# Neutrosophic parametric study of piecewise quadratic fuzzy multi-objective dynamic programming problems

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**Abstract**

The goal of this research is to investigate fuzzy multiobjective dynamic programming issues with fuzzy parameters in the objective functions and single valued trapezoidal neutrosophic numbers in the left hand side of the constraints. Piecewise quadratic fuzzy numbers characterize these fuzzy parameters. In addition, applying the score function of the neutrosophic numbers to convert the constraints parameters into its crisp . Some basic notions in the problem under the $\alpha$ −pareto optimal solution concept is redefined and analyzed to study the stability of the problem. Furthermore, a technique is presented for obtaining a subset of the parametric space that has the same $\alpha$ −pareto optimal solution. For a better understanding and comprehension of the suggested concept, a numerical example is provided.

**Keywords**: Optimization; Multiobjective dynamic programming; Fuzzy set; Piecewise quadratic fuzzy numbers; Close interval approximation; $\alpha$ −pareto optimal solution; Decision making; Stability; Neutrosophic numbers; Score function

## 1. Introduction

One of the most essential methodologies for solving optimization problems is dynamic programming (DP), where the so-called principle of optimality, as defined by Bellman, is used to create its methods [1]. MODP (multiobjective dynamic programming) is a method for resolving problems with competing objectives that follows the DP characteristics ( Mine and Fukushima, [2]; Carraway et al., [3]; and Abo- Sinna and Hussein, [4] ).

Osman [6-7] introduces the ideas of solvability set, first-kind stability set, and second-kind stability set, as well as the analysis of these concepts for parametric convex nonlinear programming problems. For a certain class of multiobjective convex programming problems, Osman and Dauer [8] dedicated themselves to discovering the stability set of the first kind and provided an algorithm for determining this set and the related pareto optimum solution.



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First and foremost, Zadeh [9] proposed the philosophy of fuzzy sets in literature. Bellman and Zadeh [10] created a method for decision-making in a fuzzy environment that improved and aided managerial decision-making. Fuzzy programming and linear programming with numerous objective functions were presented by Zimmermann [11]. Several people afterwards worked in the field of fuzzy set theory. Many authors have investigated the theory and applications of fuzzy sets, systems, and fuzzy mathematical models ( Dubois and Prade, [12]; Kaufmann and Gupta , [13]). In the literature, fuzzy dynamic programming models in particular have gotten a lot of attention (see, Zimmermann,[14]; Esogbue and Bellman, [16]; Hussein and Abo- Sinna, [17]). Tanaka and Asai [18] introduced fuzzy parameters to multiobjective linear programming (MOLP) problems. General fuzzy multiobjective nonlinear programming (MONLP) problems were formulated by Orlovski [19]. Al-Quran et al [20-23] employed some based SVNS techniques to solve real-life issues. Sakawa and Yano [24-25] looked into the idea of -pareto optimum optimality and proposed a new interactive fuzzy approach for MOLP and MONLP issues with fuzzy parameters. For fuzzy MONLP situations, Osman and El-Banna [26] proposed a qualitative analysis and stability. Al-Sharqi et al. [27-30] introduced several algebraic contributions supported by algorithms that employed these algebraic structures in solving decision-making real-life problems. There are enormous researches developed in MODP (for instance, Moghaddam and Ghoseiri, [31]; Muruganantham et al., [32]; Li et al., [33]; Deng et al., [34]; Besheli et al., [35]; Peraza et al., [36]; Azevedo et al., [37];Ni et al., [38]; Wu et al., [39]; Liu et al., [40]; Zou et al., [41]; Zhang et al., [42]; Mena et al. [43]; and Wu et al. [44).

The core notions of stability in nonlinear programming problems parameters are redefined and studied for multiobjective dynamic programming problems with fuzzy parameters in the objective functions, as a result of the above literature.In addition, an algorithm for obtaining the subset of the parametric space which has the same corresponding −pareto optimal solution is developed.

The rest of the paper is outlined as in Fig. 1 below

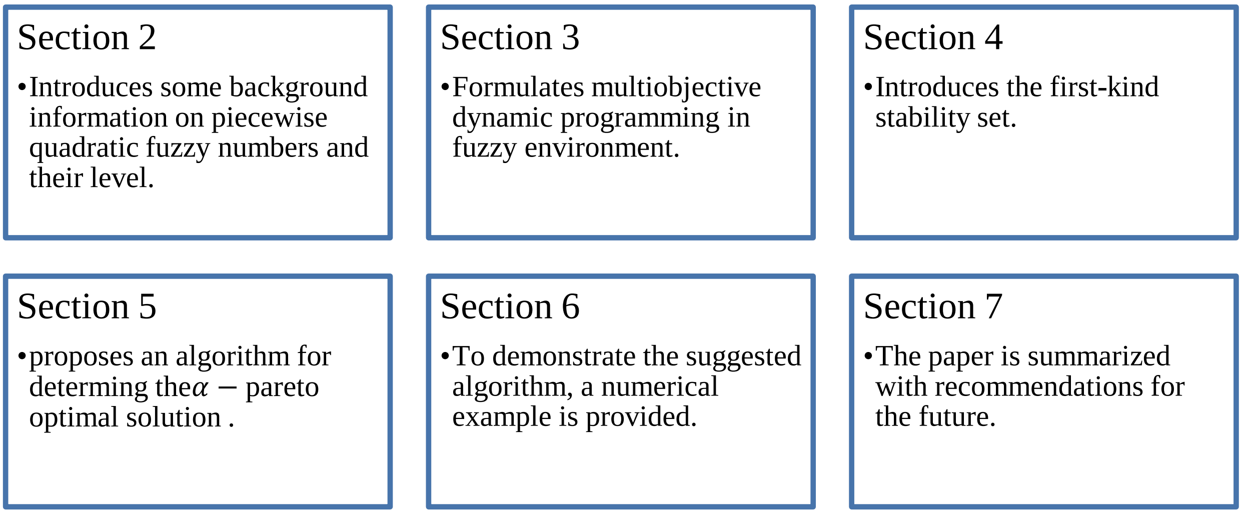


Figure 1: Layout of Remaining Paper

1. **PRELIMINARIES**

In this section, some essential definitions and terminologies are recalled from fuzzy-like literature for proper understanding of the proposed work (see, Jain, [45]; Atanason, [46]; and Thamaraiselvi, and Santhi, [47])

**2.1. Piecewise quadratic fuzzy numbers**

**Definition 1** (Jain, [37]). **A piecewise quadratic fuzzy number** (PQFN) is denoted by $\widetilde{A}_{PQ} = (a_1, a_2, a_3, a_4, a_5)$, where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ are real numbers, and its membership function $\mu_{\widetilde{A}_{PQ}}$ is given by (see,Fig. 2)



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$$\mu_{\tilde{A}_{PQ}} = \begin{cases} 0, \quad x < a_1; \\ \frac{1}{2}\frac{1}{(a_2-a_1)^2}(x-a_1)^2, \quad a_1 \le x \le a_2; \\ \frac{1}{2}\frac{1}{(a_3-a_2)^2}(x-a_3)^2+1, a_2 \le x \le a_3; \\ \frac{1}{2}\frac{1}{(a_4-a_3)^2}(x-a_3)^2+1, \quad a_3 \le x \le a_4; \\ \frac{1}{2}\frac{1}{(a_5-a_4)^2}(x-a_5)^2, a_4 \le x \le a_5; \\ 0, \; x > a_5. \end{cases}$$

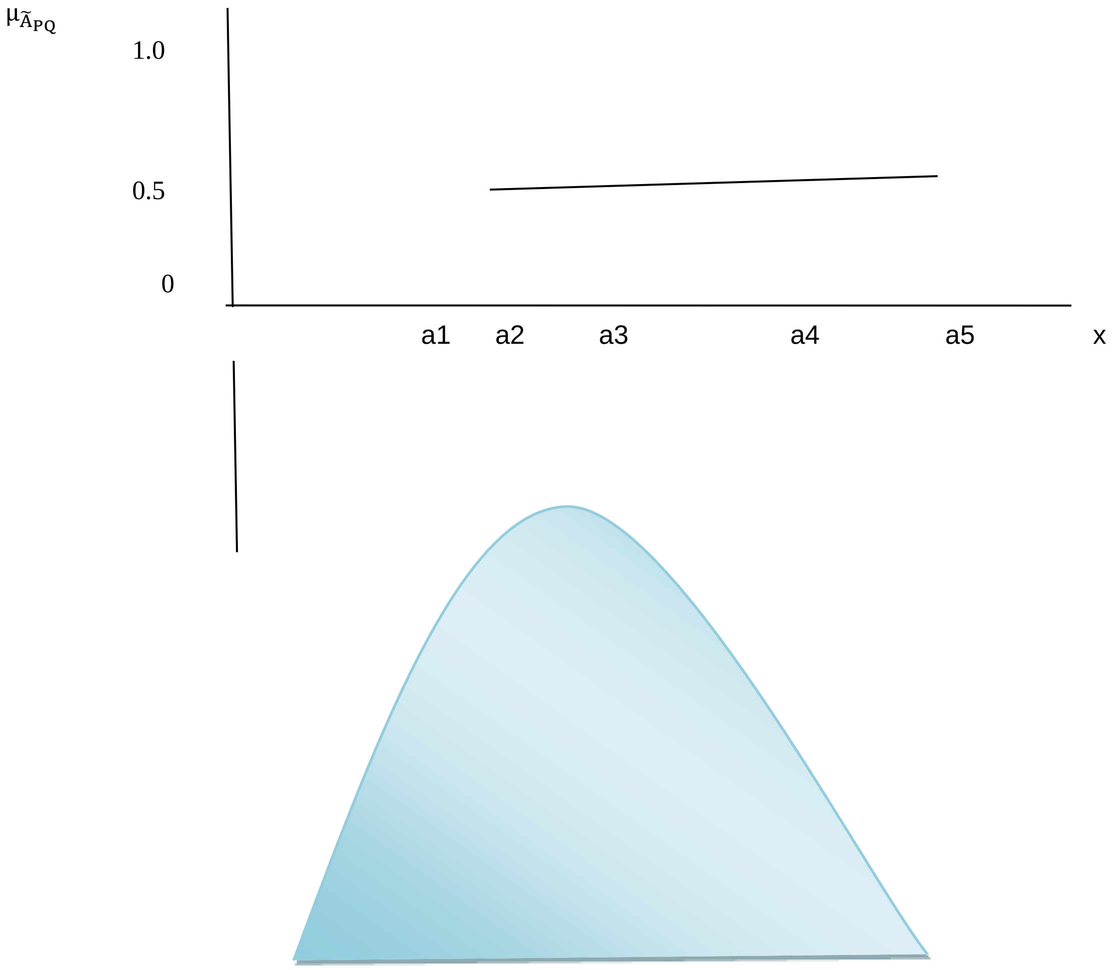


Figure 2: Graphical Representation of a Piecewise Quadratic Fuzzy Number (P.Q.F.N.)

**Definition 2**. (Jain, [37]). An interval approximation $[\tilde{A}_{PQ}] = [p^L_\alpha, p^U_\alpha]$ of a PQFN $\tilde{A}$ is called closed interval approximation if:

$p^L_\alpha = \inf\{x \in \mathfrak{R}: \mu_{\tilde{A}_{PQ}} \ge 0.5\}$, and $p^U_\alpha = \sup\{x \in \mathfrak{R}: \mu_{\tilde{A}_{PQ}} \ge 0.5\}$.

**Definition 3**. (Jain, [37]). Suppose that $\tilde{a}_{PQ} = (p_1, p_2, p_3, p_4, p_5)$ and $\tilde{b}_{PQ} = (b_1, b_2, b_3, b_4, b_5)$ be two P.Q.F.N.s. Then

1. Addition: $\tilde{a}_{PQ} \oplus \tilde{b}_{PQ} = (p_1 + b_1, p_2 + b_2, p_3 + b_3, p_4 + b_4, p_5 + b_5)$,
2. Subtraction: $\tilde{a}_{PQ} \ominus \tilde{b}_{PQ} = (p_1 - b_5, p_2 - b_4, p_3 - b_3, p_4 - b_2, p_5 - b_1)$.
3. Scalar multiplication: $\alpha.\tilde{a}_{PQ} = \begin{cases} (\alpha\, p_1, \alpha\, p_2, \alpha\, p_3, \alpha\, p_4, \alpha\, p_5), \alpha > 0, \\ (\alpha\, p_5, \alpha\, p_4, \alpha\, p_3, \alpha\, p_2, \alpha\, p_1), \alpha < 0. \end{cases}$

**Definition 4**. (Jain, [37]). Let $[A] = [a^L_\alpha, a^U_\alpha]$, and $[B] = [b^L_\alpha, b^U_\alpha]$ be two inexact interval of PQFN. Then the arithmetic operations are:

1. Addition: $[A] \oplus [B] = [p^L_\alpha + b^L_\alpha, p^U_\alpha + b^U_\alpha]$,
2. Subtraction: $[A] \ominus [B] = [p^L_\alpha - b^U_\alpha, p^U_\alpha - b^L_\alpha]$,



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3. Scalar multiplication: $\alpha[A] = \begin{cases} [\alpha\, p_\alpha^L, \alpha\, p_\alpha^U], \alpha > 0, \\ [\alpha p_\alpha^U, \alpha p_\alpha^L], \alpha < 0. \end{cases}$
4. Multiplication: $[A] \odot [B] = \left[\frac{p_\alpha^U b_\alpha^L + p_\alpha^L b_\alpha^U}{2}, \frac{p_\alpha^L b_\alpha^L + p_\alpha^U b_\alpha^U}{2}\right]$.
5. Division:

$$\frac{[A]}{[B]} = \begin{cases} \left[\frac{2p_\alpha^L}{b_\alpha^L + b_\alpha^U}, \frac{2p_\alpha^U}{b_\alpha^L + b_\alpha^U}\right], [B] > 0 \ and\ b_\alpha^L + b_\alpha^U \neq 0, \\ \left[\frac{2p_\alpha^U}{b_\alpha^L + b_\alpha^U}, \frac{2p_\alpha^L}{b_\alpha^L + b_\alpha^U}\right], [B] < 0 \ and\ b_\alpha^L + b_\alpha^U \neq 0. \end{cases}$$

**Definition 5**. The order relations $\{=_{LU}, \leq_{LU,}, \geq_{LU}\}$ between [A] and [B] is defined as

(i) $[A] =_{(L,U)} [B]$ iff $p_\alpha^L = b_\alpha^L$ and $p_\alpha^U = b_\alpha^U$.

(ii) $[A](\leq_{(L,U)})[B]$ iff $p_\alpha^L(\leq_{(L,U)})b_\alpha^L$ and $p_\alpha^U(\leq_{(L,U)})b_\alpha^U$ or $p_\alpha^L + p_\alpha^U(\leq_{(L,U)})b_\alpha^L + b_\alpha^U$.

(iii) $[A](\geq_{(L,U)})[B]$ iff $p_\alpha^L(\geq_{(L,U)})b_\alpha^U$ and $p_\alpha^U(\geq_{(L,U)})b_\alpha^L$ or $p_\alpha^L + p_\alpha^U(\geq_{(L,U)})b_\alpha^U + b_\alpha^L$.

### 2.2 Single valued trapezoidal fuzzy neutrosophic numbers

**Definition 6: (Neutrosophic set, )** A neutrosophic set $\widetilde{\mathcal{B}}$ of non-empty set $\mathcal{X}$ is defined as:
$\widetilde{\mathcal{B}}^N = \{\langle x;\ I_{\widetilde{\mathcal{B}}^N}(x), J_{\widetilde{\mathcal{B}}^N}(x), V_{\widetilde{\mathcal{B}}^N}(x)\rangle : x \in \mathcal{X}, I_{\widetilde{\mathcal{B}}^N}(x), J_{\widetilde{\mathcal{B}}^N}(x), V_{\widetilde{\mathcal{B}}^N}(x) \in\ ]0^-, 1^+[\}$
where $I_{\widetilde{\mathcal{B}}^N}(x), J_{\widetilde{\mathcal{B}}^N}(x)$, and $V_{\widetilde{\mathcal{B}}^N}(x)$ are functions that fulfill truth-membership, indeterminacy-membership, and falsity-membership: $0^- \leq \mathrm{Sup}\{I_{\widetilde{B}^N}(x)\} + \mathrm{Sup}\{J_{\widetilde{B}^N}(x)\} + \mathrm{Sup}\ \{V_{\widetilde{B}^N}(x)\} \leq 3^+$, where $]0^-, 1^+[$ is a nonstandard unit interval.

**Definition 7 (Single-valued neutrosophic set).** A Single-valued neutrosophic set $\widetilde{\mathcal{B}}^{\mathcal{SVN}}$ of a non-empty set $\mathcal{X}$ is described as $\widetilde{\mathcal{B}}^{\mathcal{SVN}} = \{\langle x;\ I_{\widetilde{\mathcal{B}}^N}(x), J_{\widetilde{\mathcal{B}}^N}(x), V_{\widetilde{\mathcal{B}}^N}(x)\rangle : x \in \mathcal{X}\}$ where $I_{\widetilde{\mathcal{B}}^N}(x), J_{\widetilde{\mathcal{B}}^N}(x)$ and $V_{\widetilde{\mathcal{B}}^N}(x) \in [0, 1]$ for each $x \in \mathcal{X}$ and $0 \leq I_{\widetilde{\mathcal{B}}^N}(x) + J_{\widetilde{\mathcal{B}}^N}(x) + V_{\widetilde{\mathcal{B}}^N}(x) \leq 3$.

**Definition 8:** (**Single-valued pentagonal fuzzy neutrosophic number (SVPFN)**). Let $\zeta_{\widetilde{\rho}}, \sigma_{\widetilde{\rho}}, \psi_{\widetilde{\rho}} \in [0, 1]$ and $r, s, t, u, v \in \mathcal{R}$ where, $r \leq s \leq t \leq u \leq v$. Then a SVPFN, $\widetilde{\rho}^{PN} = \langle (r, s, t, u, v);\ \zeta_{\widetilde{\rho}}, \sigma_{\widetilde{\rho}}, \psi_{\widetilde{\rho}} \rangle$ is a specific neutrosophic set on $\mathcal{R}$, with truth-membership, hesitant-membership, and falsity-membership functions are

$$\zeta_{\widetilde{\rho}^{PN}}(x) = \begin{cases} 0, & x < r; \\ \zeta_{\widetilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x - r)^2\right), & r \leq x \leq s; \\ \zeta_{\widetilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x - t)^2 + 1\right), & s \leq x \leq t; \\ \zeta_{\widetilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x - t)^2 + 1\right), & t \leq x \leq u; \\ \zeta_{\widetilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x - v)^2\right), & u \leq x \leq v; \\ 0, & x > v. \end{cases}$$

$$\sigma_{\widetilde{\rho}^{PN}}(x) = \begin{cases} 0, & x < r; \\ \sigma_{\widetilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x - r)^2\right), & r \leq x \leq s; \\ \sigma_{\widetilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x - t)^2 + 1\right), & s \leq x \leq t; \\ \sigma_{\widetilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x - t)^2 + 1\right), & t \leq x \leq u; \\ \sigma_{\widetilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x - v)^2\right), & u \leq x \leq v; \\ 0, & x > v. \end{cases}$$



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$$\psi_{\tilde{\rho}^{PN}}(x)=\begin{cases}0, & x<r;\\ \psi_{\tilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \psi_{\tilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \psi_{\tilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \psi_{\tilde{\rho}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v.\end{cases}$$

Where $\zeta_{\tilde{\rho}^{PN}}, \sigma_{\tilde{\rho}^{PN}}$, and$\psi_{\tilde{\rho}^{PN}}$ symbolise the max-truth, min-hesitant, and min-falsity membership degrees, respectively. SVPFN, $\tilde{\rho}^{PN}=\langle(r,s,t,u,v);\ \zeta_{\tilde{\rho}},\sigma_{\tilde{\rho}},\psi_{\tilde{\rho}}\rangle$ may described in an ill-defined quantity about $\rho$, that is approximately equal to $[s,u]$.

***Definition 9.*** Suppose, $\tilde{\rho}^{PN}=\langle(r,s,t,u,v);\ \zeta_{\tilde{\rho}},\sigma_{\tilde{\rho}},\psi_{\tilde{\rho}}\rangle$, and $\tilde{q}^{PN}=\langle(r',s',t',u',v');\ \zeta_{\tilde{q}},\sigma_{\tilde{q}},\psi_{\tilde{q}}\rangle$ be two pentagonal neutrosophic numbers. Then we have:

1. $\tilde{\rho}^{PN}\oplus\tilde{q}^{PN}=\langle(r+r',s+s',t+t',u+u',v+v');\ \zeta_{\tilde{\rho}^{PN}}\wedge\zeta_{\tilde{q}^{PN}},\sigma_{\tilde{\rho}^{PN}}\vee\sigma_{\tilde{q}^{PN}},\psi_{\tilde{\rho}^{PN}}\vee\psi_{\tilde{q}^{PN}}\rangle$,
2. $\tilde{\rho}^{PN}\ominus\tilde{q}^{PN}=\langle(r-v',s-u',t-t',u-s',v-r');\ \zeta_{\tilde{\rho}^{PN}}\wedge\zeta_{\tilde{q}^{PN}},\sigma_{\tilde{\rho}^{PN}}\vee\sigma_{\tilde{q}^{PN}},\psi_{\tilde{\rho}^{PN}}\vee\psi_{\tilde{q}^{PN}}\rangle$,
3. $\tilde{\rho}^{PN}\otimes\tilde{q}^{PN}=\langle(rr',ss',tt',uu',vv');\ \zeta_{\tilde{\rho}^{PN}}\zeta_{\tilde{q}^{PN}},\sigma_{\tilde{\rho}^{PN}}+\sigma_{\tilde{q}^{PN}}-\sigma_{\tilde{\rho}^{PN}}\sigma_{\tilde{q}^{PN}},\psi_{\tilde{\rho}^{PN}}+\psi_{\tilde{q}^{PN}}-\psi_{\tilde{\rho}^{PN}}\psi_{\tilde{q}^{PN}}\rangle$,
4. $m\,\tilde{\rho}^{PN}=\begin{cases}\langle(mr,ms,mt,mu,mv);\ \zeta_{\tilde{\rho}^{PN}},\sigma_{\tilde{\rho}^{PN}},\psi_{\tilde{\rho}^{PN}}\rangle, m>0,\\ \langle(mv,mu,mt,ms,mr);\ \zeta_{\tilde{\rho}^{PN}},\sigma_{\tilde{\rho}^{PN}},\psi_{\tilde{\rho}^{PN}}\rangle, m<0,\end{cases}$
5. $\tilde{\rho}^{PN^{-1}}=\langle\left(\frac{1}{v},\frac{1}{u},\frac{1}{t},\frac{1}{s},\frac{1}{r}\right);\ \zeta_{\tilde{\rho}^{PN}},\sigma_{\tilde{\rho}^{PN}},\psi_{\tilde{\rho}^{PN}}\rangle, \tilde{\rho}^{PN}\neq 0.$

**Definition 10 (Accuracy and Score function)**. Let $\tilde{\rho}^{PN}=\langle(r,s,t,u,v);\ \zeta_{\tilde{\rho}^{PN}},\sigma_{\tilde{\rho}^{PN}},\psi_{\tilde{\rho}^{PN}}\rangle$ be a SVPFN number, then

1. Accuracy $AC(\tilde{p}^{PN})=\left(\frac{1}{15}\right)(r+s+t+u+v)*[2+\zeta_{\tilde{\rho}^{PN}}-\sigma_{\tilde{\rho}^{PN}}]$.
2. Score $SC(\tilde{p}^{PN})=\left(\frac{1}{15}\right)(r+s+t+u+v)*\left[2+\zeta_{\tilde{\rho}^{PN}}-\sigma_{\tilde{\rho}^{PN}}-\psi_{\tilde{\rho}^{PN}}\right]$.

**Definition 11.** The order relations between $\tilde{\rho}^{PN}$and $\tilde{q}^{PN}$based on $SC(\tilde{\rho}^{PN})$ and $AC(\tilde{q}^{PN})$ are defined as:

1. If $SC(\tilde{\rho}^{PN})<SC(\tilde{q}^{PN})$, then $\tilde{\rho}^{PN}<\tilde{q}^{PN}$,
2. If $SC(\tilde{\rho}^{PN})=SC(\tilde{q}^{PN})$, then $\tilde{\rho}^{PN}=\tilde{q}^{PN}$,
3. If $AC(\tilde{\rho}^{PN})<AC(\tilde{q}^{PN})$, then $\tilde{\rho}^{PN}<\tilde{q}^{PN}$,
4. If $AC(\tilde{\rho}^{PN})>AC(\tilde{q}^{PN})$, then $\tilde{\rho}^{PN}>\tilde{q}^{PN}$,
5. If $AC(\tilde{\rho}^{PN})=AC(\tilde{q}^{PN})$, then $\tilde{\rho}^{PN}=\tilde{q}^{PN}$.

## 2. PROBLEM STATEMENT and SOLUTION CONCEPT

A vector- minimization problem with fuzzy parameters in the objective functions are formulated as follows

(PQF-VMP) $\min G_l\left(g_{l1}(x_1,\tilde{a}_1^{PQ}),g_{l2}(x_2,\tilde{a}_2^{PQ}),\dots,g_{lN}(x_N,\tilde{a}_N^{PQ})\right), l=\overline{1,L}, L\ge 2$

Subject to

$$H_q\left(h_{q1}(x_1),h_{q2}(x_2),\dots,h_{QN}(x_N)\right)\le\tilde{b}_i^N, q=\overline{1,Q}, i=1,2,\dots,m;$$

$x_n\in X_n, n=\overline{1,N}$.

Where, $X_n\subset\mathfrak{R}^{r_n}, n=\overline{1,N}$ ; $x_n$is a $r_n$ vector, $G_l, l=\overline{1,L}$ and $H_q, q=\overline{1,Q}$ are convex real function of class $C^{(1)}$ on $\mathfrak{R}^N$ and $g_{ln}, h_{qn}. l=\overline{1,L}; q=\overline{1,Q}; n=\overline{1,N}$are real valued functions on $X_n$, and $\tilde{a}^{PQ}=\left(\tilde{a}_{11}^{PQ},\tilde{a}_{22}^{PQ},\dots,\tilde{a}_{ln}^{PQ}\right), l=\overline{1,L}; n=\overline{1,N}$ represent a vector of fuzzy parameters $\inf_{ln}\left(x_n,\tilde{a}_{ln}^{PQ}\right)$, $\tilde{b}_i^N$ is SVTrN numbers. It is assumed that these fuzzy parameters are characterized by Jain [37]), and the PQF-VMP is stable (Rockafellar, [41]).

**Definition 6**.(Dubois and Prade, [12]). The $\alpha$ −level set of the fuzzy numbers $\tilde{a}_{ln}^{PQ}$ is the ordinary set$L_\alpha(\tilde{a}_{ln}^{PQ})$for which the degree of their membership functions exceeds the level $\alpha$:

$$L_\alpha\left(\tilde{a}_{ln}^{PQ}\right)=\left(a_{ln}:\mu_{\tilde{a}_{ln}^{PQ}}(a_{ln})\ge\alpha, l=1,2,\dots,L; n=\overline{1,N}\right)$$

For a certain $\alpha$,PQF-VMP problem is converted into (Sakawa and Yano, [20]) and based on the score function of the SVTrN numbers



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($\alpha$-VMP) $\min G_l\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{lN}(x_N,a_N)\right), l=\overline{1,L}, L\geq 2$

Subject to

$$H_q\left(h_{q1}(x_1), h_{q2}(x_2),\dots,h_{QN}(x_N)\right)\leq 0, q=\overline{1,Q}, a_{ln}\in L_\alpha(\tilde{a}_{ln}^{PQ}),$$
$$x_n\in X_n, n=\overline{1,N}.$$

Since PQF-VMP problem is stable, then the $\alpha$-VMP is stable too.

**Definition 7** (Mine and Fukushima, [2]; Abo-Sinna and Hussein, ([4]). The objective function $G_l$ is called separable if there exist functions $G_l^n, n=\overline{1,N}$ defined on $\Re^n$ and functions $\Omega_l^n$ defined on $\Re^2$ satisfies, for $n=\overline{2,N}$.

$$G_l^n\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{ln}(x_n,a_n)\right)$$
$$=\phi_l^n\left(G_l^{n-1}\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{ln-1}(x_{n-1},a_{n-1})\right), g_{ln}(x_n,a_n)\right), \text{ and}$$
$$G_l^N\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{lN}(x_N,a_N)\right)=G_l\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{ln}(x_n,a_n)\right)$$

Similarly,

$$H_q^n\left(h_{q1}(x_1), h_{q2}(x_2),\dots,h_{QN}(x_N)\right)$$
$$=\chi_q^n\left(H_q^{n-1}\left(h_{q1}(x_2), h_{q2}(x_2),\dots,h_{qn-1}(x_{n-1}), h_{Qn}(x_n)\right)\right), \text{ and}$$
$$H_q^n\left(h_{q1}(x_1), h_{q2}(x_2),\dots,h_{QN}(x_N)\right)=H_q\left(h_{q1}(x_1), h_{q2}(x_2),\dots,h_{QN}(x_N)\right)$$

In the case of all the objective functions and constraints are separable, we say the $\alpha$-VMP problem is separable. Moreover, the functions $\phi_l^n$ and $\chi_q^n$ are called the separating functions of G and H.

Furthermore, the separation of the $\alpha$-VMP is said to be monotone if all $\phi_l^n$ and $\chi_q^n$ are strictly increasing with respect to the first argument for each fixed second argument for each $y\in\Re$,

$\phi_l^n(r,y)>\phi_l^n(\hat{r},y)\Leftrightarrow r>\hat{r}$, and $\chi_l^n(r,y)>\chi_l^n(\hat{r},y)\Leftrightarrow r>\hat{r}$, for every $l=\overline{1,L}$;

$q=\overline{1,Q}$ and , $n=\overline{1,N}$.

**Definition 8.** ( $\alpha$ −pareto optimal solution). The feasible solution $x^*=(x_1^*,x_2^*,\dots,x_N^*), a^*=(a_1^*,a_2^*,\dots,a_N^*)$ to the $\alpha$–VMP is said to be $\alpha$ −pareto optimal solution if there exists no feasible $(x_1',x_2',\dots,x_N'), a^*=(a_1',a_2',\dots,a_N')\in L_\alpha(\tilde{a})$ such that

$G_l\left(g_{l1}(x_1',a_1'), g_{l2}(x_2',a_2'),\dots,g_{ln}(x_N',a_N')\right)\leq G_l\left(g_{l1}(x_1^*,a_1^*), g_{l2}(x_2^*,a_2^*),\dots,g_{ln}(x_N^*,a_N^*)\right)$ for all $l$ and

$G_r\left(g_{s1}(x_1',a_1'), g_{s2}(x_2',a_2'),\dots,g_{sn}(x_N',a_N')\right)< G_r\left(g_{s1}(x_1^*,a_1^*), g_{s2}(x_2^*,a_2^*),\dots,g_{sn}(x_N^*,a_N^*)\right)$, for at least one index $s\in\{1,2,\dots,L\}$.

**Assumption 1:** The $\alpha$-VMP problem is separable and the separation is monotone.

**Assumption 2.** For every $n, X_n\cap L_\alpha(\tilde{a})$ is compact and $G_l^n\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{ln}(x_n,a_n)\right), l=\overline{1,L}$ and $H_q^n\left(h_{q1}(x_1), h_{q2}(x_2),\dots,h_{QN}(x_N)\right), q=\overline{1,Q}$ are continuous functions of $(x_1,x_2,\dots,x_n)$ and $(a_1,a_2,\dots,a_n)$.

Based on the weighting method (Chankong and Haimes, [42]), $\alpha$-VMP problem can be treated as

($\alpha$-VMP$_w$) $\min\sum_{l=1}^{L} w_l G_l\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{lN}(x_N,a_N)\right)$

Subject to

$$H_q\left(h_{q1}(x_1), h_{q2}(x_2),\dots,h_{QN}(x_N)\right)\leq 0, q=\overline{1,Q}, a\in L_\alpha(\tilde{a}),$$
$$x_n\in X_n, n=\overline{1,N}, w\in W=\left\{w\in\Re^L: \textstyle\sum_{l=1}^L w_l=1,\ w_l\geq 0\right\}.$$

It's clear that the stability of ($\alpha$ − VMP) implies to the stability of ($\alpha$-VMP$_w$)

Suppose that every $G_l$ is additive, i. e., for $l=\overline{1,L}$,

$G_l\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{lN}(x_N,a_{nN})\right)=\overline{g}_{l1}(x_1,a_1),\overline{g}_{l2}(x_2,a_2),\dots,\overline{g}_{LN}(x_N,a_{nN})$.

So, the objective function in $\alpha$-VMP$_w$ becomes

$\sum_{n=1}^N\sum_{l=1}^L w_l\,\overline{g}_{Ln}(x_n,a_{nn})=\sum_{n=1}^N wg_n(x_n,a_n)$.

Now, let us define:

$$A_n(w,u)=\min\left\{\begin{matrix}\sum_{i=1}^n wg_i(x_i,a_i): G_l^l\left(g_{l1}(x_1), g_{l2}(x_2),\dots,g_{ln}(x_n)\right)\leq u_q,\\ q=\overline{1,Q}, x_1\in X_1,\dots, x_n\in X_n, n=\overline{1,N}, a\in L_\alpha(\tilde{a})\end{matrix}\right\}, \quad (1)$$



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$$w = (w_1, \dots, w_L) > 0, u = (u_1, \dots, u_Q)$$

The recursive relation , for $n = \overline{1, N}$ is

$$A_n(w, u) = \min_{x_n \in X_n, a \in L_\alpha(\tilde{a})} \left(A_{n-1}\left(w, u^{n-1}(x_n, u)\right) + w g_n(x_n, a_n)\right) \quad (2)$$

Where, $u^{n-1}(x_n, u) = \left(u_1^{n-1}(x_n, u), \dots, u_Q^{n-1}(x_n, u)\right)$.

Assuming the monotonicity of $G_l^l$, let $u_Q^{n-1}$ defined as

$u_Q^{n-1}(x_n, u) = \sup\{\nu \in \mathfrak{R}: u_Q^{n-1}(\nu, h_{qn}(x_n) \le u_q, q = \overline{1, Q}\}$.

**Theorem 1**. Suppose that the assumptions 1, 2 hold. Let $(x_1^*, x_2^*, \dots, x_n^*)$, $(a_1^*, a_2^*, \dots, a_n^*) \in L_\alpha(\tilde{a})$be any $\alpha$ −pareto optimal solution of $A_n(w^*, u)$ for some $w^* \in W$ . Then $(x_1^*, x_2^*, \dots, x_{n-1}^*), (a_1^*, a_2^*, \dots, a_{n-1}^*) \in L_\alpha(\tilde{a})$ is an $\alpha-$ pareto optimal solution of $A_{n-1}(w^*, u^{n-1}(x_n, u))$.

**Proof** (see, Mine and Fukushima, [2]).

## 4. STABILITY SET OF THE FIRST KIND

**Definition 9**. Given a certain $w^*$ with the corresponding $\alpha$ −pareto optimal solution$(x^*, a^*)$, than the stability set of the first kind of $(\alpha - \text{VMP})$ corresponding to $(x^*, a^*)$ is defined as

$S(x^*, a^*) = \left\{(w^*, a^*) \in \mathfrak{R}^{6l}: \text{is an } \alpha - \text{pareto optimal solution of } (\alpha - \text{VMP})\right\}$, where $\mathfrak{R}^{6l} = \mathfrak{R}^{l+5l}, w \in \mathfrak{R}^l, P = (p_1, p_2, p_3, p_4, p_5) \in \mathfrak{R}^{5l}, l = \overline{1, L}, L \ge 2$.

**Theorem 2.** If the functions G and H are convex and $\mu_{\tilde{a}}(a)$ is concave function, then the set $S(x^*, a^*)$ is convex and $S(x^*, a^*) \cup \{0\}$ is closed. Also, if $\text{int}\left(S(x^*, a^*) \cap S(x^\circ, a^\circ)\right) \ne \emptyset$, then $S(x^*, a^*) = S(x^\circ, a^\circ)$.

**Proof**. (Osman, [7]).

### 1.1. Determination of the stability set of the first kind

If a point $(x^*, a^*)$ is an $\alpha$ −pareto optimal solution of $(\alpha - \text{VMP})$, then there exists $w^* \in W$ such that $(x^*, a^*)$ is an $\alpha$ −pareto optimal solution of ($\alpha$-$\text{VMP}_w$) . Therefore, from the stability of ($\alpha$-$\text{VMP}_w$), it follows that there exists $w \in \mathfrak{R}^l, w \ge 0, E \in \mathfrak{R}^Q, E \ge 0$ and $F \in \mathfrak{R}^l, F \ge 0$ such that the following Kuhn- Tucker conditions are satisfied (Mangasarian, [43]).

$$w^T \frac{\partial G}{\partial x}(x^*, a^*) + E^T \frac{\partial H}{\partial x}(x^*) = 0, \quad (3)$$

$$w^T \frac{\partial G}{\partial a}(x^*, a^*) - F^T \frac{\partial \mu_{\tilde{a}}}{\partial a}(a^*) = 0, \quad (4)$$

$$\sum_{i=1}^n w_i = 1, \quad (5)$$

$$H(x^*) \le 0, \quad (6)$$

$$\alpha - \mu_{\tilde{a}}(a^*) \le 0, \quad (7)$$

$$E^T H(x^*) = 0, \quad (8)$$

$$F^T\left(\alpha - \mu_{\tilde{a}}(a^*)\right) = 0, \quad (9)$$

$$w \ge 0, E, \text{and} F \ge 0. \quad (10)$$

Let the two sets $B(x^*)$ and I defined by

$B(x^*) = \left\{q: H_q\left(h_{q1}(x_1^*), h_{q2}(x_2^*), \dots, h_{QN}(x_N^*)\right) = 0\right\}$, and

$I = \left\{l \in \{1, 2, \dots, L\}: \mu_{\tilde{a}_l}(a_l) = \alpha\right\}$.

As a result, we get the two linear independent systems of equations below.

$$w^T \frac{\partial G}{\partial x}(x^*, a^*) + \sum_{q \in B(x^*)} \gamma_q \frac{\partial H_q}{\partial x}(x^*) = 0, \quad (11)$$

$$\sum_{l=1}^L w_l \frac{\partial G_l}{\partial a_\beta}(x^*, a^*) - F_l \frac{\partial \mu_{\tilde{a}l}}{\partial a_l}(a_l^*) = 0, \quad (12)$$

$$\sum_{l=1}^L w_l = 1, w_l \ge 0, l = \overline{1, L}, \gamma_q \ge 0, q \in B(x^*), F_l \ge 0, l \in I, F_l = 0, l \ne I. \quad (13)$$

System (13) can be rewritten as

$$[M' \quad V'] \begin{bmatrix} W \\ \mu \end{bmatrix} = 0 \quad (14)$$

Where, $M' = [c'_{ij}]$ is $r \times L$ matrix, $V' = [v'_{ij}]$ is an $h \times k$ matrix, $w \in \mathfrak{R}^L, \mu \in \mathfrak{R}^K, w \ge 0, w \ne 0$ and $\mu \ge 0, E \in \mathfrak{R}^r, F \in \mathfrak{R}^h$, where $r, h$ are the cardinalities of $B(x^*)$ and I; respectively.

Suppose that $v'_{ij} = 0, j = \overline{1, m}, i \in J \subset \{1, 2, \dots, r\}$, where the cardinal number of $J$ is assumed to be equal $r - m$. Then, we consider the system

$$[M \quad V] \begin{bmatrix} W \\ \mu \end{bmatrix} = 0 \quad (15)$$



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Where, M and V are matrices of order $m \times L$ and $m \times K$; respectively. Therefore, system(11) with the condition $\sum_{j=1}^{L} M'_{ij} w_j = 0, i \in J$, gives system (14) which is equivalent to system (11).

**Proposition 1**. (Zeleny, [44]). If $K \geq m$, then

$$S(x^*, a^*) = \left\{(w, p) \in \Re^{6l}: w^T M^T (V_1^T)_j^{-1} \leq 0, j = \overline{1, m}, \sum_{j=1}^{L} M'_{ij} w_j = 0, i \in J\right\} \tag{16}$$

Where, $V = [V_1 \quad V_2], V_1$ and $V_2$ are $m \times m$ and $m \times k - m$ matrices.

**Proposition 2.** (Zeleny, [44].If $K \geq m$, then

$$(x^*, a^*) = \left\{(w, p) \in \Re^{6l}: (w^T M_2^T - M_1^T (V_1^T)^{-1} V_2^T) = 0, j = \overline{1, k - m}, w^T M^T (V_1^T)_j^{-1} \leq 0, \sum_{j=1}^{L} M'_{ij} w_j = 0, i \in J\right\} \text{ (17)}$$

## 2. AN ALGORITHM

In this section, an algorithm for determining the $S(x^*, a^*)$ can be summarized as in the following steps:

**Step 1:** Start with $\alpha = 0$.

**Step 2:** Define the membership function of the fuzzy number $\tilde{\alpha}$ according to the definition 2.

**Step 3:** Formulate the piecewise quadratic fuzzy dynamic multiobjective problem, i. e., (PQF-VMP)

**Step 4:** Choose $w^* \in W$, that is by usingthe relation (2) to obtain the $\alpha$ −pareto Optimal solution$(x^*, a^*)$ of ($\alpha$-$VMP_w$).

**Step 5:** Substitute with $(x^*, a^*)$ in the Kuhn- Tucker necessary conditions, we have systems (11) and (15). In addition, we can use the Gauss Elimination method for solving system (12).

**Step 6:**Based on the Lagrange multipliers values, we obtain

(i) If $r = q + k - m$, then $S(x^*, a^*) = \{\epsilon w^*: \epsilon > 0\}$,
(ii) If $k \geq m$, then $S(x^*, a^*)$is given by (16),
(iii) If $k < m$, then $S(x^*, a^*)$ is given by (17).

**Step 7:** Set $\alpha = (\alpha + \varepsilon) \in [0, 1]$, and go to step 1.

**Step 8:**Repeat interval at the steps of the algorithm until the $[0, 1]$ is fully exhausted.

Step 9: stop.

## 3. A NUMERICAL EXAMPLE

Take into account the following (PQF-VMP)

$$min \begin{pmatrix} g_1(x, \tilde{a}_1^{PQ}) = (x_1 - \tilde{a}_{11}^{pq})^2 + x_2^2 + x_3^2, \\ g_2(x, \tilde{a}_2^{PQ}) = (x_1 - 1)^2 + (x_2 + \tilde{a}_{22}^{PQ}) + (x_3 - 2)^2, \\ g_3(x, \tilde{a}_3^{PQ}) = 2x_1 + x_2^2 + (x_3 - \tilde{a}_{33}^{PQ})^2 \end{pmatrix} \tag{18}$$

Subject to

$$Q = \{x \in \Re^3: x_1 + x_2 + x_3 \leq \tilde{b}^N, \ x_j \geq 0, j = 1, 2, 3\}. \tag{19}$$

Where, $\tilde{a}_{11}^{pq} = (0, 0.2, 3, 4{,}6, 6), \tilde{a}_{22}^{PQ} = (0.2, 0.4, 4, 5.6, 7), \tilde{a}_{33}^{PQ} = (2, 3.4, 6, 9.8, 11), \tilde{b}^N = \langle(5, 7, 9, 11); 0.9, 0.7, 0.5\rangle$

The close intervals approximation for $\tilde{a}_{11}^{pq}, \tilde{a}_{22}^{PQ}$, and $\tilde{a}_{33}^{PQ}$ are:

$\tilde{a}_{11}^{pq} \in [0.2, 4.6], \tilde{a}_{22}^{PQ} \in [0.4, 5.6]$ and $\tilde{a}_{33}^{PQ} \in [3.4, 9.8]$.



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The (0.5-VMP) can be written as

$$\min\begin{pmatrix} g_1(x,a_1) = (x_1 - a_{11})^2 + x_2^2 + x_3^2, \\ g_2(x,a_2) = (x_1 - 1)^2 + (x_2 + a_{22}) + (x_3 - 2)^2, \\ g_3(x,a_3) = 2x_1 + x_2^2 + (x_3 - a_{33})^2 \end{pmatrix} \tag{21}$$

Subject to

$$\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3 \le 3, \quad \tilde{\mathrm{a}}_{11}^{\mathrm{pq}} \in [0.2, 4.6], \tilde{\mathrm{a}}_{22}^{\mathrm{PQ}} \in [0.4, 5.6] \text{ and } \tilde{\mathrm{a}}_{33}^{\mathrm{PQ}} \in [3.4, 9.8], x_1, x_2, x_2 \ge 0. \tag{22}$$

By applying the weighting method (Chankong and Haimes, [42]), then we have

$$\sum_{l=1}^{3} w_l g_1(x,a)$$

Subject to (23)

Constraints in (22).

At $w_0 = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, then the dynamic programming approach steps:

Firstly,

$$\mathrm{A}_1(\mathrm{w}^*,0) = \min\left\{\sum_{l=1}^{3} w_l^0 g_{l1}(x_1, a_{11}): x_1 \le 3, 0.2 \le a_{11} \le 4.6, x_1 \ge 0\right\}$$

$$=\min\left\{\frac{1}{3}(x_1 - a_{11})^2 + \frac{1}{3}(x_1 - 1)^2 + \frac{2}{3}x_1: x_1 \le 3, 0.2 \le a_{11} \le 4.6, x_1 \ge 0\right\}.$$

The 0.5 −pareto optimal solution is$(x_1^*, \mathrm{a}_{11}^*) = (0.1, 0.2)$.

Secondly,

$$\mathrm{A}_2(\mathrm{w}^*,0) = \min\left\{\begin{matrix}\sum_{l=1}^{3} w_l^0 g_{l1}(x_1^*, \mathrm{a}_{11}^*) + g_{l2}(x_2,a_{22}): x_1^* + x_2 \le 3, x_1^*, x_2 \ge 0, \\ 0.4 \le a_{22} \le 5.6\end{matrix}\right\}$$

$$=\min\left\{\begin{matrix}0.5 + \frac{1}{3}x_2^2 + \frac{1}{3}(x_2 + a_{22})^2 + \frac{1}{3}x_2^2: 0.1 + x_2 \le 3, \\ 0.4 \le a_{22} \le 5.6, x_2 \ge 0\end{matrix}\right\}.$$

The 0.5 −pareto optimal solution is$(x_1^*, x_2^*, \mathrm{a}_{11}^*, a_{22}^*) = (0.1, 0, 0.2, 0.4)$.

Thirdly,

$$\mathrm{A}_3(\mathrm{w}^*,0) = \min\left\{\begin{matrix}\sum_{l=1}^{3} w_l^0 g_{l1}(x_1^*, \mathrm{a}_{11}^*) + g_{l2}(x_2^*, a_{22}^*) + g_{l3}(x_3,a_{33}): \\ x_1^* + x_2^* + x_3 \le 3, x_1^*, x_2^*, x_3 \ge 0, \\ 3.4 \le a_{33} \le 9.8\end{matrix}\right\}$$

$$=\min\left\{\begin{matrix}0.39 + \frac{1}{3}x_3^2 + \frac{1}{3}(x_3 - 2)^2 + \frac{1}{3}(x_3 - a_{33})^2: 0.1 + x_3 \le 3, \\ 3.4 \le a_{33} \le 9.8, x_3 \ge 0\end{matrix}\right\}.$$

The 0.5 −pareto optimal solution is

$$(x_1^*, x_2^*, \mathrm{x}_3^* \ \mathrm{a}_{11}^*, a_{22}^*, a_{33}^*) = (0.1, 0, 1.8, \quad 0.2, 0.4, 0.34).$$

Now, let us determine $\mathrm{S}(\mathrm{x}^*, \mathrm{a}^*)$ as

Systems (11) and (12) allowed

$$-0.2w_1 - 1.8w_2 + 2w_3 + \gamma_1 = 0,$$

$$0w_1 + 0.8w_2 + 0w_3 + \gamma_2 = 0,$$

$$3.6w_1 - 0.4w_2 - 3.2w_3 + \gamma_3 = 0,$$

$$0.2w_1 + 0w_2 + 0w_3 - \omega_2 = 0,$$



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$$0w_1 + 0.8w_2 + 0w_3 - \omega_3 \ = 0,$$

$$0w_1 + 0w_2 + 3.2w_3 - \omega_4 = 0,$$

Then

$$M = \begin{bmatrix} -0.2 & -1.8 & 2 \\ 0 & 0.8 & 0 \\ 3.6 & -0.4 & -3.2 \end{bmatrix},\ V = V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M^T(V_1^T)^{-1} = \begin{bmatrix} -0.2 & 0 & 3.6 \\ -1.8 & 0.8 & -0.4 \\ 2 & 0 & -3.2 \end{bmatrix}$$

$$w^T M^T (V_1^T)^{-1} = [w_1 \quad w_2 \quad w_3] \begin{bmatrix} -0.2 & 0 & 3.6 \\ -1.8 & 0.8 & -0.4 \\ 2 & 0 & -3.2 \end{bmatrix} =$$

$$(-0.2w_1 - 1.8w_2 + 2w_3 \quad 0.8w_2 \quad 3.6w_1 - 0.4w_2 - 3.2w_3).$$

Thus,

$$S(0.1, 0, 1.8, \qquad 0.2, 0.4, 0.34) = \left\{ \begin{matrix} (w,p,s,h): w_1 \geq 9w_2 - 10w_3,\ w_2 \geq 8w_3 - 9w_1, \\ w_1 + w_2 + w_3 = 1, p_1 < p_2 = 1 - 4p_1 < p_3 < p_4, \\ 0 < p_1 < 0.2, s_1 < s_2 = 1 - 1.5s_1 < s_3 < s_4, \\ 0 < s_1 < 0.4,\ h_1 < h_2 = 17 - 4h_1 < h_3 < h_4, \\ 000 < h < 3.4 \end{matrix} \right\}$$

## 4. CONCLUSIONS AND FUTURE WORKS

The dynamic multiobjective programming issue with piecewise quadratic fuzzy parameters has been investigated in this study. The stability set of the first kind has been identified, and the algorithm allows for the decomposition of the parametric w- space. Future work may include the further extension of this study to other fuzzy-like structures i.e. Interval-valued fuzzy set, Intuitionistic fuzzy set, Pythagorean fuzzy set, Spherical fuzzy set, Picture fuzzy set, Neutrosophic set etc. with more discussion on real world problems.

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