



A New Proof of Feuerbach's Theorem

Ion Patrascu¹, Florentin Smarandache²

¹"Frații Buzești" National College, Bld. Stirbei Voda, nr. 5, Craiova, Dolj, Romania

²University of New Mexico, Mathematics, Physics and Natural Sciences Division,
705 Gurley Ave., Gallup, NM 87301, USA

Email: patrascu_ion@yahoo.com; smarand@unm.edu

Abstract

Feuerbach's theorem on the tangent of the circle of the nine points and the inscribed and exinscribed circle is considered one of the most beautiful theorems in geometry. In this paper, we offer a basic proof of this theorem starting from one of Gh. Buicliu's ideas [1].

Keywords: Feuerbach's theorem; circle of the nine points; inscribed circles; exinscribed circles

The Theorem: The circle of the nine points of a triangle is tangent to the inscribed circle and to the exinscribed circles of the triangle (K. Feuerbach – 1822).

Proof: May O_9 be the center of the circle of the 9 points of the ABC triangle and I the center of its inscribed circle. We denote by D the contact of the inscribed circle with BC and with D' its diameter within the inscribed circle. May AH_a be the height from A (see Figure 1); we will denote by K and M the intersections of the parallel going through I to AD' with AH_a , respectively with BC and L the intersection of the parallel at AD' , drawn through D , with AH_a .

We also note with P the intersection of the IL and BC lines and with Q the intersection of the half-line $(D'K$ with the inscribed circle).

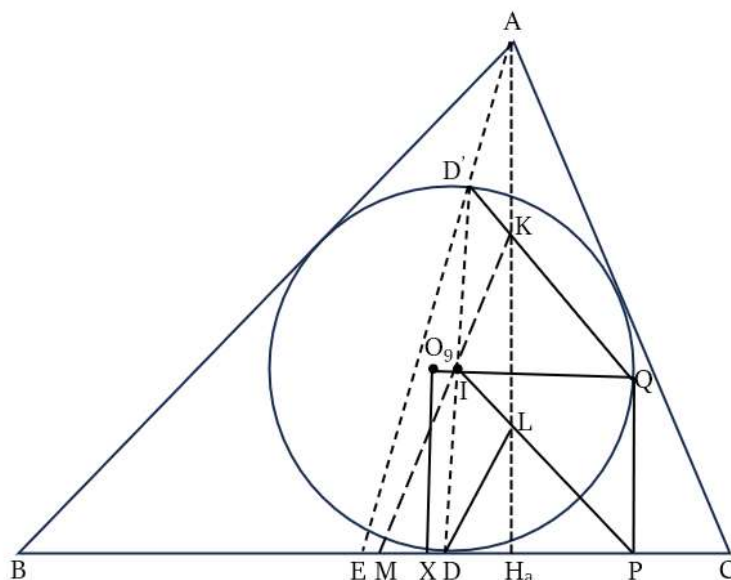


Figure 1

We will prove that Q is the tangency point of the inscribed circle with the circle of the 9 points. (Q is one of Feuerbach's points.) From the fact that the inscribed circle (I) and the exinscribed circle $A(I_a)$ are homothetical through the homothety of pole A , we know that the point E is the exinscribed (I_a) circle's contact with BC , and that

points D and E are isotomical (symmetrical to M , the middle of BC). We note that $\alpha = m(\widehat{ID'Q})$, because $ID' = IQ$, thus $m(\widehat{IQD'}) = \alpha$. The $ID'AK$ and $IDLK$ quadrilaterals are parallelograms.

$ID' \parallel AK$ și $ID' = KL = r$ (the ray of the inscribed circle), thus the $ID'KL$ quadrilateral is a parallelogram, so $m(\widehat{IQD'}) = m(\widehat{LIK}) = \alpha$. Likewise, $m(\widehat{DIL}) = m(\widehat{LIQ}) = \alpha$, which, along with $ID = IQ = r$, shows that the points D and Q are symmetrical to IP . Moreover, because PD is tangent in D to the (I) circle, we have that PQ is tangent in Q to the (I) inscribed circle, so $PD = PQ$.

The fact that the $(MI; DL)$ and $(DD'; AH_a)$ lines are parallel leads to: $\frac{PM}{PD} = \frac{PI}{PL} = \frac{PD}{PH_a}$.

We remember that $PD^2 = PM \cdot PH_a$, and seeing as how $PD = PQ$, we have that $PQ^2 = PM \cdot PH_a$, which shows PQ is tangent to the circumscribed circle of the QMH_a triangle.

We prove this circle is the circle of the 9 points of the ABC triangle.

It is sufficient to prove that Q belongs to the circle of the 9 points, so we show that $O_9Q = \frac{R}{2}$ (R being the ray of the circumscribed circle and O_9 the center of the circle of the 9 points). We denote by X the orthogonal projection of O_9 on BC . We have that $O_9X^2 + PX^2 = PO_9^2$.

On another note, $O_9M = \frac{R}{2}$ and $O_9X^2 = \frac{R^2}{4} - MX^2$. We know that X este is the middle of the MH_a segment (because O_9 is the middle of OH – where O and H are, respectively, the center of the circumscribed circle and the orthocenter of ABC).

We obtain $PO_9^2 = PX^2 - MX^2 + \frac{R^2}{4}$.

But $PX^2 - MX^2 = (PX + MX)(PX - MX) = PM \cdot (PH_a + H_aX - MX)$.

However, $MX = MH_a$, thus $PX^2 - MX^2 = PM \cdot PH_a = PD^2 = PQ^2$.

We obtain $PO_9^2 = PQ^2 + \frac{R^2}{4}$.

We interpret this relationship as follows: the triangle with the length sides PO_9 , PQ and $\frac{R}{2}$ is rectangular. Because PQ is the tangent of the circumscribed circle to the MH_aQ triangle, we obtain the $\frac{R}{2}$ length segment with an end in Q and the other on the IQ line. But the circle circumscribed to the MH_aQ triangle has its center at the intersection of the bisection of the MH_a segment with perpendiculars in Q on PQ , just as O_9 is the bisection of MH_a and is at the distance of $O_9M = \frac{R}{2}$ of M and H_a , thus $O_9Q = \frac{R}{2}$, so O_9 is the center of the circumscribed circle of the MH_aQ triangle.

We continue to prove that the circle of the 9 points is tangent to the A-exinscribed circle in a Feuerbach which we note by Q_a . We build on the E point the orthogonal projection of I_a on BC and E' its diameter in the (I_a) circle.

We draw the parallels to AE' through I_a and E and we denote by K_a respectively L_a their intersections with the height AH_a of the ABC triangle (see Figure 2). The intersection of the I_aK_a lines and BC is M – the middle of BC , but the intersection of the I_aL_a line with BC we denote by P_a . We affirm that the Q_a Feuerbach point is the intersection of the exinscribed A-circle with the $E'K_a$ segment.

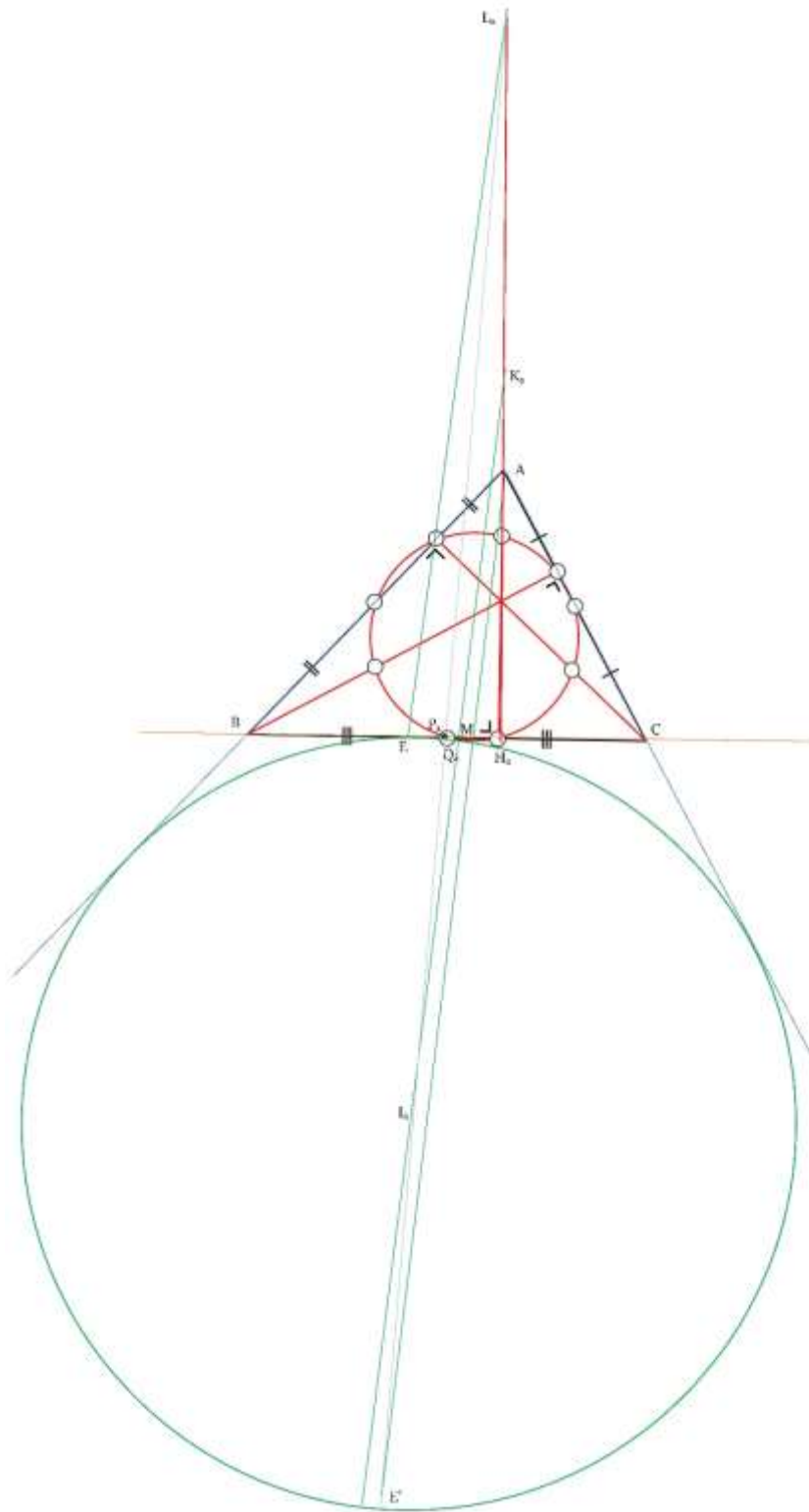


Figure 2

We prove that the Q_a point is symmetrical to P_a on the $I_a K_a$ line. From what we have so far, the $E' I_a L_a K_a$ quadrilateral is a parallelogram, so $\sphericalangle E I_a P_a = \sphericalangle I_a E' Q_a$ (alt. int.). because the $I_a F' Q_a$ triangle is isosceles we have that: $\sphericalangle I_a E' Q_a = \sphericalangle I_a Q_a E'$.

The $\sphericalangle E I_a Q_a$ angle is exterior to the $E' I_a Q_a$ triangle, so $\sphericalangle E I_a Q_a = 2 \cdot \sphericalangle I_a Q_a E'$ and seeing as $\sphericalangle E I_a P_a = \sphericalangle I_a Q_a E'$, we have that $\sphericalangle E I_a P_a = \sphericalangle I_a Q_a E'$, relationship which, together with $I_a E = I_a Q_a$ implies $\triangle P_a I_a E \equiv \triangle Q_a I_a P_a$, and

from here we obtain that $P_aQ_a = P_aE$ and $P_aQ_a \perp I_aK_a$. From the parallelism of the EL_a line with the I_aK_a line and EE' with AH_a we have that: $\frac{P_aM}{P_aE} = \frac{I_aP_a}{P_aL_a} = \frac{P_aE}{P_aH_a}$.

We remember that $P_aE^2 = P_aM \cdot P_aH_a$, thus $P_aQ_a^2 = P_aM \cdot P_aH_a$.

This relationship shows that the A-exinscribed circle and the circumscribed circle of the MH_aQ_a triangle are exterior tangents in the Q_a point.

We continue to prove that the circumscribed circle of the MH_aQ_a triangle is the circle of the 9 points (O_9).

We denote by T the orthogonal projection of O_9 on BC , we have $P_aO_9^2 = P_aT^2 + O_9T^2$.

But $O_9T^2 = O_9M^2 - MT^2 = \frac{R}{4} - MT^2$. We obtain $P_aT^2 - MT^2 + \frac{R}{4} = P_aO_9^2$. But $MT = H_aT = \frac{1}{2}MH_a$, we have $P_aT^2 - MT^2 = (P_aT + TH_a)(P_aT - MT) = P_aH_a \cdot P_aM$. We obtained $P_aO_9^2 = P_aQ_a^2 + \frac{R^2}{4}$.

We interpret this relationship as follows: we can build a rectangular triangle with the hypotenuses P_aO_9 with a P_aQ_a cathetus and a $\frac{R}{2}$ length cathetus. Because $m(\widehat{P_aQ_aI_a}) = 90^\circ$, the rectangular triangle with the P_aO_9 hypotenuses will have a 90° angle in Q_a and the third corner will be on the prolongment of the I_aQ_a segment. That corner will be on the bisection of the MH_a segment at the distance of $OM = OH_a = \frac{R}{2}$. Because $O_9M = O_9H_a = \frac{R}{2}$, we have that O_9 is the circumscribed center of the MH_aQ_a triangle, meaning this last circle is the circle of the 9 points of the ABC triangle. The theorem is proved the same way for the B-exinscribed and C-exinscribed circles.

References

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