Binary $D-$ Separation Axioms in Binary Topological Spaces

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Abstract

In this work the author introduces new separation axioms and Studies of its basic characteristics. In addition, he studied to the implication of these new separation axioms among themselves and with the well- known separation axioms $\text{binray}-T_0$, $\text{binray}-T_1$ and $\text{binray}-T_2$ are obtained previously. Finally, we explore some results implementing , binary continuous , binary contra continuous based on our defined definitions.

Keywords: $\text{binary}-D-\text{set}$; $\text{binray}-T_0$; $\text{binray}-T_1$; $\text{binray}-T_2$; $\text{binray}-D_0$; $\text{binray}-D_1$; $\text{binray}-D_2$

1.Introduction

In 1982 [1], Tong introduced the notion of $D-$sets by using open sets and used the notion to define some separation axioms namely D_i-space , $i=0,1,2$ .The impression of binary topological space from Z to Z^* is introducedin 2011 by Nithyanantha Jothi and Thangavelu [2] , we also introduced the impression of binary closed , binary closure , binary interior and binary continuity . On the other hand, the impression of base and subbase of a binary topological space are introduced and discussed. As well, in 2012, S.N. Jothi [3], [4] introduced the concept of $\text{binray}-T_0$, $\text{binray}-T_1$, , $\text{binray}-T_2$, $\text{binray}-T_3$, and $\text{binray}-T_4$ spaces .

The destination of this paper, is to exploration $\text{binray}-D_0$, $\text{binray}-D_1$, and $\text{binray}-D_2$ spaces in binary topological spaces and describe their fundamental properties, and discuss its relations with the axioms mentioned above.

2.Preliminaries

Throughout this article, a space (Y,τ) means a topological space on which no separation axioms are assumed unless otherwise mentioned, $Cl(A)$ and $int(A)$ are denoted the closure of a subset A and interior of a subset A in a topological space (Y,τ) . A^C denotes the complement of A in (Y,τ) . Any subset A of a topological space (Y,τ) be named $D-$set if there are two open set U and V such that $U\neq Y$ and $A=U-V$ [1].

Definition 2.1. [**1**]. Let Z and Z^* be any two nonempty sets. A binary topology from Z to Z^* is a binary structure $\aleph\subseteq P(Z)\times P(Z^*)$ that satisfies the axioms:

(i) (Φ,Φ) and $(Z,Z^*)\in\aleph$.
(ii) $(A_0\cap A_1, B_0\cap B_1)\in\aleph$ whereas $(A_0,B_0)\in\aleph$ and $(A_1,B_1)\in\aleph$.
(iii) If $\{(A_\gamma,B_\gamma):\gamma\in\Delta\}$ is a family of members of $\aleph$, then $(\cup_{\gamma\in\Delta}A_\gamma,\cup_{\gamma\in\Delta}B_\gamma)\in\aleph$.

Definition 2.2 [**1**]. If $\aleph$ is a binary topology from Z to Z^* , then the triplets $(Z,Z^*,\aleph)$ is said to be a binary topological space [for shorting , $B-T-S$] and the any members(A_0,B_0) of $(Z,Z^*,\aleph)$ are named binary open sets of $B-T-S$ $\aleph$. The elements of $Z\times Z^*$ are said to be the binary points of the $B-T-S$ $(Z,Z^*,\aleph)$.

Definition 2.3. [**1**]. Let Z and Z^* be any two non−empty sets and let (A_0,B_0) , $(A_1,B_1)\in P(Z)\times P(Z^*)$. We say that $(A_0,B_0)\subseteq(A_1,B_1)$ if $A_0\subseteq A_1$ and $B_0\subseteq B_1$.

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Definition 2.4. [1]. Let $(Z, Z^*, \aleph)$ be a $B-T-S$ and $A_0 \subseteq Z$, $B_0 \subseteq Z^*$. Then (A_0, B_0) is said to be binary closed in $(Z, Z^*, \aleph)$ if $(Z - A_0, Z^* - B_0) \in \aleph$.

Definition 2.5. [1]. Let $(Z, Z^*, \aleph)$ be a $B-T-S$ and $(A_0, B_0) \subseteq (Z, Z^*)$,

(i) The ordered pair $((A_0, B_0)^{1*}, (A_0, B_0)^{2*})$ is called the binary closure of (A_0, B_0) [denoted by $B-Cl(A_0, B_0)$] where
$(A_0, B_0)^{1*} = \cap \{A_\gamma : (A_\gamma, B_\gamma) \text{ is binary closed and } (A_0, B_0) \subseteq (A_\gamma, B_\gamma)\}$ and
$(A_0, B_0)^{2*} = \cap \{B_\gamma : (A_\gamma, B_\gamma) \text{ is binary closed and } (A_0, B_0) \subseteq (A_\gamma, B_\gamma)\}$.

(ii) The ordered pair $((A_0, B_0)^{1*}, (A_0, B_0)^{2*})$ is called the binary interior of (A_0, B_0) [denoted by $B-int(A_0, B_0)$] where
$(A_0, B_0)^{1^\circ} = \cup \{A_\gamma : (A_\gamma, B_\gamma) \text{ is binary open and } (A_0, B_0) \supseteq (A_\gamma, B_\gamma)\}$ and $(A_0, B_0)^{2^\circ} = \cup$
$\{B_\gamma : (A_\gamma, B_\gamma) \text{ is binary open and } (A_0, B_0) \supseteq (A_\gamma, B_\gamma)\}$

Definition 2.6. [1]. Let $(Z, Z^*, \aleph)$ be a $B-T-S$ and let $(c, c^*) \in (Z \times Z^*)$. The binary open set (A_0, B_0) is said to be a binary neighbourhood of (c, c^*) if $c \in A_0$ and $c^* \in B_0$.

Theorem 2.7. [1]. Let $(A_0, B_0) \subseteq (A_1, B_1) \subseteq (Z, Z^*)$ and $(Z, Z^*, \aleph)$ be a BTS. Then the following statements are fulfilled:

(i) $B - int(A_0, B_0) \subseteq (A_0, B_0)$.
(ii) If (A_0, B_0) is binary open if and only if $B - int(A_0, B_0) = (A_0, B_0)$.
(iii) $B - int(A_0, B_0) \subseteq B - int(A_1, B_1)$.
(iv) $B - int(B - int(A_0, B_0)) = B - int(A_0, B_0)$.
(v) $(A_0, B_0) \subseteq ClB - (A_0, B_0)$.
(vi) If (A_0, B_0) is binary closed if and only if $B - Cl(A_0, B_0) = (A_0, B_0)$.
(vii) $B - Cl(A_0, B_0) \subseteq B - Cl(A_1, B_1)$.

Definition 2.8. [3]. Let $(Z, Z^*, \aleph)$ be a $B-T-S$ and $(A_0, B_0) \subseteq (Z, Z^*, \aleph)$. The ordered pair $((A_0, B_0)^{1**}, (A_0, B_0)^{2**})$ is called the binary kernel of (A_0, B_0) [denoted by $B - Ker(A_0, B_0)$] where

$(A_0, B_0)^{1**} = \cap \{A_\gamma : (A_\gamma, B_\gamma) \text{ is binary open and } (A_0, B_0) \subseteq (A_\gamma, B_\gamma)\}$ and
$(A_0, B_0)^{2**} = \cap \{B_\gamma : (A_\gamma, B_\gamma) \text{ is binary open and } (A_0, B_0) \subseteq (A_\gamma, B_\gamma)\}$.

Theorem 2.9. Let $(Z, Z^*, \aleph)$ be a $B-T-S$ and $(z, z^*) \in Z \times Z^*$, then $B - Ker(A_0, B_0) = \{(c, c^*) \in Z \times Z^* : B - Cl(\{c\}, \{c^*\}) \cap (A_0, B_0) \neq (\Phi, \Phi)\}$.

Proof: Let $(c, c^*) \in B - Ker(A_0, B_0)$ & suppose that $\{(c, c^*) \in Z \times Z^* : B - Cl(\{c\}, \{c^*\}) \cap (A_0, B_0) = (\Phi, \Phi)\}$. Hence $(c, c^*) \notin [(Z, Z^*) - (B - cl(\{c\}, \{c^*\}))]$, since $(c, c^*) \in B - Ker(A_0, B_0)$ which is a binary open set containing (A_0, B_0). This is ridiculous. Consequently, $B - Cl(\{c\}, \{c^*\}) \cap (A_0, B_0) \neq (\Phi, \Phi)$. Next, let $B - cl(\{c\}, \{c^*\}) \cap (A_0, B_0) \neq (\Phi, \Phi)$ & suppose that $(c, c^*) \notin B - Ker(A_0, B_0)$, then there exists a binary open set (A_1, B_1) containing (A_0, B_0) & $(c, c^*) \notin (A_1, B_1)$. Let $(e, e^*) \in B - Cl(\{c\}, \{c^*\}) \cap (A_0, B_0)$. Hence, (A_1, B_1) is a binary neighborhood of (e, e^*) which $(c, c^*) \notin (A_1, B_1)$. By this contradiction $(c, c^*) \in B - Ker((A_0, B_0))$.

Theorem 2.10. Let $(Z, Z^*, \aleph)$ be a $B-T-S$ and $(c, c^*) \in Z \times Z^*$, then $(e, e^*) \in B - ker(\{c\}, \{c^*\})$ if and only if $(c, c^*) \in B - Cl(\{e\}, \{e^*\})$.

Proof: Suppose that $(e, e^*) \notin B - ker(\{c\}, \{c^*\})$, then there exists an binary open set (A_0, B_0) such that $(c, c^*) \in (A_0, B_0)$ but $(e, e^*) \notin (A_0, B_0)$; therefore, we have $(c, c^*) \notin B - Cl(\{e\}, \{e^*\})$. With similar steps, we can prove the reverse case.

Theorem 2.11. If (c, c^*) and (e, e^*) any points in a $B-T-S$ $(Z, Z^*, \aleph)$, then the following phrases are equivalent:

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(i) $B - \ker(\{c\}, \{c^*\}) \neq B - \ker(\{e\}, \{e^*\})$.
(ii) $B - Cl(\{c\},\{c^*\}) \neq B - Cl(\{e\},\{e^*\})$.

Proof: (i) ⇒ (ii) Let $B - \ker(\{c\}, \{c^*\}) \neq B - \ker(\{e\}, \{e^*\})$,then there exist $(c_0, c_0^*) \in Z \times Z^*$ such that $(c_0, c_0^*) \in B - \ker(\{c\}, \{c^*\})$ & $(c_0, c_0^*) \notin B - \ker(\{e\}, \{e^*\})$, since $(c_0, c_0^*) \in B - \ker(\{c\}, \{c^*\})$ it follows that $(\{c\}, \{c^*\}) \cap B - Cl(\{c_0\},\{c_0^*\}) \neq (\Phi,\Phi)$, from this we get $(c, c^*) \in B - Cl(\{c_0\},\{c_0^*\})$, by $(c_0, c_0^*) \notin B - \ker(\{e\}, \{e^*\})$ we have $(e, e^*) \cap B - Cl(\{c_0\},\{c_0^*\}) = (\Phi,\Phi)$, since $(c, c^*) \in B - Cl(\{c_0\},\{c_0^*\})$, $B - Cl(\{c\},\{c^*\}) \subset B - Cl(\{c_0\},\{c_0^*\})$ & $(e, e^*) \cap B - Cl(\{c\},\{c^*\}) = (\Phi,\Phi)$; therefore $B - Cl(\{c\},\{c^*\}) \neq B - Cl(\{e\},\{e^*\})$.

$(ii) \Rightarrow (i)$ Suppose $B - Cl(\{c\},\{c^*\}) \neq B - Cl(\{e\},\{e^*\})$, then there exist $(e_0, e_0^*) \in Z \times Z^*$ such that $(e_0, e_0^*) \in B - cl(\{c\},\{c^*\})$ & $(e_0, e_0^*) \notin B - Cl(\{e\},\{e^*\})$, then there exist binary open set including (e_0, e_0^*) & therefore (c, c^*) but not (e, e^*) , that means $(e, e^*) \notin B - Cl(\{c\},\{c^*\})$, hence $B - ker(\{c\}, \{z^*\}) \neq B - ker(\{e\}, \{e^*\})$.

Definition 2.12. A binary point $(c, c^*) \in Z \times Z^*$ which has only $Z \times Z^*$ as the binary neighborhoods is called a binary neat point.

Definition 2.13. A $B - T - S\ (Z, Z^*, \aleph)$ is called binary symmetric if for (c, c^*) and (e, e^*) in $Z \times Z^*$, $(c, c^*) \in B - Cl(\{e\},\{e^*\})$ implies $(e, e^*) \in B - Cl(\{c\}, \{c^*\})$.

Definition 2.14. [1]. Any two binary open sets (A_0, B_0) and (A_1, B_1) are called disjoint *if* $(A_0 \cap A_1, B_0 \cap B_1) = (\Phi,\Phi)$.

Definition 2.15. [2]. The binary points (c, c^*), $(e, e^*) \in Z \times Z^*$ are distinct if $c \neq e$ *and* $c^* \neq e^*$.

Definition 2.16. [1]. Let $\rho : Y \to Z \times Z^*$ be a function, let $A_0 \subseteq Z$ and $B_0 \subseteq Z^*$. We define $\rho^{-1}(A_0, B_0) = \{e \in Y : \rho(e) = (c, c^*) \in (A_0, B_0)\}$.

Definition 2.17. [2]. Let $(Z, Z^*, \aleph)$ be a $B - T - S$ and let (Y, τ) be a topological space, let $\rho : (Y, \tau) \to (Z, Z^*, \aleph)$ be a function, then ρ is said to be binary continuous if $\rho^{-1}(A_0, B_0)$ *is* open set in (Y, τ), for each binary open set (A_0, B_0) *in* $(Z, Z^*, \aleph)$.

For example : Consider $Y = \{e_1, e_2, e_3\}$, $Z = \{c_1, c_2\}$ and $Z^* = \{c_1^*, c_2^*\}$. Let $\tau = \{\Phi, Y, \{e_1\}\}$ and $\aleph = \{(\Phi,\Phi), (Z, Z^*), (\{c_1\},\{c_1^*\})\}$.
Then τ is a topology on Z and $\aleph$ is a binary topology from Z *to* Z^*.
Define $\rho : (Y, \tau) \to (Z, Z^*, \aleph)$, *by* $\rho(e_1) = (c_1, c_1^*)$ and $\rho(e_2) = \rho(e_3) = (c_2, c_2^*)$. Clearly ρ is binary continuous. Since $\rho^{-1}(\Phi,\Phi) = \{e \in Y : \rho(e) \in (\Phi,\Phi)\} = \Phi$, $\rho^{-1}(Z, Z^*) = \{e_1, e_2, e_3\} = Y$ and $\rho^{-1}(\{c_1\},\{c_1^*\}) = \{e_1\}$. Thus inverse image of every binary open set in $(Z, Z^*, \aleph)$ is open set in (Y, τ) .

Definition 2.18. [5]. Let (Y, τ) be a topological space and $(Z, Z^*, \aleph)$ be a binary topological space . We say that the function $\rho : (Y, \tau) \to (Z, Z^*, \aleph)$ is binary contra continuous if $\rho^{-1}(A_0, B_0)$ is open in (Y, τ) for every binary closed set (A_0, B_0) *in* $(Z \times Z^*, \aleph)$.

3. Binary D -Sets And Binary Associated Separation Axioms

Definition 3.1. [4]. Let (A_0, B_0) be a subset of a $B - T - S\ (Z, Z^*, \aleph)$, we say that (A_0, B_0) *binary* $- D -$ *set* if there are (A_1, B_1) and (A_2, B_2) two binary open sets , so that $(A_1, B_1) \neq (Z, Z^*)$ and $(A_0, B_0) = (A_1 - A_2, B_1 - B_2)$.

For Example : Let $Z = \{c, e, t\}$ *and* $Z^* = \{c^*, e^*, t^*\}$ with a BTS $\aleph = \left\{\begin{matrix}(Z, Z^*), (\Phi,\Phi), (\{e\},\{c^*\}), (\{c,e\},\{c^*,e^*\}), (\{e\},\{\Phi\}), \\ (\{c,e\},\{e^*\}), (\{c,e\},\{Z^*\})\end{matrix}\right\}$,then $(\{c\},\{e^*\})$, $(\{c\},\{c^*,e^*\})$, $(\Phi, \{t^*\})$ are *binary* $D -$ *sets* and every binary open set different from (Z, Z^*) is *binary* $- D -$ *set*.

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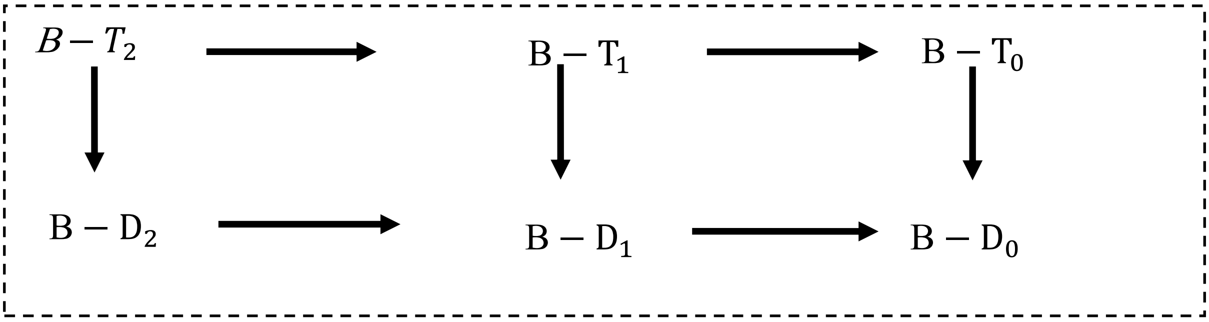
Definition 3.2. [2]. A $B-T-S\ (Z, Z^*, \aleph)$ is said to be

(i) $binray-T_0$ [$shortly$, $B-T_0$] if any pair of elements (c, c^*) and (e, e^*) of $Z \times Z^*$ are different, there is a binary open set of $(Z, Z^*, \aleph)$ containing (e, e^*) but not (c, c^*) or a binary open set of $(Z, Z^*, \aleph)$ containing (c, c^*) but not (e, e^*).
(ii) $binray-T_1$[$shortly$, $B-T_1$] if any pair of elements (c, c^*) and (e, e^*) of $Z \times Z^*$ are different, there is a binary open set of $(Z, Z^*, \aleph)$ containing (e, e^*) but not (c, c^*) and a binary open set of $(Z, Z^*, \aleph)$ containing (c, c^*) but not (e, e^*)
(iii) $Binray-T_2$ [$shortly$, $B-T_2$] if any pair of elements $(c, c^*)$$and$ (e, e^*) of $Z \times Z^*$ are different, there are two binary open sets (A_1, B_1) and (A_2, B_2) of $(Z, Z^*, \aleph)$ containing (c, c^*) and (e, e^*), respectively and $(A_1, B_1) \cap (A_2, B_2) = (\Phi, \Phi)$.
Definition 3.3. A $B-T-S\ (Z, Z^*, \aleph)$ is said to be

(i) $Binray-D_0$ [$shortly$, $B-D_0$] if any pair of elements (c, c^*) and (e, e^*) of $Z \times Z^*$ are different, there is a $binary\ D-set$ of $(Z, Z^*, \aleph)$ containing (e, e^*) but not (c, c^*) or a $binary\ D-set$ of $(Z, Z^*, \aleph)$ containing (c, c^*) but not (e, e^*),
(ii) $Binray-D_1$ [$shortly$, $B-D_1$] if any pair of elements (c, c^*) and (e, e^*) of $Z \times Z^*$ are different, there is a $binary\ D-set$ of $(Z, Z^*, \aleph)$ containing (e, e^*) but not (z, z^*) and a $binary\ D-set$ of $(Z, Z^*, \aleph)$ containing (c, c^*) but not (e, e^*)
(iii) $Binray-D_2$ [$shortly$, $B-D_2$] if any pair of elements $(c, c^*)$$and$ (e, e^*) of $Z \times Z^*$ are different, there are two $binary\ D-sets$ (A_1, B_1) and (A_2, B_2) of $(Z, Z^*, \aleph)$ containing (c, c^*) and (e, e^*), respectively and $(A_1, B_1) \cap (A_2, B_2) = (\Phi, \Phi)$.
Remark 3.4. the following diagram is explain relationship between above two definitions



For Example : $Z = \{c, e\}$ and $Z^* = \{c^*, e^*\}$ $\aleph = \{(Z, Z^*), (\Phi, \Phi), (\{c\}, \{c^*\}), (\{e\}, \{c^*\}), (Z, \{c^*\})\}$ is $B-D_0$ but not $B-D_1$.

Theorem 3.5. A $B-T-S\ (Z, Z^*, \aleph)$ is said to be $B-T_0$ if and only if it is $B-D_0$

Proof : Necessarily, let $(Z, Z^*, \aleph)$ be $B-D_0$, for any pair of elements $(c, c^*)$$and$ (e, e^*) of $Z \times Z^*$ are different, at least one of (c, c^*), (e, e^*) say (c, c^*) belongs to $binary\ D-set$ (A_0, B_0) but $(e, e^*) \notin (A_0, B_0)$, let $(A_0, B_0) = (A_1 - A_2, B_1 - B_2)$ where $(A_1, B_1) \neq (Z, Z^*)$ and $(A_1, B_1), (A_2, B_2)$ are binary open sets, then $(c, c^*) \in (A_1, B_1)$ and for $(e, e^*) \notin (A_0, B_0)$ we have two cases : first $(c, c^*) \in (A_1, B_1)$ but $(e, e^*) \notin (A_1, B_1)$, second $(e, e^*) \in (A_2, B_2)$ but $(c, c^*) \notin (A_2, B_2)$. Hence $X\ is$ $B-T_0$.

The sufficiency is stated in Remark (2.3).

Theorem 3.6. A $B-T-S\ (Z, Z^*, \aleph)$ is said to be $B-D_2$ if and only if it is $B-D_1$

Proof: Necessarily, suppose that (Z, Z^*) be $B-D_1$, then for any pair of elements $(c, c^*)$$and$ (e, e^*) of $Z \times Z^*$ are different, we have $binary\ D-sets$ (A_1, B_1), (A_2, B_2) such that $(c, c^*) \in (A_1, B_1)$ & $(e,$

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$e^*) \notin (A_1, B_1)$; $(e, e^*) \in (A_2, B_2)$ & $(c, c^*) \notin (A_2, B_2)$. Let $(A_1, B_1) = (H_1 - H_2, G_1 - G_2)$ & $(A_2, B_2) = (U_1 - U_2, V_1 - V_2)$ & (H_1, G_1) , (H_2, G_2) , (U_1, V_1) , (U_2, V_2) are binary open sets such that $(H_1, G_1), (U_1, V_1) \neq (Z, Z^*)$. From $(c, c^*) \notin (A_2, B_2)$, it follows that either $(c, c^*) \notin (U_1, V_1)$ or $(c, c^*) \in (U_1, V_1)$ & $(z, z^*) \in (U_2, V_2)$. We have two cases separately . First case : $(c, c^*) \notin (U_1, V_1)$. By $(e, e^*) \notin (A_1, B_1)$, we have two subcases:

a) $(e, e^*) \notin (H_1, G_1)$. From $(c, c^*) \in (H_1 - H_2, G_1 - G_2)$, it follows that $(c, c^*) \in (H_1 - (H_2 \cup U_1), G_1 - (G_2 \cup V_1))$ & by $(e, e^*) \in (U_1 - U_2, V_1 - V_2)$ we have $(e, e^*) \in (U_1 - (H_1 \cup U_2), V_1 - (G_1 \cup V_1))$. Therefore , $(H_1 - (H_2 \cup U_1), G_1 - (G_2 \cup V_1)) \cap (U_1 - (H_1 \cup U_2), V_1 - (G_1 \cup V_1)) = (\Phi, \Phi)$.
b) $(e, e^*) \in (H_1, G_1)$ & $(e, e^*) \in (H_2, G_2)$. With $(c, c^*) \in (H_1 - H_2, G_1 - G_2)$, $(e, e^*) \in (H_2, G_2)$ & $(H_1 - H_2, G_1 - G_2) \cap (H_2, G_2) = (\Phi, \Phi)$.
Second case : $(c, c^*) \in (U_1, V_1)$ & $(z, z^*) \in (U_2, V_2)$. We have $(e, e^*) \in (U_1 - U_2, V_1 - V_2)$, $(c, c^*) \in (U_2, V_2)$ & $(U_1 - U_2, V_1 - V_2) \cap (U_2, V_2) = (\Phi, \Phi)$ .Therefore, (c, c^*) is $B - D_2$.

The sufficiency is stated in Remark (2.3).

Theorem 3.7. Let $(Z, Z^*, \aleph)$ be an $B - T - S$ *and* $B - T_0$; then $(Z, Z^*, \aleph)$ is $B - D_1$ if and only if has no binary neat point.

Proof: Necessity ,if $(Z, Z^*, \aleph)$ is $B - T_0$, for any pair of elements (c, c^*) *and* (e, e^*) of $Z \times Z^*$ are different , at least one of them, the elements (c, c^*) has a binary neighborhood (A_0, B_0) consisting (c, c^*) & *not* (e, e^*) . Thus (A_0, B_0) Which varies from (Z, Z^*) is an $D - set$. If (Z, Z^*) has no binary neat point ; then (e, e^*) is not a binary neat point . This leads to a binary neighborhood (A_1, B_1) of (e, e^*) such that $(A_1, B_1) \neq (Z, Z^*)$. Thus $(e, e^*) \in (A_1 - A_0, B_1 - B_0)$ but not (Z, Z^*) and $(A_1 - A_0, B_1 - B_1)$ is an *binary* $D - set$. Hence $(Z, Z^*, \aleph)$ *is an* $B - D_1$.

Sufficiency , since $(Z, Z^*, \aleph)$ is $B - D_1$ & $B - T_0$, so each point $(c, c^*)$of (Z, Z^*) is contained in a *binary* $D - set$ $(A_0, B_0) = (A_1 - A_2, B_1 - B_2)$ and thus in (A_1, B_1). By definition $(A_1, B_1) \neq (Z, Z^*)$. This implies that (c, c^*) is not a binary neat point.

Theorem 3.8. A $B - T - S$ $(Z, Z^*, \aleph)$ is $B - T_0$ if and only if for any pair of elements (c, c^*) *and* (e, e^*) of $Z \times Z^*$ are different , $B - cl(\{c\}, \{c^*\}) \neq B - cl(\{e\}, \{e^*\})$.

Proof: Necessity , suppose that $(c, c^*), (e, e^*) \in Z \times Z^*$ & $(c, c^*) \neq (e, e^*)$ *with* $B - cl(\{c\}, \{c^*\}) \neq B - cl(\{e\}, \{e^*\})$, let $(s, s^*) \in Z \times Z^*$ such that $(s, s^*) \in B - cl(\{c\}, \{c^*\})$ but $(s, s^*) \notin B - cl(\{e\}, \{e^*\})$. We Claim that $(c, c^*) \notin B - cl(\{e\}, \{e^*\})$, if $(c, c^*) \in B - cl(\{e\}, \{e^*\})$, then $B - cl(\{c\}, \{c^*\}) \subset B - cl(\{e\}, \{e^*\})$, this contradicts with $(s, s^*) \notin B - cl(\{e\}, \{e^*\})$, consequently $(c, c^*) \in [B - cl(\{e\}, \{e^*\})]^C$ which $(\{e\}, \{e^*\}) \notin [B - cl(\{e\}, \{e^*\})]^C$, that's mean $(Z, Z^*, \aleph)$ is $B - T_0$.

Sufficiency , let $(Z, Z^*, \aleph)$ be $B - T_0$ space , $(c, c^*), (e, e^*) \in Z \times Z^*$ & $(c, c^*), (e, e^*)$, there exists an *binary open* (A_0, B_0) containing (z, z^*) or (e, e^*) , then $[A_0, B_0]^C$ is an *binary* closed set which$(c, c^*) \notin [A_0, B_0]^C$ & $(e, e^*) \in [A_0, B_0]^C$, since $B - cl(\{e\}, \{e^*\})$ is the smallest *binary* closed set containing (e, e^*) , $B - cl(\{e\}, \{e^*\}) \subset [A_0, B_0]^C$ & $(z, z^*) \notin B - cl(\{e\}, \{e^*\})$, hence $B - cl(\{c\}, \{c^*\}) \neq B - cl(\{e\}, \{e^*\})$.

Corollary 3.9. A $B - T - S$ $(Z, Z^*, \aleph)$ is $B - D_0$ if and only if for any pair of elements (c, c^*) *and* (e, e^*) of $Z \times Z^*$ are different , $B - ker(\{c\}, \{c^*\}) \neq B - ker(\{e\}, \{e^*\})$.

Theorem 3.10. A $B - T - S$ $(Z, Z^*, \aleph)$ is binary symmetric if and only if $(\{z\}, \{z\}^*)$ is binary closed for each $(z, z^*) \in Z \times Z^*$

Proof: Assume that $(c, c^*) \in B - cl(\{e\}, \{e^*\})$, but $(e, e^*) \notin B - cl(\{c\}, \{c^*\})$. This means that $[B - cl(\{c\}, \{c^*\})]^C$ contains (e, e^*) , this implies that $B - cl(\{e\}, \{e^*\})$ is a subset of $[B - cl(\{z\}, \{z^*\})]^C$. Now $[B - cl(\{c\}, \{c^*\})]^C$contains $(\{c\}, \{c^*\})$ which is a contradiction.

Conversely, suppose that $(\{c\}, \{c^*\}) \subset (A_0, B_0) \subset (Z, Z^*, \aleph)$ but $B - cl(\{c\}, \{c^*\}) \not\subset (A_0, B_0)$. This means that $B - cl(\{z\}, \{z^*\}) \cap (A_0, B_0) \neq (\Phi, \Phi)$, let $(e, e^*) \in \{B - cl(\{c\}, \{c^*\}) \cap (A_0, B_0)\}$. Now we have

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$(\{c\}, \{c^*\}) \in B - cl(\{e\}, \{e^*\})$ which is a subset of $(A_0, B_0)^C$ and $(c, c^*) \notin (A_0, B_0)$. But this is a contradiction.

Corollary 3.11. If A $B - T - S(Z, Z^*, \aleph)$ is $B - T_1$, then it is binary symmetric.

Corollary 3.12. A $B - T - S(Z, Z^*, \aleph)$ is binary symmetric and $-T_0$, $B - D_1$.

Theorem 2.13. For a $B - T - S$ $(Z, Z^*, \aleph)$ with at least two elements ,if $(Z, Z^*, \aleph)$ is $B - D_1$, then $B - ker(\{c\}, \{c^*\}) \neq (Z, Z^*, \aleph)$.

Proof: Let be $(Z, Z^*, \aleph)$ $B - D_1$ & $(c, c^*) \in (Z, Z^*)$, for a point $(c, c^*) \neq (e, e^*)$, there exists a $binary\ D - set$ (A_0, B_0) such that $(c, c^*) \in (A_0, B_0)$ & $(e, e^*) \notin (A_0, B_0)$, say $(A_0, B_0) = (A_1 - A_2, B_1 - B_2)$, where $(A_1, B_1), (A_2, B_2)$ are binary open sets for & $(A_1, B_1) \neq (Z, Z^*)$, thus for the point (z, z^*), we have a binary open set (A_1, B_1) such that $(\{c\}, \{c\}^*) \subset (A_1, B_1)$ & $(A_1, B_1) \neq (Z, Z^*)$, hence, $B - ker(\{z\}, \{z^*\}) \neq (Z, Z^*, \aleph)$.

4. Applications

Theorem 4.1. If function $\rho : (Y, \tau) \to (Z, Z^*, \aleph)$ is a continuous surjective function and (A_0, B_0) is a *binary* $D - set$ of BTS $(Z, Z^*, \aleph)$, then the inverse image of a *binary* (A_0, B_0) is $D - set\ of\ (Y, \tau)$.

Proof: Let (A_1, B_1) and (A_2, B_2) be two *binary* open sets of BTS $(Z, Z^*, \aleph)$ & $(A_0, B_0) = (A_1 - A_2, B_1 - B_2)$ be a *binary* D-set and $(A_1, B_1) \neq (Z, Z^*)$. We have $\rho^{-1}((A_1, B_1))$ & $\rho^{-1}((A_2, B_2))$ be open sets *of* (Y, τ) such that $\rho^{-1}((A_1, B_1)) \neq Y$. Wherefore $\rho^{-1}((A_0, B_0)) = \rho^{-1}((A_1 - A_2, B_1 - B_2)) = \rho^{-1}((A_1, B_1)) - \rho^{-1}((A_2, B_2))$, this is implies to $\rho^{-1}((A_0, B_0))$ is an $D - set$ in (Y, τ).

Theorem 4.2. If $\rho : (Y, \tau) \to (Z, Z^*, \aleph)$ is an injective continuous function and $(Z, Z^*, \aleph)$ is a $B - D_1 - space$, then (Y, τ) is a $D_1 - space$.

Proof : suppose that $(Z, Z^*, \aleph)$ is a $B - D_1 - space$ & h_1, h_2 any pair of different elements in Y, since $\rho : (Y, \tau) \to (Z, Z^*, \aleph)$ is injective & $(Z, Z^*, \aleph)$ is a $B - D_1 - space$, then there exists two *binary* $D - sets$ (A_{h1}, B_{h1}) & (A_{h2}, B_{h2}) of (Z, Z^*) containing $\rho(h_1)$ & (A_{h1}, B_{h1}) respectively such that $\rho(h_1) \notin (A_{h2}, B_{h2})$ & $(A_{h1}, B_{h1}) \notin (A_{h1}, B_{h1})$. Hence $\rho^{-1}((A_{h1}, B_{h1}))$ & $\rho^{-1}((A_{h2}, B_{h2}))$ are $D - sets\ in$ a space (Y, τ) containing h_1 & h_2 respectively such that $h_1 \notin \rho^{-1}((A_{h2}, B_{h2}))$ & $h_2 \notin \rho^{-1}((A_{h1}, B_{h1}))$,therefore a space (Y, τ) is a $D_1 - space$.

Theorem 4.3. Let (Y, τ) be a D_1 – topological space , if for all pair of different elements $h_1, h_2 \in Y$, then there is a surjective continuous function $\rho: (Y, \tau) \to (Z, Z^*, \aleph)$, wherever $(Z, Z^*, \aleph)$ is a $B - D_1 - space$ such that $\rho(h_1)\ and\ \rho(h_2)$ are different.

Proof: assume that h_1 & $h_2 \in Y$ & $h_1 \neq h_2\ in\ Y$, from hypothesis, then there is a surjective continuous function ρ from a D_1 – topological space onto a $B - D_1 - space$ $(Z, Z^*, \aleph)$ such that $\rho(h_1) \neq \rho(h_2)$. Hence there exists disjoint $binary\ D - sets$ (A_{h1}, B_{h1}) & (A_{h2}, B_{h2}) in $(Z, Z^*, \aleph)$ $such\ that$ $\rho(h_1) \in (A_{h1}, B_{h1})$ & $\rho(h_2) \in (A_{h2}, B_{h2})$. Since ρ is continuous and surjective , then $\rho^{-1}((A_{h1}, B_{h1}))$ & $\rho^{-1}((A_{h2}, B_{h2}))$ are disjoint $binary\ D - sets$ in a space (Y, τ) containing h_1 & h_2 respectively. Hence (Y, τ) is an $D_1 - space$.

Theorem 4.4. If a function $\rho: (Y, \tau) \to (Z, Z^*, \aleph)$, is injective, contra – continuous and $(Z, Z^*, \aleph)$ is a Urysohn space, then (Y, τ) is $B - D_2 - space$.

Proof: Let h_1 & $h_2 \in (Y, \tau)$ with $h_1 \neq h_2$, since ρ is injective , then $\rho(h_1) \neq \rho(h_2)$, since $(Z, Z^*, \aleph)$ is a Urysohn space, there exists *Binray* open sets (A_{h1}, B_{h1}) & (A_{h2}, B_{h2}) in $(Z, Z^*, \aleph)$ such that $\rho(h_1) \in (A_{h1}, B_{h1})$ & $\rho(h_2) \in (A_{h2}, B_{h2})$ & $B - cl((A_{h1}, B_{h1})) \cap B - cl((A_{h2}, B_{h2})) = (\emptyset, \emptyset)$, since ρ is contra – continuous, there exists two open set H_1 & $H_2\ in\ (Y, \tau)$ such that $h_1 \in H_1$, $h_2 \in H_2$ & $\rho(H_1) \subset B -$

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$cl\big((A_{h1}, B_{h1})\big)$, $\rho(H_1) \subset B - cl((A_{h2}, B_{h2}))$, then $\rho(H_1) \cap \rho(H_2) = (\emptyset, \emptyset)$ & so $\rho(H_1 \cap H_2) = (\emptyset, \emptyset)$ & , this implies that $H_1 \cap H_2 = \emptyset$ and hence (Y, τ) is $B - D_2 - space$.

5. Conclusion

In this manuscript, our primary goal was for exploration a new type of binary separation axioms in binary topological spaces , which we called the D-binary separation axioms . The case in the study of the characteristics of this type of binary separation axioms was fully as a topological spaces in classical environments, we cannot urge the importance of some basic definitions in binary topological spaces , many properties of separation axioms of this type in binary topological spaces are associated with these definitions .Therefore, before delving into the dualistic D-binary separation axioms, we recalled and presented in the second section of this article some definitions and characteristics that represent the basis for establishing some results that played a decisive role in studying of these separation axioms . At the beginning of the third section, we mentioned the definitions of the binary topological spaces $T_0, T_1, and\ T_2$, and similarly we defined the binary topological spaces $D_0, D_1, and\ D_2$ and examined the various properties and relationships associated with these axioms of separation. In the fourth section, we discussed the applications of binary continuous functions on this type of binary topological spaces. We hope that our findings in this manuscript will be useful to the research community and contribute to the advancement of some different aspects of binary topology.

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