrak{I}}(\breve{\mathfrak{v}})\right) : \breve{\mathfrak{v}} \in \Xi, 0 \le \dot{\mathfrak{I}}(\breve{\mathfrak{v}}) \le 1\right\} \text{or} \dot{\mathfrak{I}} = \left\{\frac{\dot{\mathfrak{I}}(\breve{\mathfrak{v}})}{\breve{\mathfrak{v}}} : \breve{\mathfrak{v}} \in \Xi\right\}$$

Here the function of membership for the $FS\dot{\mathfrak{I}}$ is referred as $\dot{\mathfrak{I}}(\breve{\mathfrak{v}})$. I^{Ξ} denoted the all FSs family in Ξ.

Definition (2.2): "Let Ξ be a set that is non-empty. A Neutrosophic Set (Shortly, NS) an object $\mathfrak{A}$ to have the form: $\mathfrak{A} = \{\langle \breve{\mathfrak{v}}, \dot{\mathfrak{I}}_{\mathfrak{A}}(\breve{\mathfrak{v}}), \ddot{\mathfrak{S}}_{\mathfrak{A}}(\breve{\mathfrak{v}}), \dddot{\mathfrak{M}}_{\mathfrak{A}}(\breve{\mathfrak{v}}) \rangle, \breve{\mathfrak{v}} \in \Xi\}$, here the functions $\dot{\mathfrak{I}}_{\mathfrak{A}}: \Xi \to I, \ddot{\mathfrak{S}}_{\mathfrak{A}}: \Xi \to I$ and $\dddot{\mathfrak{M}}_{\mathfrak{A}}: \Xi \to I$ indicates the membership degree along with the non-membership degree of each $\breve{\mathfrak{v}} \in \Xi$ element towards the set $\mathfrak{A}$ (*respectively*) along with $0 \le \dot{\mathfrak{I}}_{\mathfrak{A}}(\breve{\mathfrak{v}}) + \ddot{\mathfrak{S}}_{\mathfrak{A}}(\breve{\mathfrak{v}}) + \dddot{\mathfrak{M}}_{\mathfrak{A}}(\breve{\mathfrak{v}}) \le 1$ for all $\breve{\mathfrak{v}} \in \Xi$. Consider the family of all NS denoted by N (Ξ). Furthermore, we call: $\pi_{\mathfrak{A}}(\breve{\mathfrak{v}}) = 1 - \dot{\mathfrak{I}}_{\mathfrak{A}}(\breve{\mathfrak{v}}) - \ddot{\mathfrak{S}}_{\mathfrak{A}}(\breve{\mathfrak{v}}) - \dddot{\mathfrak{M}}_{\mathfrak{A}}(\breve{\mathfrak{v}}), \breve{\mathfrak{v}} \in \Xi$, the Neutrosophic index or $\breve{\mathfrak{v}}$ in $\mathfrak{A}$ for degree of hesitancy. It is clear that $0 \le \pi_{\mathfrak{A}}(\breve{\mathfrak{v}}) \le 1$ for all $\breve{\mathfrak{v}} \in \Xi$."

Definition (2.3): The 7-tuple $\left(\Xi, \dot{\mathfrak{I}}, \ddot{\mathfrak{S}}, \dddot{\mathfrak{M}}, \circledast, \Theta, \Delta\right)$ is said to be a Neutrosophic Normed Linear Space (Shortly called, NNLS) when Ξ be a space with linear throughout a field $\mathfrak{F}$, $\circledast$ belongs to t-norm which is continuous, Θ as well as Δ is a t-conorm which is continuous and $\dot{\mathfrak{I}}, \ddot{\mathfrak{S}}, \dddot{\mathfrak{M}}$ belong FSs in $\Xi \times (0, \infty)$ $\left(i.e. \dot{\mathfrak{I}}, \ddot{\mathfrak{S}}, \dddot{\mathfrak{M}} : \Xi \times (0, \infty) \to [0, 1]\right)$ satisfying the criteria as follow: for every $\breve{\mathfrak{v}}, \breve{\mathfrak{w}} \in \Xi$ and $\vartheta, \varsigma > 0$,"

(*NN*. 1) $\dot{\mathfrak{I}}(\breve{\mathfrak{v}}, \vartheta) + \ddot{\mathfrak{S}}(\breve{\mathfrak{v}}, \vartheta) + \dddot{\mathfrak{M}}(\breve{\mathfrak{v}}, \vartheta) \le 3$,

(*NN*. 2) $0 \le \dot{\mathfrak{I}}(\breve{\mathfrak{v}}, \vartheta) \le 1; 0 \le \ddot{\mathfrak{S}}(\breve{\mathfrak{v}}, \vartheta) \le 1 \ and \ 0 \le \dddot{\mathfrak{M}}(\breve{\mathfrak{v}}, \vartheta) \le 1$,

(*NN*. 3) $\dot{\mathfrak{I}}(\breve{\mathfrak{v}}, \vartheta) > 0$,

(*NN*. 4) $\dot{\mathfrak{I}}(\breve{\mathfrak{v}}, \vartheta) = 1 \Leftrightarrow \breve{\mathfrak{v}} = 0$,

(*NN*. 5) $\dot{\mathfrak{I}}(\grave{\gamma}\breve{\mathfrak{v}}, \vartheta) = \dot{\mathfrak{I}}\left(\breve{\mathfrak{v}}, \frac{\vartheta}{|\grave{\gamma}|}\right)$ for each $\mathfrak{F}|\{0\}$,

(*NN*. 6) $\dot{\mathfrak{I}}(\breve{\mathfrak{v}} + \breve{\mathfrak{w}}, \vartheta + \varsigma) \ge \dot{\mathfrak{I}}(\breve{\mathfrak{v}}, \vartheta) \circledast \dot{\mathfrak{I}}(\breve{\mathfrak{w}}, \varsigma)$,

(*NN*. 7) $\dot{\mathfrak{I}}(\breve{\mathfrak{v}}, .): (0, \infty) \to [0,1]$ is a continuous function;

(*NN*. 8) $\lim_{\vartheta \to \infty} \dot{\mathfrak{I}}(\breve{\mathfrak{v}}, \vartheta) = 1$ as well as $\lim_{\vartheta \to 0} \dot{\mathfrak{I}}(\breve{\mathfrak{v}}, \vartheta) = 0$;

(*NN*. 9) $\ddot{\mathfrak{S}}(\breve{\mathfrak{v}}, \vartheta) < 1$;

(*NN*. 10) $\ddot{\mathfrak{S}}(\breve{\mathfrak{v}}, \vartheta) = 0 \Leftrightarrow \breve{\mathfrak{v}} = 0$,

(*NN*. 11) $\ddot{\mathfrak{S}}(\grave{\gamma}\breve{\mathfrak{v}}, \vartheta) = \ddot{\mathfrak{S}}\left(\breve{\mathfrak{v}}, \frac{\vartheta}{|\grave{\gamma}|}\right)$, for each $\mathfrak{F}|\{0\}$,

(*NN*. 12) $\ddot{\mathfrak{S}}(\breve{\mathfrak{v}} + \breve{\mathfrak{w}}, \vartheta + \varsigma) \le \ddot{\mathfrak{S}}(\breve{\mathfrak{v}}, \vartheta) \lozenge \ddot{\mathfrak{S}}(\breve{\mathfrak{w}}, \varsigma)$;

(*NN*. 13) $\ddot{\mathfrak{S}}(\breve{\mathfrak{v}}, .): (0, \infty) \to [0,1]$ is a continuous function;

(*NN*. 14) $\lim_{\vartheta \to \infty} \ddot{\mathfrak{S}}(\breve{\mathfrak{v}}, \vartheta) = 0$ as well as $\lim_{\vartheta \to 0} \ddot{\mathfrak{S}}(\breve{\mathfrak{v}}, \vartheta) = 1$;

(*NN*. 15) $\dddot{\mathfrak{M}}(\breve{\mathfrak{v}}, \vartheta) < 1$;

(*NN*. 16) $\dddot{\mathfrak{M}}(\breve{\mathfrak{v}}, \vartheta) = 0 \Leftrightarrow \breve{\mathfrak{v}} = 0$,

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(NN. 17) $\ddot{\mathfrak{M}}(\gamma\breve{\mathfrak{v}},\vartheta) = \ddot{\mathfrak{M}}\left(\breve{\mathfrak{v}},\frac{\vartheta}{|\gamma|}\right)$for each $\mathfrak{F}|\{0\}$,

(NN. 18) $\ddot{\mathfrak{M}}(\breve{\mathfrak{v}} + \mathfrak{w}, \vartheta + \varsigma) \leq \ddot{\mathfrak{M}}(\breve{\mathfrak{v}},\vartheta) \odot \ddot{\mathfrak{M}}(\mathfrak{w},\varsigma)$,

(NN. 19) $\ddot{\mathfrak{M}}(\breve{\mathfrak{v}},.) : (0,\infty) \to [0,1]$continuous and

(NN. 20) $\lim_{\vartheta\to\infty} \ddot{\mathfrak{M}}(\breve{\mathfrak{v}},\vartheta) = 0$ and $\lim_{\vartheta\to 0} \ddot{\mathfrak{M}}(\breve{\mathfrak{v}},\vartheta) = 1$;

Definition (2.4):Let Ξ be a universe along with $\mathfrak{E}$ is a parameters set. Consider $\mathfrak{P}(\Xi)$ indicate theΞ with power set. A $(\mathfrak{F},\mathfrak{E})$ pair which can be known as the set with soft throughoutΞ, in which $\mathfrak{F}$ provides a mapping given by $\mathfrak{F} : \mathfrak{E} \to \mathfrak{P}(\Xi)$. Alternatively, a soft set throughout Ξ might be the family with parameterized belongs to a subsets of the universeΞ."

Definition (2.5):

(1) Let $(\mathfrak{F},\mathfrak{E})$ a soft set throughout Ξ is said to be a null set with soft indicated by $\emptyset$, when $\mathfrak{F}(e) = \emptyset$ for every $e \in \mathfrak{E}$.

(2) "Let $(\mathfrak{F},\mathfrak{E})$ a soft set throughout Ξ is said to be a set with an Absolute Soft (AS) indicated by$\tilde{\Xi}$, when $\mathfrak{F}(e) = \Xi$ for every $e \in \mathfrak{E}$."

Definition (2.6):Let $\mathfrak{R}$ be the real set along with $\mathfrak{B}(\mathfrak{R})$ be the every bounded subsets collection belongs to $\mathfrak{R}$ with non-empty as well as $\mathfrak{E}$ taken as parameters set. After that a mapping $\mathfrak{F}: \mathfrak{E} \to \mathfrak{B}(\mathfrak{R})$ can be referred to as a real set with soft. When the set of real number with soft is a singleton set with soft, it shall be known as a real number with soft as well as indicated by $\tilde{\varrho}, \tilde{\varsigma}, \tilde{\vartheta}$etc."

Remarks (2.7):

(1) $\tilde{0}$ as well as$\tilde{1}$are two real numbers with soft where $\tilde{0}(e) = 0, \tilde{1}(e) = 1, \forall e \in \mathfrak{E}$ (*respectively*).

(2) The set of real numbers with all soft indicated by $\mathfrak{R}(\mathfrak{E})$ as well as the set of every non-negative real numbers with soft represented by $\mathfrak{R}(\mathfrak{E})^*$.

Definition(2.8):Let $\tilde{\varrho}$and$\tilde{\varsigma}$are two soft real numbers, and thereafter consider the statements that follow:

1) $\tilde{\varrho} \,\tilde{\leq}\, \tilde{\varsigma}$if $\tilde{\varrho}(e) \leq \tilde{\varsigma}(e)$ for all $e \in \mathfrak{E}$;
2) $\tilde{\varrho} \,\tilde{\geq}\, \tilde{\varsigma}$if $\tilde{\varrho}(e) \geq \tilde{\varsigma}(e)$ for all $e \in \mathfrak{E}$;
3) $\tilde{\varrho} \,\tilde{<}\, \tilde{\varsigma}$if $\tilde{\varrho}(e) < \tilde{\varsigma}(e)$ for all $e \in \mathfrak{E}$;
4) $\tilde{\varrho} \,\tilde{>}\, \tilde{\varsigma}$if $\tilde{\varrho}(e) > \tilde{\varsigma}(e)$ for all $e \in \mathfrak{E}$;

hold.

Definition(2.9):Let Ξbe a vector space throughout a field $\mathfrak{K}$ as well as take $\mathfrak{E}$ be a parameter set. Taking $(\mathfrak{F},\mathfrak{E})$ be a soft set throughoutΞ. The set $(\mathfrak{F},\mathfrak{E})$ with soft is known to be a soft vector, also it is indicated by $\tilde{\breve{\mathfrak{v}}}_e$when $e \in \mathfrak{E}$ that is only one, which gives $\mathfrak{F}(e) = \{\breve{\mathfrak{v}}\}$ for certain $\breve{\mathfrak{v}} \in \Xi$ along with $\mathfrak{F}(e') = \emptyset, \forall e' \in \frac{\mathfrak{E}}{\{e\}}$. Every soft vectors set throughout $\tilde{\Xi}$shall be indicated by $SV(\tilde{\Xi})$.

Definition(2.10):Let $\tilde{\breve{\mathfrak{v}}}_e, \tilde{\mathfrak{w}}_{e'}$be two soft vectorsare said to be equivalent when $e = e'$ as well as$\breve{\mathfrak{v}} = \mathfrak{w}$. Therefore $\tilde{\breve{\mathfrak{v}}}_e \neq \tilde{\mathfrak{w}}_{e'} \Leftrightarrow \breve{\mathfrak{v}} \neq \mathfrak{w}$ or $e \neq e'$

Proposition(2.11):The vector space $SV(\tilde{\Xi})$ can be characterized by the following operations:

1) $\tilde{\breve{\mathfrak{v}}}_e + \tilde{\mathfrak{w}}_{e'} = \widetilde{(\breve{\mathfrak{v}} + \mathfrak{w})}_{(e+e')}$for every $\tilde{\breve{\mathfrak{v}}}_e, \tilde{\mathfrak{w}}_{e'} \,\tilde{\in}\, SV(\tilde{\Xi})$.
2) $\tilde{\varrho}.\tilde{\breve{\mathfrak{v}}}_e = \widetilde{(\varrho\breve{\mathfrak{v}})}_{(\varrho e)}$for all $\tilde{\breve{\mathfrak{v}}}_e \,\tilde{\in}\, SV(\tilde{\Xi})$ along with for all real number$\tilde{\varrho}$ with soft set.

Definition(2.12):Let $SV(\tilde{\Xi})$ be a soft vector space. Thereafter a mapping $\|.\|: SV(\tilde{\Xi}) \to \mathfrak{R}^+(\mathfrak{E})$is said to be a norm on $SV(\tilde{\Xi})$ with soft, when $\|.\|$satisfying this criteria follows:

1) $\|\tilde{\breve{\mathfrak{v}}}_e\| \,\tilde{\geq}\, \tilde{0}$as well as $\|\tilde{\breve{\mathfrak{v}}}_e\| = \tilde{0} \Leftrightarrow \tilde{\breve{\mathfrak{v}}}_e = \tilde{\theta}_0$.
2) $\|\tilde{\varrho}.\tilde{\breve{\mathfrak{v}}}_e\| = |\tilde{\varrho}|\|\tilde{\breve{\mathfrak{v}}}_e\|$for all $\tilde{\breve{\mathfrak{v}}}_e \,\tilde{\in}\, SV(\tilde{\Xi})$ along with for all soft scalar $\tilde{\varrho}$.
3) $\|\tilde{\breve{\mathfrak{v}}}_e + \tilde{\mathfrak{w}}_{e'}\| \,\tilde{\leq}\, \|\tilde{\breve{\mathfrak{v}}}_e\| + \|\tilde{\mathfrak{w}}_{e'}\|$for all $\tilde{\breve{\mathfrak{v}}}_e, \tilde{\mathfrak{w}}_{e'} \,\tilde{\in}\, SV(\tilde{\Xi})$.

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The space $SV(\tilde{\Xi})$ be a soft vector with a soft norm $\|.\|$on $\tilde{\Xi}$is called a Soft Normed Linear Space (Shortly SNLS) can be indicated by$(\tilde{\Xi}, \|.\|)$.

3. Main Results

Definition (3.1):The 7-tuple $(\tilde{\Xi}, \Delta, \nabla, \npreceq, *, \diamond, \star)$ is said to be anNeutrosophic Soft Metric Space (Shortly, NSMS) when $\tilde{\Xi}$is a *AS* set with an arbitrary, $*, \diamond, \star$ is a t-norm which is continuous, $\Re(\mathfrak{E})^*$ is the every positive real numbers set with soft, $SSP(\tilde{\Xi})$ indicate the every soft points set on $\tilde{\Xi}$and $\Delta, \nabla, \npreceq$ is a *FS* in $SSP(\tilde{\Xi}) \times SSP(\tilde{\Xi}) \times \Re(\mathfrak{E})^*(i.e. \Delta, \nabla, \npreceq: SSP(\tilde{\Xi}) \times SSP(\tilde{\Xi}) \times \Re(\mathfrak{E})^* \to [0,1])$satisfying the criteria follows: for every $\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\mathfrak{x}}_{e_3} \tilde{\in} SSP(\tilde{\Xi}), \tilde{\vartheta}, \tilde{\varsigma} \tilde{>} \tilde{0}$,

(NSMS. 1) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) + \nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) + \npreceq (\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \leq 3$

(NSMS. 2)$0 \leq \Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \leq 1; 0 \leq \nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \leq 1 \ and \ 0 \leq \npreceq (\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \leq 1$,

(NSMS. 3) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) > 0$,

(NSMS. 4)$\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = 1 \Leftrightarrow \tilde{\mathfrak{v}}_{e_1} = \tilde{\mathfrak{w}}_{e_2}$,

(NSMS. 5) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \Delta(\tilde{\mathfrak{w}}_{e_2}, \tilde{\mathfrak{v}}_{e_1}, \tilde{\vartheta})$,

(NSMS. 6) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta} + \tilde{\varsigma}) \tilde{\geq} \Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{x}}_{e_3}, \tilde{\vartheta}) * \Delta(\tilde{\mathfrak{x}}_{e_3}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\varsigma})$,

(NSMS. 7) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \cdot): \Re(\mathfrak{E})^* \to [0,1]$ is continuous,

(NSMS. 8) $\nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) < 1$

(NSMS. 9)$\nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = 0 \Leftrightarrow \tilde{\mathfrak{v}}_{e_1} = \tilde{\mathfrak{w}}_{e_2}$,

(NSMS. 10) $\nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \nabla(\tilde{\mathfrak{w}}_{e_2}, \tilde{\mathfrak{v}}_{e_1}, \tilde{\vartheta})$,

(NSMS. 11) $\nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta} + \tilde{\varsigma}) \tilde{\leq} \nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{x}}_{e_3}, \tilde{\vartheta}) \diamond \nabla(\tilde{\mathfrak{x}}_{e_3}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\varsigma})$,

(NSMS. 12) $\nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \cdot): \Re(\mathfrak{E})^* \to [0,1]$ is continuous, and

(NSMS. 13) $\npreceq (\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) < 1$

(NSMS. 14)$\npreceq (\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = 0 \Leftrightarrow \tilde{\mathfrak{v}}_{e_1} = \tilde{\mathfrak{w}}_{e_2}$,

(NSMS. 15) $\npreceq (\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \npreceq (\tilde{\mathfrak{w}}_{e_2}, \tilde{\mathfrak{v}}_{e_1}, \tilde{\vartheta})$,

(NSMS. 16) $\npreceq (\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta} + \tilde{\varsigma}) \tilde{\leq} \npreceq (\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{x}}_{e_3}, \tilde{\vartheta}) \star \npreceq (\tilde{\mathfrak{x}}_{e_3}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\varsigma})$,

(NSMS. 17) $\npreceq (\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \cdot): \Re(\mathfrak{E})^* \to [0,1]$ is continuous,

Definition (3.2):The 7-tuple$(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ is said to be an Neutrosophic Soft Normed Linear Space (Shortly called, NSNLS) when $\tilde{\Xi}$be an *AS* linear space throughout the field $\mathfrak{K}$, $*, \diamond, \star$ is a t-norm which is continuous, $\Re(\mathfrak{E})^*$ indicates every positive real numbers set with soft, $SSP(\tilde{\Xi})$indicate the all soft points set on $\tilde{\Xi}$ and $\dot{\phi}, \dot{\varphi}, \dot{\psi}$ be a set with fuzzy in $\tilde{\Xi} \times \Re(\mathfrak{E})^*(i.e. \dot{\phi}, \dot{\varphi}, \dot{\psi} : SSP(\tilde{\Xi}) \times \Re(\mathfrak{E})^* \to [0,1])$satisfying the criteria follows: for every $\tilde{\mathfrak{v}}_e, \tilde{\mathfrak{w}}_{e'}, \tilde{\in} SSP(\tilde{\Xi}), \tilde{\vartheta}, \tilde{\varsigma} \tilde{>} \tilde{0}$and $\tilde{\hbar} \tilde{\in} \mathfrak{K}$,

(NSN. 1) $\dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) + \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) + \dot{\psi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leq 3$;

(*NSN*. 2)$0 \leq \dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leq 1; 0 \leq \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leq 1 \ and \ 0 \leq \dot{\psi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leq 1$;

(*NSN*. 3)$\dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) > 0$;

(*NSN*. 4)$\dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) = 1 \Leftrightarrow \tilde{\mathfrak{v}}_e = \tilde{\theta}_0$,

(*NSN*. 5) $\dot{\phi}\left(\tilde{\hbar}\tilde{\mathfrak{v}}_e, \tilde{\vartheta}\right) = \dot{\phi}\left(\tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{|\tilde{\hbar}|}\right)$and $\tilde{\hbar} \neq \tilde{0}$,

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(NSN. 6) $\dot{\phi}(\tilde{\breve{\mathfrak{v}}}_e + \tilde{\breve{\mathfrak{w}}}_{e'}, \tilde{\vartheta} + \tilde{\varsigma}) \gtrsim \dot{\phi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) * \dot{\phi}(\tilde{\breve{\mathfrak{w}}}_{e'}, \tilde{\varsigma})$,

(NSN. 7) $\dot{\phi}(\tilde{\breve{\mathfrak{v}}}_e, \cdot)$ be a continuous in $\mathfrak{R}(\mathfrak{E})^*$ with non-decreasing function as well as

$$\lim_{\tilde{\vartheta}\to\infty} \dot{\phi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = 1$$

(NSN. 8) $\dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) < 1$,

(NSN. 9) $\dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = 0 \Leftrightarrow \tilde{\breve{\mathfrak{v}}}_e = \tilde{\theta}_0$,

(NSN. 10) $\dot{\varphi}\left(\tilde{\breve{\hbar}}\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}\right) = \dot{\varphi}\left(\tilde{\breve{\mathfrak{v}}}_e, \frac{\tilde{\vartheta}}{|\tilde{\breve{\hbar}}|}\right)$ and $\tilde{\breve{\hbar}} \neq \tilde{0}$,

(NSN. 11) $\dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e + \tilde{\breve{\mathfrak{w}}}_{e'}, \tilde{\vartheta} + \tilde{\varsigma}) \lesssim \dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) \diamond \dot{\varphi}(\tilde{\breve{\mathfrak{w}}}_{e'}, \tilde{\varsigma})$,

(NSN. 12) $\dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e, \cdot)$ be a continuous in $\mathfrak{R}(\mathfrak{E})^*$ with non-increasing function as well as

$\lim_{\tilde{\vartheta}\to\infty} \dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = 0$

(NSN. 13) $\dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) < 1$;

(NSN. 14) $\dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = 0 \Leftrightarrow \tilde{\breve{\mathfrak{v}}}_e = \tilde{\theta}_0$,

(NSN. 15) $\dot{\psi}\left(\tilde{\breve{\hbar}}\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}\right) = \dot{\psi}\left(\tilde{\breve{\mathfrak{v}}}_e, \frac{\tilde{\vartheta}}{|\tilde{\breve{\hbar}}|}\right)$ and $\tilde{\breve{\hbar}} \neq \tilde{0}$,

(NSN. 16) $\dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e + \tilde{\breve{\mathfrak{w}}}_{e'}, \tilde{\vartheta} + \tilde{\varsigma}) \lesssim \dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) \star \dot{\psi}(\tilde{\breve{\mathfrak{w}}}_{e'}, \tilde{\varsigma})$,

(NSN. 17) $\dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e, \cdot)$ be a continuous in $\mathfrak{R}(\mathfrak{E})^*$ with non-increasing function as well as

$\lim_{\tilde{\vartheta}\to\infty} \dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = 0$

Furthermore, assume that $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ satisfying the following conditions:

(NSN. 18) $\alpha * \alpha = \alpha, \alpha \diamond \alpha = \alpha$ *and* $\alpha \star \alpha = \alpha, \forall \alpha \in [0,1]$,

(NSN. 19) $\dot{\phi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) > 0, \dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) < 1$ and $\dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) < 1, \forall\, \tilde{\vartheta} \gtrsim \tilde{0} \Rightarrow \tilde{\breve{\mathfrak{v}}}_e = \tilde{\theta}_0$."

Example(3.3): Let $(\tilde{\Xi}, \|.\|)$ be a SNLS along with consider that $\breve{\mathfrak{a}} * \breve{\mathfrak{b}} = \breve{\mathfrak{a}}.\breve{\mathfrak{b}}, \breve{\mathfrak{a}} \diamond \breve{\mathfrak{b}} = \min\{1, \breve{\mathfrak{a}} + \breve{\mathfrak{b}}\}$ and $\breve{\mathfrak{a}} \star \breve{\mathfrak{b}} = \min\{1, \breve{\mathfrak{a}} + \breve{\mathfrak{b}}\}$ for all $\breve{\mathfrak{a}}, \breve{\mathfrak{b}} \in [0,1], \dot{\phi}, \dot{\varphi}, \dot{\psi}$ be a *FS* within $SSP(\tilde{\Xi}) \times \mathfrak{R}(\mathfrak{E})^*$ defined as:

$\dot{\phi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = \frac{\hbar(\tilde{\vartheta})^n}{\hbar(\tilde{\vartheta})^n + m\|\tilde{\breve{\mathfrak{v}}}_e\|}$, $\dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = \frac{m\|\tilde{\breve{\mathfrak{v}}}_e\|}{k(\tilde{\vartheta})^n + m\|\tilde{\breve{\mathfrak{v}}}_e\|}$ and $\dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = \frac{m\|\tilde{\breve{\mathfrak{v}}}_e\|}{k(\tilde{\vartheta})^n}$ for every $\hbar, k, m, n \in \mathbb{R}^+$. After that $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ be an NSNLS. In particular, when $\hbar = k = m = n = 1$, we get $\dot{\phi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = \frac{\tilde{\vartheta}}{\tilde{\vartheta} + \|\tilde{\breve{\mathfrak{v}}}_e\|}$, $\dot{\varphi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = \frac{\|\tilde{\breve{\mathfrak{v}}}_e\|}{\tilde{\vartheta} + \|\tilde{\breve{\mathfrak{v}}}_e\|}$ and $\dot{\psi}(\tilde{\breve{\mathfrak{v}}}_e, \tilde{\vartheta}) = \frac{\|\tilde{\breve{\mathfrak{v}}}_e\|}{\tilde{\vartheta}}$ for all $\tilde{\vartheta} \gtrsim \tilde{0}$. In this case, we said $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ standard *NSNLS.*

Definition(3.4): Let $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ be a *NSNLS* as well as consider $\left\{\tilde{\breve{\mathfrak{v}}}^n_{e_j}\right\}$ denotes a sequence of soft vectors within $\tilde{\Xi}$, after that:

1) A sequence $\left\{\tilde{\breve{\mathfrak{v}}}^n_{e_j}\right\}$ is said to be converges to $\tilde{\breve{\mathfrak{v}}}^0_{e_j}$ with respect to $(\dot{\phi}, \dot{\varphi}, \dot{\psi})$, if for each $\alpha \in (0,1)$ and $\tilde{\vartheta} \gtrsim \tilde{0}$, then exists $n_0 \in \mathbb{Z}^+$ such that $\dot{\phi}\left(\tilde{\breve{\mathfrak{v}}}^n_{e_j} - \tilde{\breve{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) \gtrsim 1 - \alpha, \dot{\varphi}\left(\tilde{\breve{\mathfrak{v}}}^n_{e_j} - \tilde{\breve{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\breve{\mathfrak{v}}}^n_{e_j} - \tilde{\breve{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) \lesssim \alpha$ for every $n \geq n_0$. (orequivalently $\lim_{\tilde{\vartheta}\to\infty} \dot{\phi}\left(\tilde{\breve{\mathfrak{v}}}^n_{e_j} - \tilde{\breve{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) = 1$, $\lim_{\tilde{\vartheta}\to\infty} \dot{\varphi}\left(\tilde{\breve{\mathfrak{v}}}^n_{e_j} - \tilde{\breve{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) = 0$ and $\lim_{\tilde{\vartheta}\to\infty} \dot{\psi}\left(\tilde{\breve{\mathfrak{v}}}^n_{e_j} - \tilde{\breve{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) = 0$, when $\tilde{\vartheta} \to \infty$).
2) A sequence $\left\{\tilde{\breve{\mathfrak{v}}}^n_{e_j}\right\}$ is said to be a Cauchy Sequence (CS) w.r.t $(\dot{\phi}, \dot{\varphi}, \dot{\psi})$, when for each $\alpha \in (0,1)$ as well as $\tilde{\vartheta} \gtrsim \tilde{0}$, there is $n_0 \in \mathbb{Z}^+$ exist which means $\dot{\phi}\left(\tilde{\breve{\mathfrak{v}}}^n_{e_j} - \tilde{\breve{\mathfrak{v}}}^m_{e_j}, \tilde{\vartheta}\right) \gtrsim 1 - \alpha$, $\dot{\varphi}\left(\tilde{\breve{\mathfrak{v}}}^n_{e_j} - \tilde{\breve{\mathfrak{v}}}^m_{e_j}, \tilde{\vartheta}\right) \lesssim \alpha$

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and $\dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{m}, \vartheta\right) \lessapprox \alpha$ for every $n, m \geq n_0$. (or equivalently $\lim_{\vartheta\to\infty} \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{m}, \vartheta\right) = 1$, $\lim_{\vartheta\to\infty} \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{m}, \vartheta\right) = 0$ and $\lim_{\vartheta\to\infty} \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{m}, \vartheta\right) = 0$, as $\tilde{\vartheta} \to \infty$).

Definition(3.5): Let $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ be a *NSNLS*. Then $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ is said to be complete when every *CS* within $SSP(\tilde{\Xi})$ converge towards the soft vector of $SSP(\tilde{\Xi})$.

Definition(3.6): Let $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ be a *NSNLS*. After that the open ball $\mathfrak{B}(\tilde{\mathfrak{v}}_{e_1}, \varrho, \tilde{\vartheta})$, the closed ball $\mathfrak{B}[\tilde{\mathfrak{v}}_{e_1}, \varrho, \tilde{\vartheta}]$ and a sphere $\mathfrak{S}(\tilde{\mathfrak{v}}_{e_1}, \varrho, \tilde{\vartheta})$ with center at $\tilde{\mathfrak{v}}_{e_1} \tilde{\in} SSP(\tilde{\Xi})$ as well as radius $0 < \varrho < 1$, $\tilde{\vartheta} \gtrapprox \tilde{0}$ are defined as follows:

$$\mathfrak{B}(\tilde{\mathfrak{v}}_{e_1}, \varrho, \tilde{\vartheta}) = \left\{ \begin{matrix} \tilde{\mathfrak{w}}_{e_2} \tilde{\in} SSP(\tilde{\Xi}): \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \gtrapprox 1 - \varrho, \dot{\varphi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \lessapprox \varrho \\ \text{and } \dot{\psi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \lessapprox \varrho \end{matrix} \right\}$$

$\mathfrak{B}[\tilde{\mathfrak{v}}_{e_1}, \varrho, \tilde{\vartheta}] = \{\tilde{\mathfrak{w}}_{e_2} \tilde{\in} SSP(\tilde{\Xi}): \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \gtrapprox 1 - \varrho, \dot{\varphi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \lessapprox \varrho$

and $\dot{\psi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \lessapprox \varrho\}$

$$\mathfrak{S}(\tilde{\mathfrak{v}}_{e_1}, \varrho, \tilde{\vartheta}) = \left\{ \begin{matrix} \tilde{\mathfrak{w}}_{e_2} \tilde{\in} SSP(\tilde{\Xi}): \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = 1 - \varrho, \dot{\varphi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \varrho \\ \text{and } \dot{\psi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \varrho \end{matrix} \right\}.$$

Definition (3.7): Let $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ be a *NSNLS* and $\mathfrak{A}$ be a subset of $SSP(\tilde{\Xi})$. Afterwards:

1) $\mathfrak{A}$ is said to be open set when for each $\tilde{\mathfrak{v}}_{e_1} \tilde{\in} \mathfrak{A}$, after that $\tilde{\vartheta} \gtrapprox \tilde{0}$ exists along with $0 < \varrho < 1$ which means $\mathfrak{B}(\tilde{\mathfrak{v}}_{e_1}, \varrho, \tilde{\vartheta}) \tilde{\subseteq} \mathfrak{A}$.
2) $\mathfrak{A}$ is said to be closed set when for any sequence $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$ within $\mathfrak{A}$ converges towards $\tilde{\mathfrak{v}}_{e_j}^{0} \tilde{\in} \mathfrak{A}$.
3) $\mathfrak{A}$ is said to be bounded set when there is $\tilde{\vartheta} \gtrapprox \tilde{0}$ exists along with $0 < \varrho < 1$ which means $\dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \gtrapprox 1 - \varrho, \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \lessapprox \varrho$ and $\dot{\psi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \lessapprox \varrho, \forall\, \tilde{\mathfrak{v}}_e \tilde{\in} \mathfrak{A}$.
4) $\mathfrak{A}$ is said to be compact set when for any sequence $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$ within $\mathfrak{A}$ has a subsequence which is converges towards an element of $\mathfrak{A}$.

Theorem (3.8): In *NSNLS* there is an open sets with finite numbers of intersection is open.

Proof

Let $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ be an *NSNLS* and let open set with finite collection $\{\mathfrak{B}_i: i = 1, 2, 3, \dots, n\}$ in NSNLS, Let $\mathfrak{H} = \{\mathfrak{B}_i: i = 1, 2, 3, \dots, n\}$ we must to demonstrate $\mathfrak{H}$ is an open set. Consider $\tilde{\mathfrak{v}}_e \tilde{\in} \mathfrak{H}$

$\Rightarrow \tilde{\mathfrak{v}}_e \tilde{\in} \mathfrak{B}_i, \forall i = 1, 2, 3, \dots, n$ and $\mathfrak{B}_i$ be an open set $\forall i$

$\Rightarrow \exists \varrho_i \in (0, 1)$ and $\tilde{\vartheta}_i \gtrapprox \tilde{0}$ $s.t.\, \mathfrak{B}(\tilde{\mathfrak{v}}_e, \varrho_i, \tilde{\vartheta}_i) \tilde{\subseteq} \mathfrak{B}_i, i = 1, 2, 3, \dots, n.$

Let $\tilde{\vartheta}_k = \max\{\tilde{\vartheta}_1, \tilde{\vartheta}_2, \tilde{\vartheta}_3, \dots, \tilde{\vartheta}_n\}$ and $\varrho_k = \min\{\varrho_1, \varrho_2, \varrho_3, \dots, \varrho_n\}$.

$\Rightarrow \mathfrak{B}(\tilde{\mathfrak{v}}_e, \varrho_k, \tilde{\vartheta}_k) \tilde{\subseteq} \mathfrak{B}_i$ for all $i = 1, 2, 3, \dots, n$

$\Rightarrow \mathfrak{B}(\tilde{\mathfrak{v}}_e, \varrho_k, \tilde{\vartheta}_k) \tilde{\subseteq} \cap\, \mathfrak{B}_i$

$\Rightarrow \mathfrak{B}(\tilde{\mathfrak{v}}_e, \varrho_k, \tilde{\vartheta}_k) \tilde{\subseteq} \mathfrak{H}$

$\Rightarrow \mathfrak{H}$ is an open set.

Theorem (3.9):

In *NSNLS*, union of collection of open sets with an arbitrary is open.

Proof

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Let $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$be an *NSNLS* in addition let $\{\mathfrak{G}_\lambda : \lambda \in \Lambda\}$be a collection of an open set with an arbitrary within$\tilde{\Xi}$. Let $\mathfrak{G} = \cup \{\mathfrak{G}_\lambda : \lambda \in \Lambda\}$we must demonstrate that $\mathfrak{G}$ is an open set. By, consider $\tilde{\tilde{\mathfrak{v}}}_e \tilde{\in} \mathfrak{G}$ after that $\tilde{\tilde{\mathfrak{v}}}_e \tilde{\in} \mathfrak{G}_\lambda$ for certain $\lambda \in \Lambda$.Given that $\mathfrak{G}_\lambda$ provides an open set

$\Rightarrow$ there is $\varrho \in (0,1)$ exist, $\tilde{\vartheta} \gtrsim \tilde{0}$ which means $\mathfrak{B}(\tilde{\tilde{\mathfrak{v}}}_e, \varrho, \tilde{\vartheta}) \tilde{\subseteq} \mathfrak{G}_\lambda$as well as since $\mathfrak{G}_\lambda \tilde{\subseteq} \mathfrak{G}$

$\Rightarrow \mathfrak{B}(\tilde{\tilde{\mathfrak{v}}}_e, \varrho, \tilde{\vartheta}) \tilde{\subseteq} \mathfrak{G}$

$\Rightarrow$ $\mathfrak{G}$ is an open set.

Theorem (3.10):

Let $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$be an *NSNLS* when $\mathfrak{A}$ is open set within soft linear space $\tilde{\Xi}$as well as $\mathfrak{B} \tilde{\subseteq} \tilde{\Xi}$ after that $\mathfrak{A} + \mathfrak{B}$ is an open set within$\tilde{\Xi}$.

Proof

Let $\tilde{\tilde{\mathfrak{v}}}_e \tilde{\in}$ $SSP(\tilde{\Xi})$ as well as $\tilde{a}_{e'} \tilde{\in} \mathfrak{A}$.Since$\mathfrak{A}$ is an open set

$\Rightarrow$ there is $\varrho \in (0,1)$exist, $\tilde{\vartheta} \gtrsim \tilde{0}$ which means $\mathfrak{B}(\tilde{a}_{e'}, \varrho, \tilde{\vartheta}) \tilde{\subseteq} \mathfrak{A}$

$\Rightarrow \mathfrak{B}(\tilde{a}_{e'}, \varrho, \tilde{\vartheta}) + \tilde{\tilde{\mathfrak{v}}}_e \tilde{\subseteq} \mathfrak{A} + \tilde{\tilde{\mathfrak{v}}}_e$

$\Rightarrow \mathfrak{B}(\tilde{a}_{e'} + \tilde{\tilde{\mathfrak{v}}}_e, \varrho, \tilde{\vartheta}) \tilde{\subseteq} \mathfrak{A} + \tilde{\tilde{\mathfrak{v}}}_e$

$\Rightarrow \mathfrak{A} + \tilde{\tilde{\mathfrak{v}}}_e$which is an open set within $\tilde{\Xi}$for every $\tilde{\tilde{\mathfrak{v}}}_e \tilde{\in} SSP(\tilde{\Xi})$ as well as since

$\mathfrak{A} + \mathfrak{B} = \cup \{\mathfrak{A} + b : b \in \mathfrak{B}\}$

$\Rightarrow \mathfrak{A} + \mathfrak{B}$whichis an open set within $\tilde{\Xi}$.

Theorem (3.11):

Every convergent sequence is Cauchy sequence.

Proof

Let a sequence $\{\tilde{\tilde{\mathfrak{v}}}^n_{e_j}\}$within an $NSNLS(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$. Consider $\{\tilde{\tilde{\mathfrak{v}}}^n_{e_j}\}$converges to $\tilde{\tilde{\mathfrak{v}}}^0_{e_j}$

$\Rightarrow$when for each $\alpha \in (0,1)$and$\tilde{\vartheta} \gtrsim \tilde{0}$, there is $n_0 \in \mathbb{z}^+$ exist which means

$\dot{\phi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) \gtrsim 1 - \alpha$, $\dot{\varphi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right) \lesssim \alpha$ for every $n \geq n_0$.

$$\Rightarrow \dot{\phi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j}, \tilde{\vartheta}\right) = \dot{\phi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j} + \tilde{\tilde{\mathfrak{v}}}^0_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right)$$

$$\Rightarrow \dot{\phi}\left(\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}\right) + \left(\tilde{\tilde{\mathfrak{v}}}^m_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}\right), \tilde{\vartheta}\right) \gtrapprox \dot{\phi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \frac{\tilde{\vartheta}}{2}\right) * \dot{\phi}\left(\tilde{\tilde{\mathfrak{v}}}^m_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \frac{\tilde{\vartheta}}{2}\right)$$

$\gtrsim (1-\alpha) * (1-\alpha) = 1 - \alpha$,

$$\Rightarrow \dot{\varphi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j}, \tilde{\vartheta}\right) = \dot{\varphi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j} + \tilde{\tilde{\mathfrak{v}}}^0_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right)$$

$= \dot{\varphi}\left(\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}\right) + \left(\tilde{\tilde{\mathfrak{v}}}^m_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}\right), \tilde{\vartheta}\right) \lessapprox \dot{\varphi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \frac{\tilde{\vartheta}}{2}\right) \diamond \dot{\varphi}\left(\tilde{\tilde{\mathfrak{v}}}^m_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha \diamond \alpha = \alpha$ And

$$\dot{\psi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j}, \tilde{\vartheta}\right) = \dot{\psi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j} + \tilde{\tilde{\mathfrak{v}}}^0_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \tilde{\vartheta}\right)$$

$$= \dot{\psi}\left(\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}\right) + \left(\tilde{\tilde{\mathfrak{v}}}^m_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}\right), \tilde{\vartheta}\right) \lessapprox \dot{\psi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \frac{\tilde{\vartheta}}{2}\right) \star \dot{\psi}\left(\tilde{\tilde{\mathfrak{v}}}^m_{e_j} - \tilde{\tilde{\mathfrak{v}}}^0_{e_j}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha \diamond \alpha = \alpha$$

$\Rightarrow \dot{\phi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j}, \tilde{\vartheta}\right) \gtrsim 1 - \alpha, \dot{\varphi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j}, \tilde{\vartheta}\right) \lesssim \alpha$and$\dot{\psi}\left(\tilde{\tilde{\mathfrak{v}}}^n_{e_j} - \tilde{\tilde{\mathfrak{v}}}^m_{e_j}, \tilde{\vartheta}\right) \lesssim \alpha$ for all $n, m \geq n_0$

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and $\alpha \in (0,1)$,Hence, the sequence $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$ is CS.

Theorem (3.12): In *NSNLS* has a sequence, if a limit exists that is unique.

Proof

Let sequence $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$ within a $NSNLS(\tilde{\Xi},\dot{\phi},\dot{\varphi},\dot{\psi},*,\diamond,\star)$, such that $\lim_{\tilde{\vartheta}\to\infty} \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right)=1$, $\lim_{\tilde{\vartheta}\to\infty} \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right)=0$, and $\lim_{\tilde{\vartheta}\to\infty} \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right)=0$ and $\lim_{\tilde{\vartheta}\to\infty} \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right)=1$, $\lim_{\tilde{\vartheta}\to\infty} \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right)=0$, and $\lim_{\tilde{\vartheta}\to\infty} \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right)=0$ these are three limits to a sequence $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$.After that by definition there are positive integers n_1, n_2 exists which means $\dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right) \gtrsim 1-\alpha$, $\dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right) \lesssim \alpha$ for all $n \geq n_1$ as well as $\alpha \in (0,1)$ and

$\dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right) \gtrsim 1-\alpha$, $\dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right) \lesssim \alpha$ for all $n \geq n_2$ as well as $\alpha \in (0,1)$.

Choose $n \geq n_0, n_0 = min\{n_1, n_2\}$

$$\dot{\phi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = \dot{\phi}\left(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e_j}^{n}+\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right) = \dot{\phi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e}\right)+\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'}\right),\tilde{\vartheta}\right)$$

$$\gtrapprox \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\frac{\tilde{\vartheta}}{2}\right) * \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\frac{\tilde{\vartheta}}{2}\right) \gtrsim (1-\alpha)*(1-\alpha) = 1-\alpha,$$

$$\dot{\varphi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e_j}^{n}+\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right) = \dot{\varphi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e}\right)+\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'}\right),\tilde{\vartheta}\right)$$

$$\lessapprox \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\frac{\tilde{\vartheta}}{2}\right) \diamond \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha \diamond \alpha = \alpha, \text{and}$$

$$\dot{\psi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = \dot{\psi}\left(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e_j}^{n}+\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}\right) = \dot{\psi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e}\right)+\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'}\right),\tilde{\vartheta}\right)$$

$$\lessapprox \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\frac{\tilde{\vartheta}}{2}\right) \star \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e'},\frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha \star \alpha = \alpha.$$

$\Rightarrow \dot{\phi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) \gtrsim 1-\alpha, \dot{\varphi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) \lesssim \alpha$ and $\dot{\psi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) \lesssim \alpha$

$\Rightarrow \lim_{n\to\infty} \dot{\phi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = 1$, $\lim_{n\to\infty} \dot{\varphi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = 0$, and $\lim_{n\to\infty} \dot{\psi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = 0$

$\Rightarrow \dot{\phi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = 1, \dot{\varphi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = 0$ and $\dot{\psi}(\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'},\tilde{\vartheta}) = 0$

By using the collections (***NSN*. 4**) and (***NSN*. 9**), we get

$$\tilde{\mathfrak{v}}_{e}-\tilde{\mathfrak{v}}_{e'} = \tilde{\theta}_0 \Rightarrow \tilde{\mathfrak{v}}_{e} = \tilde{\mathfrak{v}}_{e'}.$$

Theorem (3.13): Let $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$, $\{\tilde{\mathfrak{w}}_{e_i}^{n}\}$ be a sequences in $NSNLS(\tilde{\Xi},\dot{\phi},\dot{\varphi},\dot{\psi},*,\diamond,\star)$ along with for every $\alpha_1 \in (0,1)$ it exists $\alpha \in (0,1)$ which means $\alpha * \alpha \geq \alpha_1$, $\alpha \diamond \alpha \geq \alpha_1$ and $\alpha \star \alpha \geq \alpha_1$:

1) If $\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{\mathfrak{v}}_{e}$ then $\tilde{c}\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{c}\tilde{\mathfrak{v}}_{e}, \tilde{c} \,\tilde{\in}\, \mathfrak{K}\backslash\{\tilde{0}\}$ ($\mathfrak{K}$ is field).
2) If $\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{\mathfrak{v}}_{e}$ and $\tilde{\mathfrak{w}}_{e_i}^{n} \to \tilde{\mathfrak{w}}_{e'}$ then $\tilde{\mathfrak{v}}_{e_j}^{n} + \tilde{\mathfrak{w}}_{e_i}^{n} \to \tilde{\mathfrak{v}}_{e} + \tilde{\mathfrak{w}}_{e'}$

Proof

1) Since $\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{\mathfrak{v}}_{e}$
$\Rightarrow$ for each $\alpha \in (0,1)$ as well as $\tilde{\vartheta} \gtrsim \tilde{0}$,after that $n_0 \in \mathbb{Z}^+$ exists which gives $\dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right) \gtrsim 1-\alpha$, $\dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n}-\tilde{\mathfrak{v}}_{e},\tilde{\vartheta}\right) \lesssim \alpha$ for all $n \geq n_0$. Put $\tilde{\vartheta} = \frac{\tilde{\vartheta}_1}{|\tilde{c}|}$ such that $\tilde{\vartheta}_1 \gtrsim \tilde{0}, \tilde{c} \,\tilde{\in}\, \mathfrak{K}\backslash\{\tilde{0}\}$

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$\dot{\phi}\left(\tilde{c}\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{c}\tilde{\mathfrak{v}}_e, \tilde{\vartheta}_1\right) = \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}_1}{|\tilde{c}|}\right) = \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \tilde{\vartheta}\right) \gtrsim 1-\alpha$, $\dot{\varphi}\left(\tilde{c}\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{c}\tilde{\mathfrak{v}}_e, \tilde{\vartheta}_1\right) = \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\vartheta}{|\tilde{c}|}\right) = \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \tilde{\vartheta}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{c}\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{c}\tilde{\mathfrak{v}}_e, \tilde{\vartheta}_1\right) = \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}_1}{|\tilde{c}|}\right) = \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \tilde{\vartheta}\right) \lesssim \alpha \Rightarrow \tilde{c}\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{c}\tilde{\mathfrak{v}}_e$.

2) Since $\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{\mathfrak{v}}_e$

$\Rightarrow$ for each $\alpha \in (0,1)$and $\tilde{\vartheta} \gtrsim \tilde{0}$,after that $n_1 \in \mathbb{Z}^+$ exists which gives

$\dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2}\right) \gtrsim 1-\alpha$, $\dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha$ for all $n \geq n_1$.

And since $\tilde{\mathfrak{w}}_{e_i}^{n} \to \tilde{\mathfrak{w}}_{e'}$

$\Rightarrow$ for each $\alpha \in (0,1)$and $\tilde{\vartheta} \gtrsim \tilde{0}$, after that $n_2 \in \mathbb{Z}^+$ exists which gives

$\dot{\phi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2}\right) \gtrsim 1-\alpha$, $\dot{\varphi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha$ for all $n \geq n_2$.

Take $n_0 = min\{n_1, n_2\}$for all $\alpha \in (0,1)$, there exists $n_0 \in \mathbb{Z}^+$, (α is arbitrary)

$$\dot{\phi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} + \tilde{\mathfrak{w}}_{e_i}^{n}\right) - \left(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right) = \dot{\phi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e\right) + \left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right)$$

$$\gtrapprox \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2}\right) * \dot{\phi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2}\right) \gtrsim (1-\alpha) * (1-\alpha) = 1-\alpha,$$

$$\dot{\varphi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} + \tilde{\mathfrak{w}}_{e_i}^{n}\right) - \left(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right) = \dot{\varphi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e\right) + \left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right)$$

$\lessapprox \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2}\right) \diamond \dot{\varphi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha \diamond \alpha = \alpha$,and

$$\dot{\psi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} + \tilde{\mathfrak{w}}_{e_i}^{n}\right) - \left(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right) = \dot{\psi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e\right) + \left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right)$$

$$\lessapprox \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2}\right) \star \dot{\psi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha \star \alpha = \alpha.$$

By taking $\lim_{n\to\infty}$,

$\lim_{n\to\infty} \dot{\phi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} + \tilde{\mathfrak{w}}_{e_i}^{n}\right) - \left(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right) = 1$, $\lim_{n\to\infty} \dot{\varphi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} + \tilde{\mathfrak{w}}_{e_i}^{n}\right) - \left(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right) = 0$ and $\lim_{n\to\infty} \dot{\psi}\left(\left(\tilde{\mathfrak{v}}_{e_j}^{n} + \tilde{\mathfrak{w}}_{e_i}^{n}\right) - \left(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}\right), \tilde{\vartheta}\right) = 0$

$$\Rightarrow \tilde{\mathfrak{v}}_{e_j}^{n} + \tilde{\mathfrak{w}}_{e_i}^{n} \to \tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}.$$

Theorem (3.14):Let $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}, \left\{\tilde{\mathfrak{w}}_{e_i}^{n}\right\}$be a sequence in $NSNLS(\Xi, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$such that $\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{\mathfrak{v}}_e$and $\tilde{\mathfrak{w}}_{e_i}^{n} \to \tilde{\mathfrak{w}}_{e'}$. $\tilde{\alpha}, \tilde{\beta} \,\tilde{\in}\, \mathfrak{K}\backslash\{\tilde{0}\}$ then $\tilde{\alpha}\mathscr{f}\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) + \tilde{\beta}\mathscr{g}\left(\tilde{\mathfrak{w}}_{e_i}^{n}\right) \to \tilde{\alpha}\mathscr{f}\left(\tilde{\mathfrak{v}}_e\right) + \tilde{\beta}\mathscr{g}\left(\tilde{\mathfrak{w}}_{e'}\right)$ whenever $\mathscr{f}$ and $\mathscr{g}$ are two identify function.

Proof

Since $\tilde{\mathfrak{v}}_{e_j}^{n} \to \tilde{\mathfrak{v}}_e$

$\Rightarrow$ for each $\epsilon \in (0,1)$and $\tilde{\vartheta} \gtrsim \tilde{0}$, there exists $n_1 \in \mathbb{Z}^+$ such that $\dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) \gtrsim 1-\epsilon$, $\dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) \lesssim \epsilon$ and $\dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) \lesssim \epsilon$ for all $n \geq n_1$.

And since $\tilde{\mathfrak{w}}_{e_i}^{n} \to \tilde{\mathfrak{w}}_{e'}$

$\Rightarrow$ for each $\epsilon \in (0,1)$and $\tilde{\vartheta} \gtrsim \tilde{0}$, there exists $n_2 \in \mathbb{Z}^+$ such that $\dot{\phi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right) \gtrsim 1-\epsilon$, $\dot{\varphi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right) \lesssim \epsilon$ and $\dot{\psi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right) \lesssim \epsilon$ for all $n \geq n_2$.

Take $n_0 = min\{n_1, n_2\}$for all $\epsilon \in (0,1)$, there exists $n_0 \in \mathbb{Z}^+$, (ϵ is arbitrary)

$$\dot{\phi}\left(\left(\tilde{\alpha}\mathscr{f}\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) + \tilde{\beta}\mathscr{g}\left(\tilde{\mathfrak{w}}_{e_i}^{n}\right)\right) - \left(\tilde{\alpha}\mathscr{f}\left(\tilde{\mathfrak{v}}_e\right) + \tilde{\beta}\mathscr{g}\left(\tilde{\mathfrak{w}}_{e'}\right)\right), \tilde{\vartheta}\right)$$

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$$= \dot{\phi}\left(\tilde{\alpha}\left(f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) - f(\tilde{\mathfrak{v}}_e)\right) + \tilde{\beta}\left(g(\tilde{\mathfrak{w}}_{e_i}^{n}) - g(\tilde{\mathfrak{w}}_{e'})\right), \tilde{\vartheta}\right)$$

$$\gtrsim \dot{\phi}\left(f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) - f(\tilde{\mathfrak{v}}_e), \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) * \dot{\phi}\left(g(\tilde{\mathfrak{w}}_{e_i}^{n}) - g(\tilde{\mathfrak{w}}_{e'}), \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right)$$

$$= \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) * \dot{\phi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right) \gtrsim (1-\epsilon) * (1-\epsilon)$$

$$= 1-\epsilon,$$

$$\dot{\varphi}\left(\left(\tilde{\alpha}f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) + \tilde{\beta}g(\tilde{\mathfrak{w}}_{e_i}^{n})\right) - \left(\tilde{\alpha}f(\tilde{\mathfrak{v}}_e) + \tilde{\beta}g(\tilde{\mathfrak{w}}_{e'})\right), \tilde{\vartheta}\right)$$

$$= \dot{\varphi}\left(\tilde{\alpha}\left(f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) - f(\tilde{\mathfrak{v}}_e)\right) + \tilde{\beta}\left(g(\tilde{\mathfrak{w}}_{e_i}^{n}) - g(\tilde{\mathfrak{w}}_{e'})\right), \tilde{\vartheta}\right)$$

$$\lesssim \dot{\varphi}\left(f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) - f(\tilde{\mathfrak{v}}_e), \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) \diamond \dot{\varphi}\left(g(\tilde{\mathfrak{w}}_{e_i}^{n}) - g(\tilde{\mathfrak{w}}_{e'}), \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right)$$

$= \dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) \diamond \dot{\varphi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right) \lesssim \epsilon \diamond \epsilon = \epsilon$,and

$$\dot{\psi}\left(\left(\tilde{\alpha}f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) + \tilde{\beta}g(\tilde{\mathfrak{w}}_{e_i}^{n})\right) - \left(\tilde{\alpha}f(\tilde{\mathfrak{v}}_e) + \tilde{\beta}g(\tilde{\mathfrak{w}}_{e'})\right), \tilde{\vartheta}\right)$$

$$= \dot{\psi}\left(\tilde{\alpha}\left(f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) - f(\tilde{\mathfrak{v}}_e)\right) + \tilde{\beta}\left(g(\tilde{\mathfrak{w}}_{e_i}^{n}) - g(\tilde{\mathfrak{w}}_{e'})\right), \tilde{\vartheta}\right)$$

$$\lesssim \dot{\psi}\left(f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) - f(\tilde{\mathfrak{v}}_e), \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) \star \dot{\psi}\left(g(\tilde{\mathfrak{w}}_{e_i}^{n}) - g(\tilde{\mathfrak{w}}_{e'}), \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right)$$

$$= \dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_e, \frac{\tilde{\vartheta}}{2|\tilde{\alpha}|}\right) \star \dot{\psi}\left(\tilde{\mathfrak{w}}_{e_i}^{n} - \tilde{\mathfrak{w}}_{e'}, \frac{\tilde{\vartheta}}{2|\tilde{\beta}|}\right) \lesssim \epsilon \star \epsilon = \epsilon$$

Therefore, $\tilde{\alpha}f\left(\tilde{\mathfrak{v}}_{e_j}^{n}\right) + \tilde{\beta}g(\tilde{\mathfrak{w}}_{e_i}^{n}) \to \tilde{\alpha}f(\tilde{\mathfrak{v}}_e) + \tilde{\beta}g(\tilde{\mathfrak{w}}_{e'})$.

Theorem (3.15): An *NSNLS*$(\Xi, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ in which every *CS* has a convergent subsequence is complete.

Proof

Let $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$ be a *CS* within an *NSNLS*$(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$ and $\left\{\tilde{\mathfrak{v}}_{e_j}^{n_k}\right\}$ be a subsequence that converges to $\tilde{\mathfrak{v}}_{e_j}^{0}$. We prove that $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$ converges to $\tilde{\mathfrak{v}}_{e_j}^{0}$.

Let $\tilde{\vartheta} \gtrsim \tilde{0}$ and $\alpha \in (0,1)$.

Since $\left\{\tilde{\mathfrak{v}}_{e_j}^{n}\right\}$ be a *CS*, $n_0 \in \mathbb{Z}^+$ exists likewise $\dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{k}, \frac{\tilde{\vartheta}}{2}\right) \gtrsim 1-\alpha$, $\dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{k}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{k}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha$ for all $n, k \geq n_0$.

Since $\left\{\tilde{\mathfrak{v}}_{e_j}^{n_k}\right\}$ converges to $\tilde{\mathfrak{v}}_{e_j}^{0}$, $i_k > n_0$ is a positive integer exist which gives,
$\dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{i_k} - \tilde{\mathfrak{v}}_{e_j}^{0}, \frac{\tilde{\vartheta}}{2}\right) \gtrsim 1-\alpha$, $\dot{\varphi}\left(\tilde{\mathfrak{v}}_{e_j}^{i_k} - \tilde{\mathfrak{v}}_{e_j}^{0}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha$ and $\dot{\psi}\left(\tilde{\mathfrak{v}}_{e_j}^{i_k} - \tilde{\mathfrak{v}}_{e_j}^{0}, \frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha$

Now

$$\dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{0}, \frac{\tilde{\vartheta}}{2}\right) = \dot{\phi}\left(\tilde{\mathfrak{v}}_{e_j}^{n} - \tilde{\mathfrak{v}}_{e_j}^{i_k} + \tilde{\mathfrak{v}}_{e_j}^{i_k} - \tilde{\mathfrak{v}}_{e_j}^{0}, \frac{\tilde{\vartheta}}{2} + \frac{\tilde{\vartheta}}{2}\right)$$

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$\gtrsim \dot{\phi}\left(\tilde{\mathfrak{v}}^{n}_{e_j}-\tilde{\mathfrak{v}}^{i_k}_{e_j},\frac{\tilde{\vartheta}}{2}\right) * \dot{\phi}\left(\tilde{\mathfrak{v}}^{i_k}_{e_j}-\tilde{\mathfrak{v}}^{0}_{e_j},\frac{\tilde{\vartheta}}{2}\right) \gtrsim (1-\alpha)*(1-\alpha) > 1-\alpha,$

$\dot{\varphi}\left(\tilde{\mathfrak{v}}^{n}_{e_j}-\tilde{\mathfrak{v}}^{0}_{e_j},\frac{\tilde{\vartheta}}{2}\right) = \dot{\varphi}\left(\tilde{\mathfrak{v}}^{n}_{e_j}-\tilde{\mathfrak{v}}^{i_k}_{e_j}+\tilde{\mathfrak{v}}^{i_k}_{e_j}-\tilde{\mathfrak{v}}^{0}_{e_j},\frac{\tilde{\vartheta}}{2}+\frac{\tilde{\vartheta}}{2}\right)$

$\lesssim \dot{\varphi}\left(\tilde{\mathfrak{v}}^{n}_{e_j}-\tilde{\mathfrak{v}}^{i_k}_{e_j},\frac{\tilde{\vartheta}}{2}\right) \diamond \dot{\varphi}\left(\tilde{\mathfrak{v}}^{i_k}_{e_j}-\tilde{\mathfrak{v}}^{0}_{e_j},\frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha \diamond \alpha < \alpha$,and

$\dot{\psi}\left(\tilde{\mathfrak{v}}^{n}_{e_j}-\tilde{\mathfrak{v}}^{0}_{e_j},\frac{\tilde{\vartheta}}{2}\right) = \dot{\psi}\left(\tilde{\mathfrak{v}}^{n}_{e_j}-\tilde{\mathfrak{v}}^{i_k}_{e_j}+\tilde{\mathfrak{v}}^{i_k}_{e_j}-\tilde{\mathfrak{v}}^{0}_{e_j},\frac{\tilde{\vartheta}}{2}+\frac{\tilde{\vartheta}}{2}\right)$

$\lesssim \dot{\psi}\left(\tilde{\mathfrak{v}}^{n}_{e_j}-\tilde{\mathfrak{v}}^{k}_{e_j},\frac{\tilde{\vartheta}}{2}\right) \star \dot{\psi}\left(\tilde{\mathfrak{v}}^{i_k}_{e_j}-\tilde{\mathfrak{v}}^{0}_{e_j},\frac{\tilde{\vartheta}}{2}\right) \lesssim \alpha \star \alpha < \alpha.$

Therefore$\left\{\tilde{\mathfrak{v}}^{n}_{e_j}\right\}$ converges to $\tilde{\mathfrak{v}}^{0}_{e_j}$ in $(\tilde{\Xi},\dot{\phi},\dot{\varphi},\dot{\psi},*,\diamond,\star)$ and hence it is complete.

Definition (3.16):Let $(\tilde{\Xi},\dot{\phi},\dot{\varphi},\dot{\psi},*,\diamond,\star)$be an *NSNLS* satisfying the condition **(NSNLS.18) and (NSNLS.19).**Define$\|\tilde{\mathfrak{v}}_e\|_\alpha = \inf\{\tilde{\vartheta} \gtrsim \tilde{0}{:}\dot{\phi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \gtrsim \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha$ and$\dot{\psi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha, \alpha \in (0,1)\}$. $\|\cdot\|_\alpha$are called α −soft norms on $SSP(\tilde{\Xi})$ corresponding to the NSNLS.

Theorem (3.17):Let$\|\cdot\|_\alpha$, $\alpha \in (0,1)$, be defined in definition **(3.16)**, then $\{\|\cdot\|_\alpha{:}\ \alpha \in (0,1)\}$ provides ascending family as part of α −soft norms depending on $SSP(\tilde{\Xi})$.

Proof

(1) From definition $\|\tilde{\mathfrak{v}}_e\|_\alpha \gtrsim \tilde{0}$and let $\|\tilde{\mathfrak{v}}_e\|_\alpha = \tilde{0}$

$\Rightarrow\ inf\ \{\tilde{\vartheta} \gtrsim \tilde{0}{:}\ \dot{\phi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \gtrsim \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha$ and$\dot{\psi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha, \alpha \in (0,1)\} = \tilde{0}$

⇒ For all$\tilde{\vartheta} \gtrsim \tilde{0}$: $\dot{\phi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \gtrsim \alpha \gtrsim 0$, $\dot{\varphi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha \lesssim 1$ and $\dot{\psi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha \lesssim 1, \alpha \in (0,1)$.

By condition **(NSNLS.19),** we get$\tilde{\mathfrak{v}}_e = \tilde{\theta}_0$.

Conversely, let$\tilde{\mathfrak{v}}_e = \tilde{\theta}_0$.

$\Rightarrow \dot{\mathfrak{I}}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) = 1, \ddot{\mathfrak{S}}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) = 0$ and$\dddot{\mathfrak{M}}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) = 0, \forall\ \tilde{\vartheta} \gtrsim \tilde{0}$

⇒ For all $\alpha \in (0,1)$

$inf\ \{\tilde{\vartheta} \gtrsim \tilde{0}{:}\ \dot{\phi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \gtrsim \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha$ and$\dot{\psi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha, \alpha \in (0,1)\} = \tilde{0}$

$\Rightarrow \|\tilde{\mathfrak{v}}_e\|_\alpha = \tilde{0}$

(2) For all $\tilde{c} \neq \tilde{0}$

$\|\tilde{c}.\tilde{\mathfrak{v}}_e\|_\alpha = inf\ \{\tilde{\varsigma} \gtrsim \tilde{0}{:}\ \dot{\phi}(\tilde{c}\tilde{\mathfrak{v}}_e,\tilde{\varsigma}) \gtrsim \alpha,\ \dot{\varphi}(\tilde{c}\tilde{\mathfrak{v}}_e,\tilde{\varsigma}) \lesssim 1-\alpha$ and$\dot{\psi}(\tilde{c}\tilde{\mathfrak{v}}_e,\tilde{\varsigma}) \lesssim 1-\alpha, \alpha \in (0,1)\}$

$= inf\{\tilde{\varsigma} \gtrsim \tilde{0}{:}\ \dot{\phi}\left(\tilde{\mathfrak{v}}_e,\frac{\tilde{\varsigma}}{|\tilde{c}|}\right) \gtrsim \alpha,\ \dot{\varphi}\left(\tilde{\mathfrak{v}}_e,\frac{\tilde{\varsigma}}{|\tilde{c}|}\right) \lesssim 1-\alpha$ and$\dot{\psi}\left(\tilde{\mathfrak{v}}_e,\frac{\tilde{\varsigma}}{|\tilde{c}|}\right) \lesssim 1-\alpha, \alpha \in (0,1)\}$

Let $\tilde{\vartheta} = \frac{\tilde{\varsigma}}{|\tilde{c}|} \gtrsim \tilde{0}$,

$\Rightarrow \|\tilde{c}.\tilde{\mathfrak{v}}_e\|_\alpha$

$=\ inf\{|\tilde{c}|\tilde{\vartheta} \gtrsim \tilde{0}{:}\ \dot{\phi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \gtrsim \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha$ and$\dot{\psi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha, \alpha \in (0,1)\}$

$= |\tilde{c}|inf\{\tilde{\vartheta} \gtrsim \tilde{0}{:}\ \dot{\phi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \gtrsim \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha$ and$\dot{\psi}(\tilde{\mathfrak{v}}_e,\tilde{\vartheta}) \lesssim 1-\alpha, \alpha \in (0,1)\} = |\tilde{c}|\|\tilde{\mathfrak{v}}_e\|_\alpha$

If $\tilde{c} = \tilde{0} \Rightarrow \|\tilde{c}.\tilde{\mathfrak{v}}_e\|_\alpha = \|\tilde{0}.\tilde{\mathfrak{v}}_e\|_\alpha = \tilde{0} = \tilde{0}\|\tilde{\mathfrak{v}}_e\|_\alpha = |\tilde{c}|\|\tilde{\mathfrak{v}}_e\|_\alpha, \forall\ \tilde{c} \tilde{\in} \mathfrak{K}$.

(3) $\|\tilde{\mathfrak{v}}_e\|_\alpha + \|\tilde{\mathfrak{w}}_{e'}\|_\alpha$

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$= inf\{\tilde{\vartheta} \gtrsim \tilde{0}: \dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \geqq \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha$ and $\dot{\psi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha, \alpha \in (0,1)\}$

$+ inf\{\tilde{\varsigma} \gtrsim \tilde{0}: \dot{\phi}(\tilde{\mathfrak{w}}_{e'}, \tilde{\varsigma}) \geqq \alpha,\ \dot{\varphi}(\tilde{\mathfrak{w}}_{e'}, \tilde{\varsigma}) \leqq 1-\alpha$ and $\dot{\psi}(\tilde{\mathfrak{w}}_{e'}, \tilde{\varsigma}) \leqq 1-\alpha, \alpha \in (0,1)\}$

$= inf\{\tilde{\vartheta}+\tilde{\varsigma} \gtrsim \tilde{0}: \dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) * \dot{\phi}(\tilde{\mathfrak{w}}_{e'}, \tilde{\varsigma}) \geqq \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \diamond \dot{\varphi}(\tilde{\mathfrak{w}}_{e'}, \tilde{\varsigma}) \leqq 1-\alpha$ and

$\dot{\psi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \star \dot{\psi}(\tilde{\mathfrak{w}}_{e'}, \tilde{\varsigma}) \leqq 1-\alpha, \alpha \in (0,1)\}$

$\geqq inf\ \{\tilde{\vartheta}+\tilde{\varsigma} \gtrsim \tilde{0}: \dot{\phi}(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}, \tilde{\vartheta}+\tilde{\varsigma}) \geqq \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}, \tilde{\vartheta}+\tilde{\varsigma}) \leqq 1-\alpha$ and

$\dot{\psi}(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}, \tilde{\vartheta}+\tilde{\varsigma}) \leqq 1-\alpha, \alpha \in (0,1)\}$

$= inf\ \{\tilde{\varrho} \gtrsim \tilde{0}: \dot{\phi}(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}, \tilde{\varrho}) \geqq \alpha,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}, \tilde{\varrho}) \leqq 1-\alpha$ and

$\dot{\psi}(\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}, \tilde{\varrho}) \leqq 1-\alpha, \alpha \in (0,1)\} = \|\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}\|_\alpha$.Thus $\|\tilde{\mathfrak{v}}_e + \tilde{\mathfrak{w}}_{e'}\|_\alpha \leqq \|\tilde{\mathfrak{v}}_e\|_\alpha + \|\tilde{\mathfrak{w}}_{e'}\|_\alpha$.

Then $\|\cdot\|_\alpha$is α −soft norm on $SSP(\tilde{\Xi})$. We show that for any $0 < \alpha_1 < \alpha_2 < 1$.

Then$\|\tilde{\mathfrak{v}}_e\|_{\alpha_1} \leqq \|\tilde{\mathfrak{v}}_e\|_{\alpha_2}$.

Since $\alpha_1 < \alpha_2$

$\Rightarrow \{\tilde{\vartheta} \gtrsim \tilde{0}: \dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \geqq \alpha_2,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha_2and\dot{\psi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha_2, \alpha_2 \in (0,1)\}$

$\tilde{\subset} \{\tilde{\vartheta} \gtrsim \tilde{0} : \dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \geqq \alpha_1,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha_1and\dot{\psi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha_1, \alpha_1 \in (0,1)\}$

$\Rightarrow inf\ \{\tilde{\vartheta} \gtrsim \tilde{0}: \dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \geqq \alpha_2,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha_2and\dot{\psi}\left(\tilde{\mathfrak{v}}_e, \tilde{\tilde{\vartheta}}\right) \leqq 1-\alpha_2, \alpha_2 \in (0,1)\}$

$\geqq inf\{\tilde{\vartheta} \gtrsim \tilde{0}: \dot{\phi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \geqq \alpha_1,\ \dot{\varphi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha_1and\dot{\psi}(\tilde{\mathfrak{v}}_e, \tilde{\vartheta}) \leqq 1-\alpha_1, \alpha_1 \in (0,1)\}$

$\Rightarrow \|\tilde{\mathfrak{v}}_e\|_{\alpha_1} \leqq \|\tilde{\mathfrak{v}}_e\|_{\alpha_2}$.Then $\{\|\cdot\|_\alpha: \alpha \in (0,1)\}$ provides ascending family as part of α −soft norms depending on $SSP(\tilde{\Xi})$corresponding *NSNLS*."

Theorem (3.18):Every *NSNLS* is a Neutrosophic soft metric space.

Proof

Let $(\tilde{\Xi}, \dot{\phi}, \dot{\varphi}, \dot{\psi}, *, \diamond, \star)$be an *NSNLS*. Define the *NSMS* by $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta})$, $\nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta})= \dot{\varphi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta})$ and $\not\leq(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta})= \dot{\psi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta})$ for every $\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2} \tilde{\in} SSP(\tilde{\Xi})$. After that it is obvious to demonstrate that the axioms of the *NSMS* are satisfied.

(***NSM.1***) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) + \nabla(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) + \not\leq(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) + \dot{\varphi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) +$

$\dot{\psi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) \leq 1$

(***NSM.*** **2**) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) > 0$

(***NSM.*** **3**) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = 1$ if and only if$\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2} = \tilde{\theta}_0$if and only if

$\tilde{\mathfrak{v}}_{e_1} = \tilde{\mathfrak{w}}_{e_2}$.

(***NSM.*** **4**) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}) = \dot{\phi}(\tilde{\mathfrak{w}}_{e_2} - \tilde{\mathfrak{v}}_{e_1}, \tilde{\vartheta}) = \Delta(\tilde{\mathfrak{w}}_{e_2}, \tilde{\mathfrak{v}}_{e_1}, \tilde{\vartheta})$

(***NSM.*** **5**) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}+\tilde{\varsigma}) = \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}+\tilde{\varsigma}) = \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{x}}_{e_3} + \tilde{\mathfrak{x}}_{e_3} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\vartheta}+\tilde{\varsigma})$

$\geqq \dot{\phi}(\tilde{\mathfrak{v}}_{e_1} - \tilde{\mathfrak{x}}_{e_3}, \tilde{\vartheta}) * \dot{\phi}(\tilde{\mathfrak{x}}_{e_3} - \tilde{\mathfrak{w}}_{e_2}, \tilde{\varsigma}) = \Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{x}}_{e_3}, \tilde{\vartheta}) * \Delta(\tilde{\mathfrak{x}}_{e_3}, \tilde{\mathfrak{w}}_{e_2}, \tilde{\varsigma})$

From definition of *NSNLS*, we get

(***NSM.*** **6**) $\Delta(\tilde{\mathfrak{v}}_{e_1}, \tilde{\mathfrak{w}}_{e_2}, \cdot): \mathfrak{R}(\mathfrak{E})^* \to [0,1]$is continuous.

Similarly with respect to$\dot{\varphi}$ and$\dot{\psi}$, we get conditions

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(***NSM***. 7) $\nabla(\breve{\mathfrak{v}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = \dot{\varphi}(\breve{\mathfrak{v}} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) < 1$

(***NSM***. **8**) $\nabla(\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = \dot{\varphi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = 0$ if and only if $\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2} = \tilde{\theta}_0$if and only if $\breve{\tilde{\mathfrak{v}}}_{e_1} = \tilde{\tilde{\mathfrak{w}}}_{e_2}$.

(***NSM***. **9**) $\nabla(\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = \dot{\varphi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = \dot{\varphi}(\tilde{\tilde{\mathfrak{w}}}_{e_2} - \breve{\tilde{\mathfrak{v}}}_{e_1}, \vartheta) = \nabla(\tilde{\tilde{\mathfrak{w}}}_{e_2}\breve{\tilde{\mathfrak{v}}}_{e_1}, \vartheta)$ and

(***NSM***. **10**) $\nabla(\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta + \varsigma) = \dot{\varphi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta + \varsigma) = \dot{\varphi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{x}}}_{e_3} + \tilde{\tilde{\mathfrak{x}}}_{e_3} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta + \varsigma)$
$\lesseqqgtr \dot{\varphi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{x}}}_{e_3}, \vartheta) \diamond \dot{\varphi}(\tilde{\tilde{\mathfrak{x}}}_{e_3} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \varsigma) = \nabla(\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{x}}}_{e_3}, \vartheta) \diamond \nabla(\tilde{\tilde{\mathfrak{x}}}_{e_3}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \varsigma)$.

(***NSM***. **11**)$\nabla(\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \cdot): \mathfrak{R}(\mathfrak{E})^* \to [0,1]$

(***NSM***. **12**) $\npreceq (\breve{\mathfrak{v}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = \dot{\psi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) < 1$

(***NSM***. **13**)$\npreceq (\breve{\mathfrak{v}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = \dot{\psi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = 0$ if and only if $\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2} = \tilde{\theta}_0$if and only if $\breve{\tilde{\mathfrak{v}}}_{e_1} = \tilde{\tilde{\mathfrak{w}}}_{e_2}$.

(***NSM***. **14**) $\npreceq (\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = \dot{\psi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta) = \dot{\psi}(\tilde{\tilde{\mathfrak{w}}}_{e_2} - \breve{\tilde{\mathfrak{v}}}_{e_1}, \vartheta) = \npreceq (\tilde{\tilde{\mathfrak{w}}}_{e_2}, \breve{\tilde{\mathfrak{v}}}_{e_1}, \vartheta)$ and

(***NSM***. **15**) $\npreceq (\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta + \varsigma) = \dot{\psi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta + \varsigma) = \dot{\psi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{x}}}_{e_3} + \tilde{\tilde{\mathfrak{x}}}_{e_3} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \vartheta + \varsigma)$
$\lesseqqgtr \dot{\psi}(\breve{\tilde{\mathfrak{v}}}_{e_1} - \tilde{\tilde{\mathfrak{x}}}_{e_3}, \vartheta) \star \dot{\psi}(\tilde{\tilde{\mathfrak{x}}}_{e_3} - \tilde{\tilde{\mathfrak{w}}}_{e_2}, \varsigma) = \npreceq (\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{x}}}_{e_3}, \vartheta) \star \npreceq (\tilde{\tilde{\mathfrak{x}}}_{e_3}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \varsigma)$.

(***NSM***. **16**)$\npreceq (\breve{\tilde{\mathfrak{v}}}_{e_1}, \tilde{\tilde{\mathfrak{w}}}_{e_2}, \cdot): \mathfrak{R}(\mathfrak{E})^* \to [0,1]$

Therefore $(\tilde{\Xi}, \Delta, \nabla, \npreceq, *, \diamond, \star)$ is an *NSMS*.

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