On The Foundations of Fuzzy Number Theory and Fuzzy Diophantine Equations

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Abstract

Despite the great and rapid progress in the study of Fuzzy Logic and its applications in various scientific fields, it has not yet been used to build a consistent number theory like classical number theory. This research provides for the first time a conception of the concepts of number theory based on fuzzy logic and fuzzy membership functions, where it defines the division process, the fuzzy congruence, the greatest common divisor between integers with a fuzzy membership function. On the other hand, it presents many famous Diophantine equations formulated using fuzzy sets, in addition to many properties of fuzzy number theoretical systems, through many related theorems and accompanying illustrative examples. Also, in this research, we are raising many open research questions related to fuzzy number theory, which we believe will represent the future of progress in the study of this new mathematical branch.

Keywords: Fuzzy Number Theory; fuzzy Diophantine equation; standard fuzzy number theoretical system; fuzzy divisor; fuzzy prime; fuzzy gcd.

1. Introduction

Fuzzy sets have been distinguished throughout their history as a fertile field for finding applications in various branches of mathematics, practical and real life. Fuzzy theory has been widely used in decision-making applications, abstract algebra, in mathematical analysis, and also in engineering and computer science [21-50].
Although the study of fuzzy algebraic structures has diversified throughout its history through the study of rings, groups, vector spaces as well as matrices, it becomes clear that until now no consistent theory of fuzzy integer numbers has been formulated similar to classical number theory that is concerned with the properties of integers, as concepts such as fuzzy Diophantine equations, or fuzzy congruencies, or other central concepts in number theory have not been formulated, due to the lack of concepts such as division, common divisors, prime numbers, and other basic concepts among fuzzy integer numbers.

Based on our awareness of this large research gap, we are trying in this research to formulate an abstract concept of the fuzzy number theory system, where it becomes possible to use the concept of a fuzzy membership function in constructing a definition of the division process, and for congruencies and even Diophantine equations built on the set of integers with a fuzzy membership function.
The main objective of this work is to open a door to the study of fuzzy number theory and to pose a large number of open research questions that can contribute to the development of the study of both fuzzy and classical number theoretical systems.
In addition, this research work proposes some new models through which researchers interested in generalizing classical number theory can study the same familiar concepts from the perspective of a fuzzy membership function.
This work is considered the first and foundational research aimed at opening a new research field, we call it fuzzy number theory.

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2. General versions of number theory

Classical number theory is concerned with the study of the properties of integers and their applications in various fields, and due to the breadth of this branch of theoretical mathematics, it has been generalized through many numbers theoretical systems that expand the set of integers, among which we mention:
Neutrosophic number theory and neutrosophic Diophantine equations [1-3,7-9,11-14], split-complex number theory [4,18], weak fuzzy number theory and its Diophantine equations [19-20], and Plithogenic number theory [5-6,10,15-17].
Our work on fuzzy number theory is a new attempt to generalize the traditional view of integers, and an attempt to open a new research field for researchers interested in this, accompanied by many open problems that are strongly related to the problems of classical number theory.

3. Main Discussion

Definition:
Let $\mathbb{Z}$ be the ring of integers, $\mu: \mathbb{Z} \to]0,1]$ be a membership function, we say that $(\mathbb{Z}, \mu)$ is a fuzzy number theoretical system.
Definition:
Consider the following membership function: $\mu: \mathbb{Z} \to]0,1]$ such that:

$$\mu(x) = \begin{cases} \frac{1}{|x|}; & x \neq 0 \\ 1; & x = 0 \end{cases}$$

Then $(\mathbb{Z}, \mu)$ is called the standard fuzzy number theoretical system (SFNTS)
For a fixed natural number $n \geq 2$, we define the n-standard fuzzy number theoretical system as $(\mathbb{Z}, \mu)$,with:

$$\mu: \mathbb{Z} \to]0,1]; \mu(x) = \begin{cases} \frac{1}{|x|^n}; & x \neq 0 \\ 1; & x = 0 \end{cases}$$

Definition:
Let $(\mathbb{Z}, \mu)$ be a fuzzy number theoretical system (FNTS), then for $a, b \in \mathbb{Z}$, we say that $a|b$ if and only if $\frac{\mu(b)}{\mu(a)} \in \mathbb{Z}^+$.
Example:
Take $\mu: \mathbb{Z} \to]0,1]$ such that:
$\mu(1) = \mu(3) = \mu(5) = \frac{3}{4}$, $\mu(2) = \mu(4) = \frac{1}{4}$, $\mu(7) = \mu(6) = \frac{1}{3}$,
$\mu(x) = 1; x \geq 8$ or $x \leq 0$, then we have:
$6|7$, that is because $\frac{\mu(7)}{\mu(6)} = 1 \in \mathbb{Z}^+$,
$2|1$, that is because $\frac{\mu(1)}{\mu(2)} = 3 \in \mathbb{Z}^+$
Theorem:
Let $\mu: \mathbb{Z} \to]0,1]$, then:
1] If $a|b$, then $\mu(b) \geq \mu(a)$ $\forall\, a, b \in \mathbb{Z}$
2] $a|a$ for all $a \in \mathbb{Z}$.
3] If $a|b$ and $b|c$, then $a|c$ $\forall\, a, b, c \in \mathbb{Z}$.
Proof:
1] $a|b$ if and only if $\frac{\mu(b)}{\mu(a)} \in \mathbb{Z}^+$, thus $\frac{\mu(b)}{\mu(a)} \geq 1$, so that $\mu(b) \geq \mu(a)$.
2] $\frac{\mu(a)}{\mu(a)} = 1 \in \mathbb{Z}^+$, hence $a|a$.
3] Assume that $a|b$ and $b|c$, we get:
$\frac{\mu(b)}{\mu(a)}, \frac{\mu(c)}{\mu(b)} \in \mathbb{Z}^+$, hence: $\frac{\mu(b)}{\mu(a)} . \frac{\mu(c)}{\mu(b)} = \frac{\mu(c)}{\mu(a)} \in \mathbb{Z}^+$, thus $a|c$.
Definition:
Let $(\mathbb{Z}, \mu)$ be a (FNTS), and $a, b, c \in \mathbb{Z}$, we say that $\mathrm{c} = \gcd(\mathrm{a}, \mathrm{b})$ if and only if:

1) $c|a$ and $c|b$.
2) For any $t|a$ and $t|b$, then $t|c$; $t \in \mathbb{Z}$.

Example:
Define $\mu: \mathbb{Z} \to]0,1]$ such that:
$\mu(1) = \frac{1}{2}$, $\mu(2) = \frac{1}{4}$, $\mu(3) = \mu(4) = \frac{3}{4}$, $\mu(5) = \frac{1}{4}$, $\mu(x) = 1$ for all other $x \in \mathbb{Z}$.
We can see that: $2|3$, $2|4$, $5|3$, $5|4$, $5|2$, $2|5$, hence $\gcd(3,4) \in \{2,5\}$.
Remark:
In fuzzy number theory, the (gcd) may not be unique in general.

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Definition:

1] Let $a, b \in (\mathbb{Z}, \mu)$, we say (a,b) are relatively prime if and only if they not have a common divisor.

2] $a \in \mathbb{Z}$ is called a fuzzy prime element if and only if it has not any divisor different from itself.

Example:

Define $\mu: \mathbb{Z} \to]0,1]$ such that: $\mu(x) = 1; x \geq 4$ or $x \leq 0$ and: $\mu(1) = \frac{1}{3}, \mu(2) = \frac{1}{4}, \mu(3) = \frac{1}{5}$.

It is clear that: x is not a divisor of 1, 2, 3 for all different x in Z, hence 1, 2, 3 are fuzzy prime elements in $(\mathbb{Z}, \mu)$.

Remark:

In a (FNTS) $(\mathbb{Z}, \mu)$, if a, b are fuzzy prime elements, then (a, b) are relatively prime.

Remark:

If $a \in \mathbb{Z}$ is a fuzzy prime element, then $\mu(a) \neq \mu(b)$ for all $b \in \mathbb{Z}$ and $b \neq a$.

Theorem:

Let $(\mathbb{Z}, \mu)$ be a (FNTS), $a, b, c \in \mathbb{Z}$ such that:

$c|a$ and $c|b$, then there exists $r_1, r_2 \in \mathbb{Z}$ such that:

$r_1\mu(a) + r_2\mu(b) = 0$.

Proof:

$c|a$, $c|b$ implies that: $\frac{\mu(a)}{\mu(c)} = \alpha \in \mathbb{Z}^+, \frac{\mu(b)}{\mu(c)} = \beta \in \mathbb{Z}^+$, hence

$$\begin{cases} \mu(a) = \alpha\, \mu(c) \\ \mu(b) = \beta\, \mu(c) \end{cases}$$

Assume that $l = lcm(\alpha, \beta)$, then $l = r_1\alpha$, $l = r_2\beta; r_1, r_1 \in \mathbb{Z}$,

So that:

$$\begin{cases} r_1\mu(a) = l\, \mu(c) \\ r_2\mu(b) = l\, \mu(c) \end{cases}$$

Hence: $r_1\mu(a) + (-r_2)\mu(b) = 0$

Remark:

If $c|a$ and $c|b$, then $\frac{\mu(a)}{\mu(b)} = \frac{r_2}{r_1} \in Q^+$

Also, $\mu(c) = \frac{1}{\alpha+\beta}(\mu(a) + \mu(b)); \alpha, \beta \in \mathbb{Z}^+$.

Remark:

If $c|a$ and $c|b$ and $gcd(\frac{\mu(a)}{\mu(c)}, \frac{\mu(b)}{\mu(c)}) = l$, then by classical Bezout's theorem, we can find $\alpha, \beta \in \mathbb{Z}$ such that: $\alpha\, \mu(a) + \beta(\mu(b)) = l.\mu(c)$.

Definition:

Let $(\mathbb{Z}, \mu)$ be a (FNTS), then $p \in \mathbb{Z}$ is called completely fuzzy prime element if and only if:

1) p is a fuzzy prime.
2) If $p|ab$, then $p|a$ or $p|b$.

It is clear that if $p \in \mathbb{Z}$ is a completely fuzzy prime element with $p|ab$, then:

$\frac{\mu(a)\mu(ab)}{\mu^2(p)}$ or $\frac{\mu(b)\mu(ab)}{\mu^2(p)} \in \mathbb{Z}^+$

Definition:

Let $(\mathbb{Z}, \mu)$ be a (FNTS), and $a, b, c \in \mathbb{Z}$, we define:

$a \equiv b(mod\ c)$ if and only if:

$\frac{|\mu(a)-\mu(b)|}{\mu(c)} \in \mathbb{Z}^+$

Theorem:

Let $(\mathbb{Z}, \mu)$ be a (F N T S), then:

1] $a \equiv a(mod\ b)$

2] If $a \equiv b(mod\ c)$, then $b \equiv a(mod\ c)$

3] If $a \equiv b(mod\ c)$, $b \equiv d(mod\ c)$, then $a \equiv d(mod\ c), \forall\ a, b, c \in \mathbb{Z}$.

Proof:

1] $\frac{|\mu(a)-\mu(a)|}{\mu(b)} = 0 \in \mathbb{Z}^+$, hence $a \equiv a(mod\ b)$

2] Assume that $a \equiv b(mod\ c)$, then $\frac{|\mu(a)-\mu(b)|}{\mu(c)} \in \mathbb{Z}^+$, this implies that $\frac{|\mu(b)-\mu(a)|}{\mu(c)} \in \mathbb{Z}^+$, thus $b \equiv a(mod\ c)$

3] Suppose that $a \equiv b(mod\ c)$, $b \equiv d(mod\ c)$, then

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$$\begin{cases} \frac{|\mu(a)-\mu(b)|}{\mu(c)} \in \mathbb{Z}^+ \\ \frac{|\mu(b)-\mu(d)|}{\mu(c)} \in \mathbb{Z}^+ \end{cases} ; \quad a,b,c,d \in \mathbb{Z}$$

We have the following cases:
Case(1):
If $\mu(b) \le \mu(a), \mu(d) \le \mu(b)$, then $|\mu(a)-\mu(b)| = \mu(a)-\mu(b), |\mu(b)-\mu(d)| = \mu(b)-\mu(d)$, hence:

$$\frac{|\mu(a)-\mu(d)|}{\mu(c)} = \frac{|\mu(a)-\mu(b)|}{\mu(c)} + \frac{|\mu(b)-\mu(d)|}{\mu(c)} - \frac{|\mu(a)-\mu(d)|}{\mu(c)} \in \mathbb{Z}^+$$

Case(2):
If $\mu(a) \le \mu(b) \le \mu(d)$, then $|\mu(a)-\mu(b)| = \mu(b)-\mu(a), |\mu(b)-\mu(d)| = \mu(d)-\mu(b)$, hence:

$$\frac{|\mu(a)-\mu(d)|}{\mu(c)} = \frac{|\mu(d)-\mu(a)|}{\mu(c)} = \frac{|\mu(d)-\mu(b)|}{\mu(c)} + \frac{|\mu(b)-\mu(a)|}{\mu(c)} \in \mathbb{Z}^+$$

Case(3):
If $\mu(b) \le \mu(a) \le \mu(d)$, then $|\mu(a)-\mu(b)| = \mu(a)-\mu(b), |\mu(b)-\mu(d)| = \mu(d)-\mu(b)$, hence:

$$\frac{|\mu(a)-\mu(d)|}{\mu(c)} = \frac{|\mu(d)-\mu(a)|}{\mu(c)} = \frac{\mu(d)-\mu(b)}{\mu(c)} - \frac{\mu(a)-\mu(b)}{\mu(c)}$$

Since $\frac{\mu(d)-\mu(b)}{\mu(c)}, \frac{\mu(a)-\mu(b)}{\mu(c)} \in \mathbb{Z}^+$, hence
$\frac{\mu(d)-\mu(b)}{\mu(c)} - \frac{\mu(a)-\mu(b)}{\mu(c)} \in \mathbb{Z}^+$ that is because:

$$\frac{\mu(d)-\mu(b)}{\mu(c)} \ge \frac{\mu(a)-\mu(b)}{\mu(c)}$$

Case(4):
If $\mu(d) \le \mu(a) \le \mu(b)$, we can prove it by a similar argument of case (3).
Case (5):
If $\mu(b) \le \mu(d) \le \mu(a)$, then $|\mu(a)-\mu(b)| = \mu(a)-\mu(b)$, $|\mu(b)-\mu(d)| = \mu(d)-\mu(b)$, and it holds by a similar discussion of case (3).
Case (6):
If $\mu(a) \le \mu(d) \le \mu(b)$, then $|\mu(a)-\mu(b)| = \mu(b)-\mu(a), |\mu(b)-\mu(d)| = \mu(b)-\mu(d)$, hence

$$\frac{|\mu(a)-\mu(d)|}{\mu(c)} = \frac{\mu(d)-\mu(a)}{\mu(c)} = \frac{\mu(b)-\mu(a)}{\mu(c)} - \frac{\mu(b)-\mu(d)}{\mu(c)} \in \mathbb{Z}^+$$

So that, the proof is complete.

Remark:
According to the previous theorem, we can see that $(\equiv)$ is an equivalence relation on the fuzzy number theoretical system $(\mathbb{Z}, \mu)$.
We denote the equivalence class of $(a) \in \mathbb{Z}$ modulo (c) as follows: $[a]_c = \{b \in \mathbb{Z}; b \equiv a(modc)\}$
Theorem:
Let $(\mathbb{Z}, \mu)$ be a (FNTS), then:
1] If $\mu(a) = \mu(b)$, then $a \equiv b\ (modc)$ for all $c \in \mathbb{Z}$.
2] If $c|a$, $c|b$, then $a \equiv b\ (modc)$.
3] If $c|a$, and $a \equiv b\ (modc)$, then $c|b$.
Proof:
1]
$\frac{|\mu(a)-\mu(b)|}{\mu(c)} = 0 \in \mathbb{Z}^+$, hence $a \equiv b\ (modc)$.
2] Assume that $c|a$, $c|b$, hence $\frac{\mu(a)}{\mu(c)}, \frac{\mu(b)}{\mu(c)} \in \mathbb{Z}^+$.
If $\mu(a) \ge \mu(b)$, then $\frac{|\mu(a)-\mu(b)|}{\mu(c)} = \frac{\mu(a)}{\mu(c)} - \frac{\mu(b)}{\mu(c)} \in \mathbb{Z}^+$
If $\mu(a) \le \mu(b)$, then $\frac{|\mu(a)-\mu(b)|}{\mu(c)} = \frac{\mu(b)}{\mu(c)} - \frac{\mu(a)}{\mu(c)} \in \mathbb{Z}^+$,
Thus $a \equiv b\ (modc)$.
3] Assume that $a \equiv b\ (modc)$, $c|a$, then:
$\frac{\mu(a)}{\mu(c)}, \frac{|\mu(a)-\mu(b)|}{\mu(c)} \in \mathbb{Z}^+$, so that:
$\frac{\mu(a)}{\mu(c)} - \frac{\mu(b)}{\mu(c)} = \alpha \in \mathbb{Z}$, hence $\frac{\mu(b)}{\mu(c)} = -\alpha + \frac{\mu(a)}{\mu(c)} \in \mathbb{Z}^+$, which means that $c|b$.
Definition:
Let $(\mathbb{Z}, \mu)$ be a (FNTS), add $a, c \in \mathbb{Z}$, then:

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1] a is called a fuzzy idempotent element modulo c if and only if $a \equiv a^2 (mod\ c)$, i.e. $\frac{|\mu(a)-\mu(a^2)|}{\mu(c)} \in \mathbb{Z}^+$

2] a is called a fuzzy nilpotent element with order n if and only if $a \equiv a^n (mod\ c)$, $a \not\equiv a^k (mod\ c)$; k<n.
Which is equivalent to:

$$\begin{cases} \frac{|\mu(a)-\mu(a^n)|}{\mu(c)} \in \mathbb{Z}^+ \\ \frac{|\mu(a)-\mu(a^k)|}{\mu(c)} \notin \mathbb{Z}^+, k < n \end{cases}$$

Definition:
Let $(\mathbb{Z}, \mu)$ be a (FNTS), we define:
1] The linear Diophantine fuzzy equation in one variable $a.\mu(x) = \mu(b)$; $a \in \mathbb{Z}$, $x, b \in \mathbb{Z}$.
2] The linear fuzzy Diophantine equation in two variables:
$a.\mu(x) + b.\mu(y) = l.\mu(c)$; $a, b, l, c, x, y \in \mathbb{Z}$.
3] The fuzzy Pythagoras triple $(x\ , y, z)$:

$$(\mu(x))^2 + (\mu(y))^2 = (\mu(z))^2$$

4] The fuzzy Pythagoras quadruple (x, y, z, t):

$$(\mu(x))^2 + (\mu(y))^2 + (\mu(z))^2 = (\mu(t))^2$$

5] The fuzzy Fermat's triples (x, y, z):

$$(\mu(x))^n + (\mu(y))^n = (\mu(z))^n; \quad n \geq 3$$

4. Results in the standard system:

Consider the standard fuzzy number theoretical system $(\mathbb{Z}, \mu)$ such that: $\mu: \mathbb{Z} \to]0,1]$: $\mu(x) = \begin{cases} \frac{1}{|x|}; & x \neq 0 \\ 1; & x = 0 \end{cases}$

We will discuss the fuzzy number theoretical concepts through this system.
To distinguish between $(\mathbb{Z}, \mu)$ and $\mathbb{Z}$, we denote all concepts by (F), for example:
$a|_F\ b$, $a \equiv_F b(mod\ c)$, $gcd_F(a, b)$, …. And so on.

Theorem:
Let $(\mathbb{Z}, \mu)$ be the standard fuzzy number theoretical system, then:
1] $y|_F\ x \Leftrightarrow x|y$
2] $a \equiv_F b(modc) \Leftrightarrow ab||c(|b| - |a|)|$
3] $(\mathbb{Z}, \mu)$ has no fuzzy primes.
4] If $c = gcd_F(x, y) \Leftrightarrow c = lcm(x, y)$
5] The one variable Diophantine equation $a.\mu(x) = \mu(b)$ in $(\mathbb{Z}, \mu)$ is solvable with $x = ba$ or $x = -ba$ if $a, b, x \neq 0$.
Proof:
1] Assume that $y|_F\ x$, then $\frac{\mu(x)}{\mu(y)} = \frac{|y|}{|x|} \in \mathbb{Z}^+$, hence $x|y$. The converse is clear.
2] Assume that $a \equiv_F b(modc)$, then $\frac{|\mu(a)-\mu(b)|}{\mu(c)} \in \mathbb{Z}^+$, hence $\frac{\left|\frac{1}{|a|}-\frac{1}{|b|}\right|}{\frac{1}{|c|}} \in \mathbb{Z}^+$, then $\frac{|c|.||b|-|a||}{|ab|} \in \mathbb{Z}^+$, thus $\frac{|c(|b|-|a|)|}{|ab|} \in \mathbb{Z}^+$, hence $ab||c(|b| - |a|)|$
The converse is clear.
4] $c|_F\ x, c|_F\ y$, then $x|c, y|c$.
For any $t \in \mathbb{Z}$ such that $t|_F\ x, t|_F\ y$, we get $t|_F\ c$, thus $x|t,\ y|t,\ c|t$, hence $c = lcm(x, y)$.
The converse is clear.
3] If $p \in \mathbb{Z}$ is a fuzzy prime element, then
$\nexists\ x \in \mathbb{Z}: \frac{\mu(p)}{\mu(x)} \in \mathbb{Z}^+$, hence $\nexists\ x \in \mathbb{Z}: \frac{|x|}{|p|} \in \mathbb{Z}^+$, which is impossible.
5] $a.\mu(x) = \mu(b)$ equivalents: $\frac{a}{|x|} = \frac{1}{|b|} \Leftrightarrow |x| = a.|b| \Leftrightarrow x = ba$ or $x = -ba$ for $x, a, b \neq 0$.

Remark:
If $b = 0$, then $x = a$ or $x = -a$; $a \neq 0$.
If $a = 0$, then the equation is not solvable.
If $x = 0$, then $a = \frac{1}{|b|}$, which is possible if and only if

$$\begin{cases} b = 1\ or\ -1\ and\ a = 1 \\ b = 0, a = 1 \end{cases}$$

Theorem:
Let $(\mathbb{Z}, \mu)$ be the (SFNTS), then:
1] a is idempotent modulo (c) if and only if $a^2|c.(|a| - 1)$.
2] a is nilpotent of order n modulo (c) if and only if $a^n\big||c|.(|a^{n-1}| - 1)$, and a^k is not a divisor of $\big||c|.(|a^{k-1}| - 1)$ for k<n.
Proof:

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1] a is idempotent modulo (c) if and only if $a \equiv_F a^2 (mod\ c)$, hence $\frac{\frac{1}{|a|}-\frac{1}{|a^2|}}{\frac{1}{|c|}} = |c|.(\frac{1}{|a|}-\frac{1}{a^2}) \in \mathbb{Z}^+$, hence $\frac{|c|.(|a|-1)}{a^2} \in \mathbb{Z}^+$, which is equivalent to $a^2 \Big| |c|.(|a|-1)$.

2] $a \equiv_F a^n (mod\ c)$ if and only if: $\frac{\frac{1}{|a|}-\frac{1}{|a^n|}}{\frac{1}{|c|}} = |c|.\frac{(|a^{n-1}|-1)}{|a^n|} \in \mathbb{Z}^+$, thus $|a^n| \Big| |c|.(|a^{n-1}|-1)$. On the other hand $\frac{|\mu(a)-\mu(a^k)|}{\mu(c)} \notin \mathbb{Z}^+, k < n$, thus a^k is not a divisor of $\Big||c|.(|a^{k-1}|-1)$ for k<n.

Example:
For $c = 36$, we have $a = 3$ is idempotent modulo (36), that is because: $a^2 = 9|36.(2) = 72$.
We can see that: $\mu(a) - \mu(a^2) = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$, $\mu(c) = \frac{1}{36}$, $\frac{\mu(a)-\mu(a^2)}{\mu(c)} = 8 \in \mathbb{Z}^+$, hence $a \equiv a^2 (mod\ c)$.

5. The classical formulas of some Diophantine equations in the standard system.

1] The linear Diophantine equation in two variables:
$a.\mu(x) + b.\mu(y) = l.\mu(c) \Leftrightarrow \frac{a}{|x|} + \frac{b}{|y|} = \frac{l}{|c|} \Leftrightarrow a.|cy| + b.|cx| = l.|xy|; a, b, c, l, x, y \neq 0.$
2] Fuzzy Pythagoras triples:

$$(\mu(x))^2 + (\mu(y))^2 = (\mu(z))^2 \Leftrightarrow \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2} \Leftrightarrow z^2(y^2 + x^2) = x^2y^2; \quad x, y, z \neq 0$$

3] Fuzzy Pythagoras quadruples:

$$(\mu(x))^2 + (\mu(y))^2 + (\mu(z))^2 = (\mu(t))^2 \Leftrightarrow \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{t^2}$$

4] Fuzzy Fermat's triples:

$$(\mu(x))^n + (\mu(y))^n = (\mu(z))^n \Leftrightarrow \frac{1}{|x|^n} + \frac{1}{|y|^n} = \frac{1}{|z|^n}; \quad n \geq 3$$

Example:
In the (SFNTS) $(\mathbb{Z}, \mu)$, consider the linear Diophantine equation in two variables.
$2\mu(x) + 3\mu(y) = \mu(5)$, it is equivalent to:

$$\frac{2}{|x|} + \frac{3}{|y|} = \frac{1}{5} \Leftrightarrow 10|y| + 15|x| - |xy| = 0$$

Put $Z = 10|y| - |xy|$, then:

$$\begin{cases} 15|x| + Z = 0 & (1) \\ Z = 10|y| - |x|.|y| & (2) \end{cases}$$

The equation (1) is solvable in $\mathbb{Z}$, let us consider the solution $(x_0, z_0) = (3, -45)$, By using (2), we get: $-45 = 10|y| - 3|y| \Leftrightarrow |y| = \frac{-45}{7}$ which is contradiction.
In general, $Z = -15.|x|$, thus: $-15|x| = 10|y| - |x|.|y|$, so that $|y| = \frac{-15|x|}{10-|x|}$.
The possible solutions (x, y) should have the property $10 - |x| | -15|x|$, and $10 - |x| < 0$.
Now, we can write: $10 - |x|\ | -15|x|$, hence $\exists\ q \in \mathbb{Z}$ such that $-15|x| = q(10 - |x|)$, thus $|x| = \frac{10q}{-15+q}$.
Also, $|y| = \frac{-15\left(\frac{10q}{-15+q}\right)}{10-\frac{10q}{-15+q}}$ under the condition $-15 + q|10q$ for all $q \in \mathbb{Z}$, and $\frac{10q}{-15+q} > 0$.
For example: if q=20, then $|x| = 40$, $|y| = 20$, we get a solution.

6. Recommendations and open research problems:

This work can be considered as first work about fuzzy number theory, and the applications of fuzzy logic in Diophantine equations. We expect that this work will open a huge number of research problems that concern fuzzy number theory and Diophantine equations.
We list some of open problems.
1] Define a suitable fuzzy number theoretical system with some conditions on μ, and try to find the related formulas for congruencies, solutions for fuzzy linear Diophantine equations, fuzzy Pythagoras triples, and so on.
2] Find an algorithm to solve linear fuzzy Diophantine equation in two variables in the standard system or n-standard system. (or in any suitable system you defined).
3] Try to generalize famous Diophantine equations and number theoretical concepts in to the standard system/ n-standard system.
4] Try to use fuzzy number theory in cryptography, especially crypto-algorithms (RSA, ElGamal,…).
5] Try to find algorithms for generating Pythagoras triples, quadruples, and Fermat's triples in the standard system, n- standard system, or any other suitable fuzzy number theoretical system you defined.

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Suggestions for some interesting fuzzy number theoretical systems:

1] $\mu_1: \mathbb{Z} \to]0,1]: \mu_1(x) = \begin{cases} \frac{1}{|x|};\ x \geq 1 \\ \frac{1}{x^2};\ x \leq -1 \\ 1\ ; x = 0 \end{cases}$

2] $\mu_2: \mathbb{Z} \to]0,1]: \mu_2(x) = \begin{cases} \frac{1}{|x|};\ a \leq x \leq b\ \ a, b \in \mathbb{Z} \\ c;\ x < a, x > b\ \ c \in]0,1] \end{cases}$

3] $\mu_3: \mathbb{Z} \to]0,1]: \mu_3(x) = \begin{cases} \frac{c}{|x|}; & |x| \geq c \\ \frac{|x|}{c}; & |x| < c \end{cases} \quad c \in \mathbb{R}^+$

4] $\mu_4: \mathbb{Z} \to]0,1]: \mu_4(x) = \begin{cases} \frac{c}{|x|^n}; & |x|^n \geq c \quad c \in \mathbb{R}^+ \\ \frac{|x^n|}{c}; & |x^n| < c \quad n \in \mathbb{N} \end{cases}$

5] $\mu_5: \mathbb{Z} \to]0,1]: \mu_5(x) = \begin{cases} 1; & x \in S \leq \mathbb{Z} \\ \frac{1}{|x|^n}; & x \notin S\ \ ; n \in \mathbb{N} \end{cases}$

6] $\mu_6: \mathbb{Z} \to]0,1]: \mu_6(x) = \begin{cases} \frac{1}{\sqrt[n]{|x|}} & x \neq 0 \\ 1; & x = 0 \end{cases};\ \ n \in \mathbb{N}$

7] $\mu_7: \mathbb{Z} \to]0,1]: \mu_7(x) = \begin{cases} \frac{1}{|x|^k}; & x \in S_1 \leq \mathbb{Z} \\ 1; & x = 0 \\ l\ \frac{1}{\sqrt[n]{|x|}}; & x \notin S_1 \end{cases}$

8] $\mu_8: \mathbb{Z} \to]0,1]: \mu_8(x) = \left\{ \frac{1}{|x|!};\quad x \in \mathbb{Z} \right.$

9] $\mu_9: \mathbb{Z} \to]0,1]: \mu_9(x) = \begin{cases} \frac{c}{|x|!}; & |x|! \geq c \\ \frac{|x|!}{c}; & |x|! < c \end{cases} \quad c \in \mathbb{R}^+$

10] $\mu_{10}: \mathbb{Z} \to]0,1]: \mu_{10}(x) = \left\{ \frac{1}{(|x|!)^n};\quad x \in \mathbb{Z},\ n \in \mathbb{N} \right.$

For every system, we must study its relations, concepts, Diophantine equations..,etc.

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