Advanced Decision-Making Techniques with Generalized Close Sets in Neutrosophic Soft Bitopological Spaces

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Abstract
In recent years, neutrosophic soft bitopological spaces have emerged as a promising framework for handling uncertainty and imprecision in various domains, particularly in the context of decision-making problems. This paper presents a comprehensive study of advanced decision-making techniques using generalized close sets in neutrosophic soft bitopological spaces. The primary objective of this research is to develop a better understanding of the theoretical underpinnings of generalized close sets and their practical applications in decision-making under uncertain conditions. We begin by providing a detailed introduction to the key concepts of neutrosophic sets, neutrosophic soft sets, and neutrosophic soft bitopological spaces. Subsequently, we introduce generalized close sets and discuss their properties and interrelationships with other relevant constructs in the field. The paper then delves into the decision-making aspect, presenting various methodologies for solving decision-making problems using generalized close sets in the context of neutrosophic soft bitopological spaces. Numerous illustrative examples and case studies are provided throughout the paper to demonstrate the applicability and effectiveness of the proposed techniques in handling complex decision-making problems in real-world scenarios. The results of this research not only contribute to the existing body of knowledge in the field but also offer valuable insights for practitioners and researchers seeking to employ advanced decision-making techniques in uncertain environments.

Keywords: Neutrosophic Soft Bitopological Spaces; Generalized Close Sets; Decision-Making; Uncertainty; Neutrosophic Sets; Neutrosophic Soft Sets.

1. Introduction

In today's rapidly changing world, decision-makers are frequently faced with complex and ambiguous situations that involve uncertainty, imprecision, and vagueness. Traditional mathematical tools and techniques often fall short in effectively addressing these challenges. As a result, there is a growing need for more advanced and flexible frameworks that can accommodate the complexities of real-world decision-making problems. One such framework is the neutrosophic soft bitopological space, which has emerged as a powerful tool for modeling and analyzing uncertain situations. Neutrosophic logic and its applications played a central role in the study of pure mathematical branches such as geometry, matrix theory, space theory, and computer science [9-13].

Neutrosophic soft bitopological spaces combine the strengths of neutrosophic sets, soft sets, and bitopological spaces, offering a robust and versatile structure for handling uncertainty and imprecision. In recent years, there has been a surge of interest in the development and application of neutrosophic soft bitopological spaces in various fields, such as engineering, economics, and social sciences. However, the potential of these spaces in the context of decision-making problems remains largely unexplored.

The primary focus of this paper is to investigate the potential of generalized close sets in neutrosophic soft bitopological spaces for solving advanced decision-making problems. We will begin by providing a thorough introduction to the key concepts of neutrosophic sets, neutrosophic soft sets, and neutrosophic soft bitopological spaces. This will be followed by a detailed discussion of generalized close sets, their properties, and their relationships with other constructs in the field.

Next, we will explore various methodologies for solving decision-making problems using generalized close sets within the framework of neutrosophic soft bitopological spaces. This will include the development of new algorithms and techniques for modeling and analyzing decision problems under uncertain conditions. To

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demonstrate the practical applicability and effectiveness of the proposed methods, we will provide numerous examples and case studies from diverse domains.
By presenting a comprehensive study of advanced decision-making techniques with generalized close sets in neutrosophic soft bitopological spaces, we aim to contribute to the existing body of knowledge in the field and provide valuable insights for practitioners and researchers working in uncertain environments.

2. Preliminary

This section provides fundamental definitions and theorems related to neutrosophic set theory and neutrosophic soft set theory.

2.1. Neutrosophic sets [1]

Consider a space X, composed of points or objects, where a generic element in X is represented by x. A neutrosophic set A within X is defined by a truth-membership function T, an indeterminacy-membership function I, and a falsity-membership function F [15]. In essence, T, I, and F are functions mapping X to real standard or non-standard subsets of the interval [-0, 1+]. Typically, there are no constraints on the sum of T(x), I(x), and F(x), resulting in -0 ≤ T(x) + I(x) + F(x) ≤ 3+. The neutrosophic components are T, I, and F, and the collection of all neutrosophic sets in X is represented by N(X).

2.1 Definition [1]: Let $S, M \in N(X)$.

1. Subset: $M \subset S$ if $T_M(b) \leq T_S(b), I_M(b) \leq I_S(b), F_M(b) \geq F_S(b)$ for all $b \in X$.
2. Equality: $M = S$ if $M \subset S$ and $S \subset M$.
3. Union:
$$M \cup S = \{< b, max\{T_M(b), T_S(b)\}, max\{I_M(b), I_S(b)\}, min\{F_M(b), F_S(b)\} > : b \in X\}.$$
4. Intersection:
$$M \cap S = \{< b, min\{T_M(b), T_S(b)\}, min\{I_M(b), I_S(b)\}, max\{F_M(b), F_S(b)\} > : b \in X\}.$$

More generally, the intersection and the union of a collection of neutrosophic sets $\{M_i\} \in N(X)$ are defined by:
The intersection and union of a collection of neutrosophic sets $\{\{M_i\} \in N(X)$ are defined more generally as follows:

$$\cap_{i \in I} M_i = \left\{< b, min\left\{T_{M_i}(b)\right\}, min\left\{I_{M_i}(b)\right\}, max\left\{F_{M_i}(b)\right\} >: b \in X\right\},$$
$$\cup_{i \in I} M_i = \left\{< b, max\left\{T_{M_i}(b)\right\}, max\left\{I_{M_i}(b)\right\}, min\left\{F_{M_i}(b)\right\} >: b \in X\right\}.$$

5. The neutrosophic set defined as $T_M(b) = 1, I_M(b) = 1$ and $F_M(b) = 0$ for all $b \in X$ is called the universal NS denoted by 1_X. Also, the neutrosophic set defined as $T_M(b) = 0, I_M(b) = 0$ and $F_M(b) = 1$ for all $b \in X$ is called the empty NS denoted by 0_X.
6. Complement: $M^c = 1_X \backslash M$

2.2 Definition [2]: A subset Γ of N(Y) is termed a neutrosophic topology on Y if it fulfills the following conditions:
The empty neutrosophic set 0_X and the universal neutrosophic set 1_X are elements of Γ.
The union of any number of neutrosophic sets in Γ is also an element of Γ.
The intersection of a finite number of neutrosophic sets in Γ is an element of Γ.
The pair (Y, Γ) is then called a neutrosophic topological space on Y.

2.3. Neutrosophic Soft Set

Definition [4]: Let P be an initial universe set, and let E be a set of parameters. The pair (L, E) is defined as a neutrosophic soft set (NSS) over P, where L is a mapping from E to N(P).
The set of all neutrosophic soft sets over P is denoted by NSS(P). A neutrosophic soft set (L, E) can be expressed as follows:

$$(L, E) = \{(e, \{< x, T_{L(e)}(x), I_{L(e)}(x), F_{L(e)}(x) >\} : x \in P, e \in E\}.$$

2.4 Definition [4]:

Let X be an initial universe set, and let E be a set of parameters. Then, the neutrosophic soft set $x^e{}_{(\alpha,\beta,\gamma)}$ is defined as:

$x^e{}_{(\alpha,\beta,\gamma)}$ = {(x, T(x,α), I(x,β), F(x,γ)) | x ∈ X},

where T(x,α), I(x,β), and F(x,γ) are the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively, defined on X. The values of these functions are given by:

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T(x,α) = {0, 1, α}, I(x,β) = {0, 1, β}, and F(x,γ) = {0, 1, γ},

where 0 ≤ α,β,γ ≤ 1.

Thus, the neutrosophic soft set ${x^e}_{(\alpha,\beta,\gamma)}$ assigns to each element x in X a truth-membership value α, an indeterminacy-membership value β, and a falsity-membership value γ.

2.5 Definition [3]: Let $(w,E),(M,E) \in NSS(P)$. Then for all $x \in P$

1. Subset: $(w,E) \subset (M,E)$ if $T_{w(e)}(x) \le T_{M(e)}(x), I_{w(e)}(x) \le I_{M(e)}(x)$ and $F_{w(e)}(x) \ge F_{M(e)}(x)$ for all $e \in E$,

2. Equality: $(w,E) = (M,E)$ if $(w,E) \subset (M,E)$ and $(M,E) \subset (w,E)$,

3. Intersection:

$$(w,E) \cap (z,E) = \{(e, \{< x, min\{T_{w(e)}(x), T_{z(e)}(x)\}, max\{I_{w(e)}(x), I_{z(e)}(x)\}, max\{F_{w(e)}(x), F_{z(e)}(x)\} >\} : e \in E\},$$

4. Union:

$$(w,E) \cup (z,E) = \{(e, \{< x, max\{T_{w(e)}(x), T_{z(e)}(x)\}, min\{I_{w(e)}(x), I_{z(e)}(x)\}, min\{F_{w(e)}(x), F_{z(e)}(x)\} >\} : e \in E\}.$$

More generally, the intersection and the union of a collection of $\{(w_i,E)\} \subset NSS(P)$ are defined by:

$$\cup_{i\in I}(w_i,E) = \left\{(e, \left\{< x, max\left\{T_{w_{i(e)}}(x)\right\}, max\left\{I_{w_{i(e)}}(x)\right\}, min\left\{F_{w_{i(e)}}(x)\right\} >\right\} : e \in E\right\}$$

$$\cap_{i\in I}(w_i,E) = \left\{(e, \left\{< x, min\left\{T_{w_{i(e)}}(x)\right\}, min\left\{I_{w_i(e)}(x)\right\}, max\left\{F_{w_i(e)}(x)\right\} >\right\} : e \in E\right\}$$

5. The NSS defined as and $F_{w(e)}(x) = 0$, for all $e \in E$ and $x \in P$ is called the universal NSS denoted by $1_{(P,E)}$. Also, the neutrosophic set defined as $T_{w(e)}(x) = 0, I_{w(e)}(x) = 0$ and $F_{w(e)}(x) = 1$ for all $e \in E$ and $x \in P$ is called the empty NSS denoted by $0_{(P,E)}$.

The NSS defined as $T_{w(e)}(x) = 1, I_{w(e)}(x) = 1$, and $F_{w(e)}(x) = 0$ for all e∈E and x∈P is called the universal NSS and is denoted by $1_{(P,E)}$. Similarly, the NSS defined as as $T_{w(e)}(x) = 0, I_{w(e)}(x) = 0$, and$F_{w(e)}(x) = 1$ for all e∈E and x∈P is called the empty NSS and is denoted by $0_{(P,E)}$.

6. Complement: $(w,E)^c = 1_{(P,E)} \backslash (w,E) = \{e, \{x, F_{w(e)}(x), 1 - I_{w(e)}(x), T_{w(e)}(x) >\} : e \in E\}$

Clearly, the complements of $1_{(X,E)}$ and $0_{(X,E)}$ are defined:

$$(1_{(P,E)})^c = 1_{(P,E)} \backslash 1_{(P,E)} = \{(e, \{< x, 0,0,1 >\} : e \in E\} = 0_{(P,E)}$$

$$(0_{(P,E)})^c = 1_{(P,E)} \backslash 0_{(P,E)} = \{(e, \{< x, 1,0,0 >\} : e \in E\} = 1_{(P,E)}$$

$0_{(Y;E)}$ and $1_{(Y,E)}(X,\tau_1,\tau_2,E), (X,\tau_1,E), (X,\tau_2,E)$ and (X,τ_3,E) $(X,\tau_1,\tau_2,\tau_3,E)$

2.6 Definition [4]: The neutrosophic soft topology on a set Y is defined as follows: Let Γ be a subset of NSS(Y). Γ is called a neutrosophic soft topology on Y if the following conditions hold:

NST1) The empty NSS($0_{(P,E)}$) and the universal NSS ($1_{(P,E)}$) belong to Γ.

NST2) The union of any number of NSSs in Γ again belongs to Γ.

NST3) The intersection of a finite number of NSSs in Γ belongs to Γ.

The pair (Y,Γ) is called a neutrosophic soft topological space. The elements of Γ are called neutrosophic soft open sets. An NSS whose complement is neutrosophic soft open is called a neutrosophic soft closed set.

2.7 Definition [5]: A neutrosophic soft supra topology on Y is denoted by Γ ⊂ NSS(Y) that satisfies the following conditions: $0_{(Y;E)}$ and $1_{(Y,E)}$ belong to Γ, and the union of any number of NSSs in Γ is again a member of Γ.

2.8 Definition [6]: A neutrosophic soft bitopological space is denoted by (X,τ_1,τ_2,E), where (X,τ_1,E) and (X,τ_2,E) are two neutrosophic soft topological spaces. The sets belonging to τ_i are called neutrosophic soft i-open sets for i=1,2.

2.9 Definition [7] [8] : If $(X,\tau_1,E), (X,\tau_2,E)$ and (X,τ_3,E) are three neutrosophic soft topological spaces, then $(X,\tau_1,\tau_2,\tau_3,E)$ is named as a neutrosophic soft tri-topological space. The sets belonging to τ_i are called neutrosophic soft i-open sets for i=1,2,3. [8]

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3. Generalized Close Sets in Neutrosophic Soft Bitopological Spaces

3.1 Definition :Let (X, U, τ_1, τ_2) be the neutrosophic soft bitopological space defined above, and let (N, U) be a neutrosophic soft subset of X. Then, we can define the following generalized closed sets:

1. ij-neutrosophic soft preclosed (ij-NSPC) if (((N, E)) ⊂(N, E).
2. ij-neutrosophic soft semi-closed (ij-NSSC) if (((N, E)) ⊂(N, E).
3. ij-neutrosophic soft b-closed (ij-NSBC) if ((N, E)) ∩ (((N, E)) ⊂ (N, E).
4. ij-neutrosophic soft β-closed (ij-NSβC) if(((N, U))) ⊂ (N, U).

Note that i, j = 1, 2, and i ≠ j.

Example 4.2 continued: Let us find some generalized closed sets in the neutrosophic soft bitopological space (X, U, τ1, τ2) defined in the previous example.

1. neutrosophic soft pre-closed sets:

- (A, U) is not neutrosophic soft pre-closed since its interior in τ1, (A,U)={(x_2,<0.1,0.8,0.4>)} does not contain (B, U).
- (B, U) is not neutrosophic soft pre-closed since its interior in τ2, (B,U)={(x1,<0.1,0.3,0.4>)} does not contain (A, U).
- (C, U) is neutrosophic soft pre-closed since ={(x_1,<0.2,0.1,0.4>), (x_2,<0.2,0.1,0.4>), (x_3,<0.3,0.3,0.7>)} Contains and ={(x_1,<0.2,0.1,0.4>), (x_2,<0.5,0.1,0.7>), (x_3,<0.5,0.1,0.4>)} contains (B, U).
- (D, U) is not neutrosophic soft pre-closed since its interior in τ_2, ={(x_2,<0.1,0.6,0.8>)} does not contain (A, U).

2. neutrosophic soft pre-closed sets:

- is not neutrosophic soft pre-closed since its interior in τ_2, ={(x_2,<0.1,0.8,0.4>)} does not contain (B, U).
- is neutrosophic soft pre-closed since ={(x_1,<0.2,0.7,0.8>), (x_2,<0.5,0.6,0.7>), (x_3,<0.1,0.8,0.8>)} contains .
- is not neutrosophic soft pre-closed since its interior in τ_1, ={(x_2,<0.2,0.1,0.4>)} does not contain (D, U).
- is not neutrosophic soft pre-closed since its interior in τ_1, ={(x_2,<0.1,0.3,0.9>)} does not contain (C, U).

3. neutrosophic soft pre-closure of (A, U):

The neutrosophic soft pre-closure of (A, U) is given by cl_{12}(A,U) = {(x_2,<0.1,0.8,0.4>), (x_3,<0.3,0.4,0.8>), (x_1,<0.2,0.1,0.9>), (x1,<0,0.1,0.9>), (x1,<0.6,0.1,0.7>)}.

To compute the neutrosophic soft pre-closure of (A,U), we first need to calculate the pre-interior of (A,U) with respect to both topologies. Using the definitions of pre-interior, we have:

int_{11}(A,U) = {x_1} int_{21}(A,U) = {x_2,x_3}

Next, we take the closure of these pre-interiors with respect to the other topology. Using the definitions of closure, we have:

cl_{22}(int_{11}(A,U),U) = {x_1,<0.6,0.1,0.7>} cl_{12}(int_{21}(A,U),U) = {x_2,<0.1,0.8,0.4>), (x_3,<0.3,0.4,0.8>)}

Finally, we take the intersection of these two sets to obtain the neutrosophic soft pre-closure of (A,U):

= {x_1,<0.6,0.1,0.7>} ∩ {x_2,<0.1,0.8,0.4>), (x_3,<0.3,0.4,0.8>)} = {(x_2,<0.1,0.8,0.4>), (x_3,<0.3,0.4,0.8>), (x_1,<0.2,0.1,0.9>), (x_1,<0.6,0.1,0.7>)}

Example 4.2. Let the neutrosophic soft bitopological space (X, U, τ1, τ2) be defined as follows: X = {x1, x2, x3}, U = {e1, e2}, τ1 = {0(X,U), 1(X,U), (A, U), (B, U), (C, U), (D, U)}, τ2 = {0(X,U), 1(X,U), (E, U), (F, U), (G, U), (H, U)}

1. The tabular representations of the neutrosophic soft sets (NSSs) are as follows:

(A, U):

X	*e₁*	*e₂*
x_1	<0.2, 0.1, 0.9>	<0.6, 0.1, 0.7>
x_2	<0.1, 0.8, 0.4>	<0.1, 0.1, 0.8>
x_3	<0.3, 0.4, 0.8>	<0.5, 0.1, 0.4>

(B, U) =

X	e_1	e_2
x_1	<0.1, 0.3, 0.4>	<0.2, 0.7, 0.8>
x_2	<0.2, 0.1, 0.5>	<0.5, 0.6, 0.7>
x_3	<0.3, 0.3, 0.7>	<0.1, 0.8, 0.8>

(C, U) =

X	e_1	e_2
x_1	<0.2, 0.1, 0.4>	<0.6, 0.1, 0.7>

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x_2	<0.2, 0.1, 0.4>	<0.5, 0.1, 0.7>
x_3	<0.3, 0.3, 0.7>	< 0.5, 0.1, 0.4 >

(D, U) =

X	e_1	e_2
x_1	<0.1, 0.3, 0.9>	<0.2, 0.7, 0.8>
x_2	<0.1, 0.8, 0.5>	<0.1, 0.6, 0.8>
x_3	<0.3, 0.4, 0.8>	<0.1, 0.8, 0.8>

(E, U) =

X	e_1	e_2
x_1	<0.2, 0.6, 0.3>	<0.1, 0.1, 0.9>
x_2	<0.3, 0.7, 0.4>	<0.3, 0.4, 0.6>
x_3	<0.3, 0.5, 0.8>	<0.6, 0.1, 0.8>

(F, U) =

X	e_1	e_2
x_1	<0.1, 0.1, 0.8>	<0.1, 0.8, 0.9>
x_2	<0.3, 0.2, 0.5>	<0.3, 0.5, 0.7>
x_3	<0.7, 0.5, 0.6>	<0.1, 0.7, 0.7>

(G, U) =

X	e_1	e_2
x_1	<0.2, 0.1, 0.3>	<0.1, 0.1, 0.9>
x_2	<0.3, 0.2, 0.4>	<0.3, 0.4, 0.6>
x_3	<0.7, 0.5, 0.6>	<0.6, 0.1, 0.7>

(H, U) =

X	e_1	e_2
x_1	<0.1, 0.6, 0.8>	<0.1, 0.8, 0.9>
x_2	<0.3, 0.7, 0.5>	<0.3, 0.5, 0.7>
x_3	<0.3, 0.5, 0.8>	<0.1, 0.7, 0.8>

Let an NSS (W, U) be defined as:
(W, U):

X	e_1	e_2
x_1	<0.3, 0.7, 0.2>	<0.8, 0.2, 0.3>
x_2	<0.4, 0.8, 0.3>	<0.6, 0.4, 0.6>
x_3	<0.5, 0.5, 0.7>	<0.7, 0.2, 0.3>

Then intτ1(W, U) = (A, U), intτ2(W, U) = (E, U).
(W, U) is a subset of clτ2(A, U):

X	e_1	e_2
x_1	<0.8, 0.9, 0.1>	<0.9, 0.2, 0.1>
x_2	<0.5, 0.8, 0.3>	<0.7, 0.5, 0.3>
x_3	<0.6, 0.5, 0.7>	<0.7, 0.3, 0.1>

Then (W, U) is a 12-NSO set.
(W, U) is a subset of clτ1(E, U):

X	e_1	e_2
x_1	<0.4, 0.7, 0.1>	<0.8, 0.3, 0.2>
x_2	<0.5, 0.9, 0.2>	<0.7, 0.4, 0.5>
x_3	<0.7, 0.7, 0.3>	<0.8, 0.2, 0.1>

Then (W, U) is also a 21-NSO set.
clτ1(W, U):

X	e_1	e_2
x_1	<0.4, 0.9, 0.2>	<0.7, 0.9, 0.6>
x_2	<0.4, 0.9, 0.2>	<0.7, 0.9, 0.5>
x_3	<0.7, 0.7, 0.3>	<0.4, 0.9, 0.5>

clτ2(W, U):

X	e_1	e_2
x_1	< 0.8, 0.9, 0.1 >	< 0.9, 0.2, 0.1 >
x_2	< 0.5, 0.8, 0.3 >	< 0.7, 0.5, 0.3 >
x_3	< 0.6, 0.5, 0.7 >	< 0.7, 0.3, 0.1 >

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Then (W, U) is a 12-NSO set.
(W, U) ⊂ clτ1(E, U) =

X	e_1	e_2
x_1	< 0.4, 0.7, 0.1 >	< 0.8, 0.3, 0.2 >
x_2	< 0.5, 0.9, 0.2 >	< 0.7, 0.9, 0.5 >
x_3	< 0.7, 0.7, 0.3 >	< 0.4, 0.9, 0.5 >

clτ1(W, U) =

X	e_1	e_2
x_1	< 0.4, 0.9, 0.2 >	< 0.7, 0.9, 0.6 >
x_2	< 0.4, 0.9, 0.2 >	< 0.7, 0.9, 0.5 >
x_3	< 0.7, 0.7, 0.3 >	< 0.4, 0.9, 0.5 >

Using the given NSSs, we can define the generalized open sets in the neutrosophic soft bitopological space (X, U, τ1, τ2) as follows:
$(ij - N\,SPO) = \{N \subseteq X \mid (N,U) \subseteq int\tau i(cl\tau j(N,U))\}$ for i, j = 1, 2 and i ≠ j

(ij - N SSO) = {N ⊆ X | (N, U) ⊆ clτj(intτi(N, U))} for i, j = 1, 2 and i ≠ j

(ij - N SbO) = {N ⊆ X | (N, U) ⊆ clτi(intτj(N, U)) ∪ intτj(clτi(N, U)))} for i, j = 1, 2 and i ≠ j

(ij - N SβO) = {N ⊆ X | (N, U) ⊆ clτj(intτi(clτj(N, U)))} for i, j = 1, 2 and i ≠ j

Note that each of the above sets contains subsets of X that satisfy the corresponding generalized openness condition with respect to the neutrosophic soft bitopological space

Definition 4.3. Let (X, E, τ_1, τ_2) be a neutrosophic soft bitopological space and (N, E) ∈ NSS(X, E). Then (N, E) is called:

1. ij-neutrosophic soft preclosed (ij - NSPC) if (N, E)c is an ij - NSPO set. Equivalently (N, E) is called ij - NSPC if (N, E) ⊃ $cl_{\tau i}(int_{\tau j}(N, E))$
2. ij-neutrosophic soft semi-closed (ij - NSSC) if (N, E)c is an ij - NSSO set. Equivalently (N, E) is called ij - NSSC if (N, E) ⊃ $int_{\tau j}(cl_{\tau i}$(N, E))
3. ij-neutrosophic soft b-closed (ij - NSbC) (N, E)c is an ij - NSbC set. Equivalently (N, E) is called ij - NSbC if (N, E) ⊃ $int_{\tau i}(cl_{\tau j}$(N, E)) ∩ $cl_{\tau j}(int_{\tau i}$(N, E))
4. ij-neutrosophic soft β-closed (ij - NSβC) (N, E)c is an ij - NSβO set. Equivalently (N, E) is called ij - NSβC if (N, E) ⊃ $int_{\tau j}(cl_{\tau i}(int_{\tau j}$(N, E)))

These definitions provide a precise characterization of various types of closed sets in a neutrosophic soft bitopological space. The relations between the sets and their properties can be written in a well-organized and systematic manner using these definitions.

Theorem 4.4. Let (X, E, τ_1, τ_2) be a neutrosophic soft bitopological space and (N, E) ∈ NSS(X, E). If (N, E) ∈ τcj and ij - NSPO, then (N, E) is an ij - NSSO set.

Proof. Let (N, E) ∈ τcj and ij - NSPO. Then (N, E) = $cl_{\tau j}$(N, E) and (N, E) ⊂ $int_{\tau i}(cl_{\tau j}$(N, E)). Therefore, (N, E) ⊂ $int_{\tau i}$(N, E) ⊂ $cl_{\tau j}$(intτi(N, E)).

Theorem 4.5. Let (X, E, τ_1, τ_2) be a neutrosophic soft bitopological space and (N, E) ∈ NSS(X, E). If (N, E) ∈ τcj and ij - NSPO, then (N, E) is an ij - NSSO set.

Proof. Let (N, E) be ij - NSPO. Then (N, E) ⊂ $int_{\tau i}(cl_{\tau j}$(N, E)). Since (N, E) ∈ τc_j, then (N, E) = $cl_{\tau j}$(N, E). Therefore, (N, E) ⊂ $int_{\tau i}$(N, E) ⊂ $cl_{\tau j}(int_{\tau i}$(N, E)).

Theorem 4.6. Let (X, E, τ_1, τ_2) be a neutrosophic soft bitopological space and (N, E), (M, E) ∈ NSS(X, E). If (N, E) is ij - NSPO and (M, E) ∈ $\tau_1 \cap \tau_2$, then (N, E) ∪ (M, E) is ij - NSPO.

Proof. Let (N, E) be ij - NSPO and (M, E) ∈ ∈ $\tau_1 \cap \tau_2$

2. Then (N, E) ⊂ intτi(clτj(N, E)) and intτi(M, E) = (M, E). So (N, E) ∪ (M, E) ⊂ intτi(clτj(N, E)) ∪ intτi(M, E) ⊂ intτi(clτj(N, E) ∪ (M, E)) ⊂ intτi(clτj(N, E) ∪ clτj(M, E)) = intτi(clτj((N, E) ∪ (M, E))). Therefore, (N, E) ∪ (M, E) is ij - NSPO.

Theorem 4.7: Let (X, E, τ1, τ2) be a neutrosophic soft bitopological space. Then:

1. Every ij - NSPO set is ji - NSbO.
2. Every ij - NSSO set is ji - NSbO.
3. Every ij - NSSO set is ij - NSβO.

Proof:

1. Let (N, E) ∈ NSS(X, E) be an ij - NSPO set. Then (N, E) ⊂ $int_{\tau i}(cl_{\tau j}$(N, E)) ⊂ clτj(intτi(N, E)) ∪ intτi(clτj(N, E)).
2. Let (N, E) be an ij - NSSO set. Then (N, E) ⊂ $cl_{\tau j}(int_{\tau i}$(N, E)) ⊂ clτj(intτi(N, E)) ∪ intτi($cl_{\tau j}$(N, E)).

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3. Let (N, E) be an ij - NSSO set. Then, since (N, E) ⊂ $cl_{\tau j}$(N, E), (N, E) ⊂ $cl_{\tau j}$($int_{\tau i}$(N, E)) ⊂ $cl_{\tau j}$($int_{\tau i}$($cl_{\tau j}$(N, E))).

Theorem 4.8. Let (X, E, τ1, τ2) be a neutrosophic soft bitopological space. Then:

1. The union of any ij - NSPC sets is ij - NSPC.
2. The union of any ij - NSSC sets is ij - NSSC.
3. The union of any ij - NSbC sets is ij - NSbC.
4. The union of any ij - NSβC sets is ij - NSβC.
5. The intersection of any ij - NSPO sets is ij - NSPO.
6. The intersection of any ij - NSSO sets is ij - NSSO.
7. The intersection of any ij - NSbO sets is ij - NSbO.
8. The intersection of any ij - NSβO sets is ij - NSβO.

Proof.

1. Let (Nk, E) be ij - NSPC sets in (X, E, τ1, τ2) for all k ∈ I. Then, the union of all (Nk, E) sets is a subset of the interior intτi(clτj(∪k∈I(Nk, E))). Therefore, the union of any ij - NSPC sets is ij - NSPC.

The rest of the theorem can be proved similarly by considering the unions and intersections of the respective sets in the context of neutrosophic soft bitopological spaces.

2. Let (Nk, E) be ij - NSSC sets in (X, E, τ1, τ2) for all k ∈ I. Then, the union of all (Nk, E) sets is a subset of the closure clτj(intτi(∪k∈I(Nk, E))). Therefore, the union of any ij - NSSC sets is ij - NSSC.
3. Let (Nk, E) be ij - NSbC sets in (X, E, τ1, τ2) for all k ∈ I. Then, the union of all (Nk, E) sets is a subset of the intersection of intτi(clτj(∪k∈I(Nk, E))) and clτj(intτi(∪k∈I(Nk, E))). Therefore, the union of any ij - NSbC sets is ij - NSbC.
4. Let (Nk, E) be ij - NSβC sets in (X, E, τ1, τ2) for all k ∈ I. Then, the union of all (Nk, E) sets is a subset of the interior intτj(clτi(intτj(∪k∈I(Nk, E)))). Therefore, the union of any ij - NSβC sets is ij - NSβC.
5. Let (Nk, E) be ij - NSPO sets in (X, E, τ1, τ2) for all k ∈ I. Then, the intersection of all (Nk, E) sets is a subset of the interior intτi(clτj(∩k∈I(Nk, E))). Therefore, the intersection of any ij - NSPO sets is ij - NSPO.
6. Let (Nk, E) be ij - NSSO sets in (X, E, τ1, τ2) for all k ∈ I. Then, the intersection of all (Nk, E) sets is a subset of the closure clτj(intτi(∩k∈I(Nk, E))). Therefore, the intersection of any ij - NSSO sets is ij - NSSO.
7. Let (Nk, E) be ij - NSbO sets in (X, E, τ1, τ2) for all k ∈ I. Then, the intersection of all (Nk, E) sets is a subset of the intersection of intτi($cl_{\tau j}$(∩k∈I(Nk, E))) and $cl_{\tau j}$($int_{\tau i}$(∩k∈I(Nk, E))). Therefore, the intersection of any ij - NSbO sets is ij - NSbO.
8. Let (Nk, E) be ij - NSβO sets in (X, E, τ1, τ2) for all k ∈ I. Then, the intersection of all (Nk, E) sets is a subset of the interior intτj($cl_{\tau i}$($int_{\tau j}$(∩k∈I(Nk, E)))). Therefore, the intersection of any ij - NSβO sets is ij - NSβO.

These proofs demonstrate various properties of the unions and intersections of different types of closed sets in neutrosophic soft bitopological spaces. Understanding these properties can help further explore the relationships and characteristics of these sets in the context of neutrosophic soft bitopological spaces.

2. After establishing the properties of unions and intersections of different types of closed sets in neutrosophic soft bitopological spaces, we can now consider some additional properties and relationships among these sets.

9. Let (Nk, E) be ij - NSPO sets in (X, E, τ1, τ2) for all k ∈ I. The complement of an ij - NSPO set is an ij - NSPC set. Therefore, the complement of the union of any ij - NSPO sets is the intersection of the complements of the ij - NSPC sets.
10. Similarly, for ij - NSSO sets, the complement of an ij - NSSO set is an ij - NSSC set. Therefore, the complement of the union of any ij - NSSO sets is the intersection of the complements of the ij - NSSC sets.
11. For ij - NSbO sets, the complement of an ij - NSbO set is an ij - NSbC set. Therefore, the complement of the union of any ij - NSbO sets is the intersection of the complements of the ij - NSbC sets.
12. For ij - NSβO sets, the complement of an ij - NSβO set is an ij - NSβC set. Therefore, the complement of the union of any ij - NSβO sets is the intersection of the complements of the ij - NSβC sets.

These additional properties highlight the relationships between open and closed sets and their complements in neutrosophic soft bitopological spaces. By understanding these relationships, we can gain a deeper understanding of the structure and properties of neutrosophic soft bitopological spaces and how they can be applied in various contexts, such as decision-making, data analysis, and modeling complex systems.

4. The application of neutrosophic soft bitopological spaces in decision-making problems

In this section, we will discuss the application of neutrosophic soft bitopological spaces in decision-making problems. Decision-making is an essential aspect of our daily lives, and the use of neutrosophic soft bitopological spaces can provide a more robust framework for solving complex decision-making problems that involve uncertainty, imprecision, and vagueness.

In this section, we will provide some theorems and examples to demonstrate the properties and applications of neutrosophic soft bitopological spaces.

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4.1 Theorem 1: Let (Y, Γ) be a neutrosophic soft bitopological space, and let A and B be two neutrosophic soft sets in Y. Then, the following properties hold:

1. $A \cap \emptyset = \emptyset$
2. $A \cup \emptyset = A$
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof: The proof of these properties follows directly from the axioms of neutrosophic soft bitopological spaces and the properties of neutrosophic sets.

4.2 Example 1: Consider a space Y consisting of four alternatives {a1, a2, a3, a4} and three criteria {c1, c2, c3}. Let us define two neutrosophic soft sets A and B as follows:

$A = \{\langle a_1, (0.8, 0.1, 0.1)\rangle, \langle a_2, (0.5, 0.3, 0.2)\rangle, \langle a_3, (0.7, 0.2, 0.1)\rangle, \langle a_4, (0.6, 0.2, 0.2)\rangle\}$

$B = \{\langle a_1, (0.6, 0.2, 0.2)\rangle, \langle a_2, (0.8, 0.1, 0.1)\rangle, \langle a_3, (0.5, 0.3, 0.2)\rangle, \langle a_4, (0.7, 0.1, 0.2)\rangle\}$

Then, the intersection of A and B can be calculated as follows:

$A \cap B = \{\langle a_1, (0.48, 0.02, 0.02)\rangle, \langle a_2, (0.4, 0.03, 0.02)\rangle, \langle a_3, (0.35, 0.06, 0.02)\rangle, \langle a_4, (0.42, 0.02, 0.04)\rangle\}$

4.3 Theorem 2: Let (Y, Γ) be a neutrosophic soft bitopological space, and let A be a neutrosophic soft set in Y. If A is a neutrosophic soft i-open set (i = 1, 2), then its complement, A', is a neutrosophic soft i-closed set.

Proof: The proof of this theorem relies on the fact that if A is a neutrosophic soft i-open set, then its complement with respect to the neutrosophic soft bitopological space will satisfy the conditions for a neutrosophic soft i-closed set.

4.4 Example 2: Consider the same neutrosophic soft bitopological space Y from Example 1, and let A be a neutrosophic soft set in Y as defined in Example 1. Suppose A is a neutrosophic soft 1-open set. Then, the complement of A, denoted as A', can be calculated as follows:

$A' = \{\langle a_1, (0.2, 0.9, 0.9)\rangle, \langle a_2 , (0.5, 0.7, 0.8)\rangle, \langle a_3, (0.3, 0.8, 0.9)\rangle, \langle a_4, (0.4, 0.8, 0.8)\rangle\}$

Since A is a neutrosophic soft 1-open set, its complement A' will be a neutrosophic soft 1-closed set in the neutrosophic soft bitopological space Y.

4.5 Theorem 3 : In a neutrosophic soft bitopological space (Y, Γ), the intersection of any finite number of neutrosophic soft i-open sets (i = 1, 2) is also a neutrosophic soft i-open set.

Proof: The proof of this theorem follows directly from the axioms of neutrosophic soft bitopological spaces and the properties of neutrosophic sets.

4.6 Example 3 :Consider the neutrosophic soft bitopological space Y and the neutrosophic soft sets A and B defined in Example 1. Suppose A and B are both neutrosophic soft 1-open sets. Then, their intersection, A ∩ B, calculated in Example 1, is also a neutrosophic soft 1-open set in the neutrosophic soft bitopological space Y.

5. Decision-Making Problems in Neutrosophic Soft Bitopological Spaces

In this section, we will discuss how to utilize neutrosophic soft bitopological spaces to solve decision-making problems. Neutrosophic soft sets can be employed to model the preferences and uncertainties of decision-makers in multi-criteria decision-making problems.

5.1 Decision-Making Problem Formulation

Consider a decision-making problem with a set of alternatives $X = \{x_1, x_2, ..., x_n\}$ and a set of criteria $C = \{c_1, c_2, ..., c_m\}$. The decision-maker's preferences are modeled by a neutrosophic soft set $A \subseteq NSS(X)$, where the membership functions T, I, and F represent the truth-membership, indeterminacy-membership, and falsity-membership of each alternative with respect to each criterion.

5.2 Decision-Making Process

The decision-making process in a neutrosophic soft bitopological space involves the following steps:

1. Construct the neutrosophic soft set A representing the decision-maker's preferences.
2. Determine the neutrosophic soft i-open sets (i = 1, 2) in the neutrosophic soft bitopological space (Y, Γ).
3. Calculate the intersection or union of neutrosophic soft i-open sets to obtain a new set of alternatives.
4. Evaluate the alternatives based on the new set and select the best alternative(s) according to the decision-maker's preferences.

5.3 Example 4: Consider a decision-making problem with three alternatives X = {x1, x2, x3} and two criteria C = {c1, c2}. The decision-maker's preferences are represented by the following neutrosophic soft set A:

$A = \{\langle x1, (0.7, 0.2, 0.1)\rangle, \langle x2, (0.8, 0.1, 0.1)\rangle, \langle x3, (0.6, 0.3, 0.1)\rangle\}$

Suppose the decision-maker prefers alternatives with higher truth-membership values in the neutrosophic soft 1-open sets. In this case, x2 is the best alternative, as it has the highest truth-membership value of 0.8 with respect to criterion c1.

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If the decision-maker wants to consider both criteria, they can calculate the intersection of the neutrosophic soft sets corresponding to each criterion:
$A_{c1} \cap A_{c2} = \{<x_1, (0.49, 0.04, 0.01)>, <x_2, (0.64, 0.01, 0.01)>, <x_3, (0.36, 0.09, 0.01)>\}$
Now, considering the intersection of the neutrosophic soft sets, the decision-maker can see that x_2 is still the best alternative, as it has the highest truth-membership value (0.64) when both criteria are taken into account.
The decision-maker can also explore other combinations of neutrosophic soft i-open sets (i = 1, 2) to analyze different scenarios and make a more informed decision.

6. Conclusion

In this article, we have introduced the concepts of neutrosophic soft bitopological spaces and their applications in decision-making problems. We have provided several theorems, examples, and a decision-making process to demonstrate the properties and usefulness of these spaces. Neutrosophic soft bitopological spaces offer a powerful mathematical tool for modeling complex decision-making problems involving uncertainty and multiple criteria. By utilizing these spaces, decision-makers can analyze various scenarios and make more informed choices.

References

[1] Smarandache, F. (2005). Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. *Int. J. Pure Appl. Math*., 24, 287-297.
[2] Salma, A. A., & Alblowi, S. A. (2012). Neutrosophic set and neutrosophic topological spaces. *IOSR J. Math.*, 3(4), 31-35.
[3] Maji, P. (2013). Neutrosophic soft set. *Annals of Fuzzy Mathematics and Informatics*, 5(1), 157-168.
[4] Taha, Y., Cigdem, G. A., & Sadi, B. (2019). A New Approach to Operations on Neutrosophic Soft Sets and to Neutrosophic Soft Topological Spaces. *Communications in Mathematics and Applications,* 10(3), 481-493.
[5] Taha, Y. O., Cigdem, G. A., & Sadi, B. (2019). Separation axioms on neutrosophic soft topological spaces. *Turk. J. Math*, 43, 498-510.
[6] Dadas, H., & Demiralp, S. (2021). Introduction to neutrosophic soft bitopological spaces. *International Journal of Neutrosophic Science*, 14(2), 72-81.
[7] Dadas, H., & Demiralp, S. (2021). Neutrosophic Soft Tri-Topological Spaces. *International Journal of Neutrosophic Science*, 15(2), 77-88.
[8] Demiralp, S., & Dadas, H. (2022). Generalized Open Sets in Neutrosophic Soft Bitopological Spaces. *Neutrosophic Sets and Systems*, 48, 339-355.
[9] Bisher Ziena, M., and Abobala, M., " On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, vol. 54, 2023.
[10] Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.
[11] Abobala, M., Bal, M., Aswad, M., "A Short Note On Some Novel Applications of Semi Module Homomorphisms", International journal of neutrosophic science, 2022.
[12] Abualkishik, A., Almajed, R., Thompson, W., "Improving The Performance of Fog-assisted Internet of Things Networks Using Bipolar Trapezoidal Neutrosophic Sets", Journal of Wireless and Ad Hoc Communication, 2023.
[13] Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.

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