Bayesian Approximation Methods of the Estimation for Generalized Exponential Distribution and Neutrosophic Approximation Methods

Hanaa Saad M. Shebib[1,*], Rawaa S. AL-Saffar[2]

[1]Department of Economics, Faculty of Administration and Economics, University of Al-Kufa, Iraq
[2]Department of Statistics, Faculty of Administration and Economics, Mustanisiryah University, Iraq
Emails: hanas.mohammed@uokufa.edu.iq;Rawaaalsaffar@uomustansiriyah.edu.iq

Abstract

In this article, we used approximate methods to obtain Bayes method estimations for the shape and scale parameters of the generalized exponential distribution, as three approximation methods were employed: Lindley approximation and neutrosophic Lindley approximation, Gibbs sampling and neutrosophic Gibbs sampling, the most important samples based on the gamma informative prior under the squared error loss function. Through different simulation experiments a comparison was made between those estimators of these three approximate methods, from the simulation results we found a relative preference for the important sampling method over the other two methods. The results of simulation experiments were also confirmed by applying these approximate methods to real data representing the operating times of one of the machines of the publishing, printing, and translation house in Baghdad. On the other hand, we apply the same method to the neutrosophic exponential distribution, and the results will be compared to the classical case.

Keywords: Generalized exponential distribution; neutrosophic exponential distribution; approximation; Bayes method.

Introduction:

The two-parameters Generalized Exponential Distribution (GED) is one of the continuous distributions introduced by Gupta and Kundu (1999) [7] as a special case of the three-parameter generalized Weibull distribution, the (GED) was used very effectively in survival data analysis. In recent years, impressive papers have been devoted to the study of estimations and inferences about (GED) parameters using the Bayesian method among them Gupta and Kundu (2008) [10], Nasiri and Pazira (2011) [15], Kundu and Pradhan (2009) [11]. These papers explained that the Bayesian estimation for (GED) parameters cannot be obtained in explicit forms, therefore, most researchers dealt with the approximation methods as Lindley approximate (LA) and Gibbs sampling (GS). Moreover, the MCMC methods may suffer with the difficulty of the posterior distributions generation when dealing the (GED) data. So in this research we proposed to employing the important sampling (IS) method as an additional or alternative method, then compare it with the two other methods the (GS) and (LA), we obtain the Bayesian estimators based on the gamma prior under the squared error loss functions (SELF), Monte Carlo simulations were performed to compare the performances of the Bayes estimates under the three methods. Also, real data that represents the operating times of a machine in the House of printing, publishing and translation in Baghdad has been analyzed by using (AL) and (IS) methods for comparison purpose and confirmation the simulation results.

In [21-25], many examples of neutrosophic probability theory and distributions were presented by many authors. There are many interesting results about computing the neutrosophic probabilities, statistical approximations, and neutrosophic simulations.

Generalized Exponential Distribution: [5], [9], [15]

The p.d.f for (GED) random variable is defined as:

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$$f(y,\omega,\theta) = \omega\theta\left(1-e^{-\theta y}\right)^{\omega-1} e^{-\theta y} \quad ,\omega,\theta > 0, y \geq 0 \tag{1}$$

where ω, θ are the shape and scale parameters respectively.

For a random sample $(y_1, y_2, \dots, y_n)$ from (GED) the Likelihood estimators for (ω) and (θ) can be found as follows:

$$L(y_i,\omega,\theta) = \omega^n\theta^n\, e^{-\theta\sum y_i}\prod_{i=1}^{n}\left(1-e^{-\theta y_i}\right)^{\omega-1} \tag{2}$$

$$lnL(y_i,\omega,\theta) = nln(\omega) + nln(\theta) + (\omega-1)\sum_{i=1}^{n} ln\left(1-e^{-\theta y_i}\right) - \theta\sum_{i=1}^{n} y_i \tag{3}$$

The first derivative of both sides of equation (3) with respect to (ω) and (θ), then equaling the results to zero, we obtain the MLE of (ω) as a function of (θ)

$$\widehat{\omega}(\theta) = -\frac{n}{\sum_{i=1}^{n} ln(1-e^{-\theta y_i})} \tag{4}$$

putting $(\omega) = \widehat{\omega}(\theta)$ in (3) to get:

$$f(\theta) = D - nln\sum_{i=1}^{n} - ln\left(1-e^{-\theta y_i}\right) + nln(\theta) - \sum_{i=1}^{n} ln\left(1-e^{-\theta y_i}\right) - \theta\sum_{i=1}^{n} y_i \tag{5}$$

where D is a constant, $D = [nln(n) - n]$.

Therefore, by maximizing (5) with respect to (θ) we can obtain the MLE of (θ), say $\left(\hat{\theta}_{ML}\right)$. Gupta and Kundu proved that the function $f(\theta) = lnL[\widehat{\omega}(\theta),\theta]$ is a unimodal function, the value of $\left(\hat{\theta}_{ML}\right)$ which makes equation (3) as greatest as possible can be obtained from the fixed points of the blow solution:

$$g(\theta) = \theta = \left[\frac{\sum_{i=1}^{n}\frac{y_i e^{-\theta y_i}}{1-e^{-\theta y_i}}}{\sum_{i=1}^{n} ln(1-e^{-\theta y_i})} + \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{1-e^{-\theta y}}\right]^{-1} \tag{6}$$

then we can get the MLE for (ω) say $(\widehat{\omega}_{ML})$ form (4) as $\widehat{\omega}_{ML} = \widehat{\omega}\left(\hat{\theta}_{ML}\right)$.

Furthermore, the Fisher Information matrix is given as:

$$I(\omega,\theta) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 L}{\partial\omega^2}\right) & E\left(\frac{\partial^2 L}{\partial\omega\partial\theta}\right) \\ E\left(\frac{\partial^2 L}{\partial\theta\partial\omega}\right) & E\left(\frac{\partial^2 L}{\partial\theta^2}\right)\end{pmatrix} \quad (7)$$

$$E\left(\frac{\partial^2 L}{\partial\omega^2}\right) = -\frac{n}{\omega^2}$$

$$E\left(\frac{\partial^2 L}{\partial\omega\partial\theta}\right) = \frac{n}{\theta}\left[\frac{\omega}{\omega-1}\{\psi(\omega)-\psi(1)\} - \{\psi(\omega+1)-\psi(1)\}\right] \qquad \omega > 2$$

$$E\left(\frac{\partial^2 L}{\partial\theta^2}\right) = -\frac{n}{\theta^2}\left[1 + \frac{\omega(\omega-1)}{\omega-2}\left\{\psi'(1) - \psi'(\omega-1) + \left(\psi(\omega-1)-\psi(1)\right)^2\right\}\right] - \frac{n\omega}{\theta^2}\left[\psi'(1) - \psi(\omega) + \{\psi(\omega)-\psi(1)\}^2\right] \qquad \omega > 2$$

$$\left(\frac{\partial^2 L}{\partial\omega\partial\theta}\right) = \frac{n\omega}{\theta}\int_0^\infty y e^{-2y}(1-e^{-y})^{\omega-2}dy \qquad 0 < \omega \leq 2$$

$$E\left(\frac{\partial^2 L}{\partial\theta^2}\right) = -\frac{n}{\theta^2} - \frac{n\omega(\omega-1)}{\theta^2}\int_0^\infty y e^{-2y}(1-e^{-y})^{\omega-3}dy \qquad 0 < \omega \leq 2$$

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ψ is digamma function, $\psi(z) = \frac{d}{dz}\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, Γ is gamma function

Bayes Method: [2], [4], [6], [10], [14]

In this section, we will use the Bayes method to estimate the unknown parameters, assuming that ω and θ have the following gamma prior distributions:

$$\pi(\omega) \propto \omega^{b-1} e^{-d\omega} \qquad ,\omega, b, d > 0 \tag{8}$$

$$\pi(\theta) \propto \theta^{c-1} e^{-h\theta} \qquad ,\theta, c, h > 0 \tag{9}$$

thus the joint prior distribution for ω and θ:

$$\pi(\omega,\theta) \propto \omega^{b-1}\,\theta^{c-1} e^{-d\omega}\, e^{-h\theta} \qquad ,\theta,\omega,b,d,c,h > 0 \tag{10}$$

where the parameters b, d, c, and h are non-negative and known.

Suppose that $(y_1, y, \dots, y_n)$ is a random sample from $GE(\omega,\theta)$, then based on (2) and (10) we can obtain the joint posterior density function of the parameters ω and θ as follows:

$$Q(\omega,\theta|y) = \frac{L(y|\omega,\theta)\,\pi(\omega,\theta)}{\int\int_{\forall\omega\forall\theta} L(y|\omega,\theta)\,\pi(\omega,\theta)\,d\omega\,d\theta}$$

$$Q(\omega,\theta|y) = \frac{\omega^{n+b-1}\, e^{-\omega\left[d-\sum_{i=1}^{n}\ln\left(1-e^{-\theta y_i}\right)\right]}\,\theta^{n+c-1} e^{-\theta\left(\sum_{i=1}^{n} y_i+h\right)}}{\iint_{\forall\omega\forall\theta}\omega^{n+b-1}\, e^{-\omega\left[d-\sum_{i=1}^{n}\ln\left(1-e^{-\theta y_i}\right)\right]}\,\theta^{n+c-1} e^{-\theta\left(\sum_{i=1}^{n} y_i+h\right)}\,d\omega\,d\theta} \tag{11}$$

Then, the Bayes estimator under the squared error loss function of the $g(\omega,\theta)$ which is a function for the ω and θ can be defined as:

$$g(\hat{\omega},\hat{\theta})_B = \frac{\int\int_{\forall\omega\forall\theta} g(\omega,\theta) L(y|\omega,\theta)\,\pi(\omega,\theta)\,d\omega\,d\theta}{\int\int_{\forall\omega\forall\theta} L(y|\omega,\theta)\,\pi(\omega,\theta)\,d\omega\,d\theta}$$

$$g(\hat{\omega},\hat{\theta})_B = \frac{\iint_{\forall\omega\forall\theta}\omega^{n+b}\, e^{-\omega\left[d-\sum_{i=1}^{n}\ln\left(1-e^{-\theta y_i}\right)\right]}\,\theta^{n+c} e^{-\theta\left(\sum_{i=1}^{n} y_i+h\right)}\,d\omega\,d\theta}{\iint_{\forall\omega\forall\theta}\omega^{n+b-1}\, e^{-\omega\left[d-\sum_{i=1}^{n}\ln\left(1-e^{-\theta y_i}\right)\right]}\,\theta^{n+c-1} e^{-\theta\left(\sum_{i=1}^{n} y_i+h\right)}\,d\omega\,d\theta} \tag{12}$$

Clearly from (12), it is not easy to find an exact solution to get the estimators of ω and θ, so we must use approximation methods.

Lindley Approximation: [10], [13]

The 3rd method to solve (12) is Lindley approximation (LA), in the case of two parameters, Lindley's approximation for $\hat{g} = g(\hat{\omega},\hat{\theta})$ can be written as follows:

$$\hat{g} = g(\hat{\omega},\hat{\theta}) + \frac{1}{2}[U + l_{30}V_{12} + l_{03}V_{21} + l_{21}W_{21} + l_{12}W_{12}] + P_1U_{12} + P_2U_{21}$$

where:

$$U = \sum_{i=1}^{2}\sum_{j=1}^{2}\mu_{ij}\,\delta_{ij} \quad , \quad l_{ij} = \frac{\partial_{i+j} lnL(y_i,\omega,\theta)}{\partial\omega_i\,\partial\theta_j} \qquad ,i,j = 0,1,2,3 \quad ,i+j = 3$$

$$P_1 = \frac{\partial ln\{\pi(\omega,\theta)\}}{\partial\omega} \quad , \quad P_2 = \frac{\partial ln\{\pi(\omega,\theta)\}}{\partial\theta} \quad , \quad \mu_{ij} = \frac{\partial^2 g}{\partial\omega_i\,\partial\theta_j} \quad , \quad \mu_i = \frac{\partial g}{\partial\omega_i} \quad , \quad \mu_j = \frac{\partial g}{\partial\theta_j}$$

$$U_{ij} = \mu_i\delta_{ii} + \mu_j\delta_{ji} \;,\; V_{ij} = (\mu_i\delta_{ii} + \mu_j\delta_{ij})\delta_{ii} \;,\; W_{ij} = 3\mu_i\delta_{ii}\delta_{ij} + \mu_j(\delta_{ii}\delta_{jj} + 2\delta_{ij}^2)$$

here:

$\hat{\omega}$ and $\hat{\theta}$ the MLEs of ω and θ respectively.

δ_{ij} the $(i,j)th$ element of the inverse of the Fisher information matrix.

therefore, we have:

$$l_{30} = \frac{2n}{\hat{\omega}^3}, l_{03} = \frac{2n}{\hat{\theta}^3} + (\hat{\omega}-1)\sum_{i=1}^{n}\frac{y_i^3 e^{-\hat{\theta}y_i}\left(1+e^{-\hat{\theta}y_i}\right)}{\left(1-e^{-\hat{\theta}y_i}\right)^3}, l_{12} = -\sum_{i=1}^{n}\frac{y_i^3 e^{-\hat{\theta}y_i}}{\left(1-e^{-\hat{\theta}y_i}\right)^2}, l_{12} = 0$$

$$\delta_{11} = \frac{\left[\frac{n}{\hat{\theta}^2} + (\hat{\omega}-1)\sum_{i=1}^{n}\frac{y_i^3 e^{-\hat{\theta}y_i}\left(1+e^{-\hat{\theta}y_i}\right)}{\left(1-e^{-\hat{\theta}y_i}\right)^3}\right]}{\left[\left\{\frac{n}{\hat{\omega}^2}\right\}\left\{\frac{n}{\hat{\theta}^2} + (\hat{\omega}-1)\sum_{i=1}^{n}\frac{y_i^3 e^{-\hat{\theta}y_i}\left(1+e^{-\hat{\theta}y_i}\right)}{\left(1-e^{-\hat{\theta}y_i}\right)^3}\right\}\right] - \left[\sum_{i=1}^{n}\frac{y_i^3 e^{-\hat{\theta}y_i}\left(1+e^{-\hat{\theta}y_i}\right)}{\left(1-e^{-\hat{\theta}y_i}\right)^3}\right]^2}$$

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$$\delta_{12}=\delta_{21}=\frac{\left[\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right]}{\left[\left\{\frac{n}{\hat\omega^2}\right\}\left\{\frac{n}{\hat\theta^2}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\right]-\left[\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right]^2}$$

$$\delta_{22}=\frac{\left[\frac{n}{\hat\omega^2}\right]}{\left[\left\{\frac{n}{\hat\omega^2}\right\}\left\{\frac{n}{\hat\theta^2}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\right]-\left[\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right]^2}$$

Now when $g(\omega,\theta)=\omega$ we have $\mu_1=1,\ \mu_2=0,\ \mu_{ij}=0,\ i,j=1,2$, therefore:
$U=0, U_{12}=\delta_{11}, U_{21}=\delta_{12}, V_{12}=\delta_{11}^2, V_{21}=\delta_{21}\delta_{22}$
$W_{12}=3\delta_{11}\delta_{12}, W_{21}=\delta_{11}\delta_{22}+2\delta_{21}^2$
$P_1=\left(\frac{h-1}{\hat\omega}-c\right), P_2=\left(\frac{d-1}{\hat\omega}-b\right)$
and when $g(\omega,\theta)=\theta$ we have $\mu_1=0,\ \mu_2=1,\ \mu_{ij}=0,\ i,j=1,2$, therefor:
$U=0, U_{12}=\delta_{21}, U_{21}=\delta_{22}, V_{12}=\delta_{12}\delta_{11}, V_{21}=\delta_{22}^2$
$W_{12}=\delta_{11}\delta_{22}+2\delta_{12}^2, W_{21}=3\delta_{22}\delta_{21}$
From the above the Lindley approximate for the Bayes estimators of ω and θ under the squared errors loss function are respectively:

$$\hat\omega_{BLA}=\frac{1}{2}\left[\frac{2n}{\hat\omega^3}\delta_{11}^2\left\{\frac{2n}{\hat\theta^3}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\delta_{21}\delta_{22}-\left\{\sum_{i=1}^{n}\frac{y_i^2e^{-\hat\theta y_i}}{\left(1-e^{-\hat\theta y_i}\right)^2}\right\}(\delta_{11}\delta_{22}+2\delta_{21}^2)\right]+\left(\frac{h-1}{\hat\omega}-c\right)\delta_{11}+\left(\frac{d-1}{\hat\omega}-b\right)\delta_{12}\quad(13)$$

$$\hat\theta_{BLA}=\frac{1}{2}\left[\frac{2n}{\hat\omega^3}\delta_{11}\delta_{12}\left\{\frac{2n}{\hat\theta^3}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\delta_{22}^2-\left\{\sum_{i=1}^{n}\frac{y_i^2e^{-\hat\theta y_i}}{\left(1-e^{-\hat\theta y_i}\right)^2}\right\}\delta_{22}\delta_{21}\right]+\left(\frac{h-1}{\hat\omega}-c\right)\delta_{11}+\left(\frac{d-1}{\hat\omega}-b\right)\delta_{22}\quad(14)$$

Gibbs Sampling: [3], [10], [16], [17]

The first method that solves the ratio of the two integrals in posterior distribution is MCMC specifically Gibbs sampling (GS), since the joint posterior of ω and θ is defined in (11), then the conditional posterior densities $(\omega|\theta)$ and $(\theta|\omega)$ can be written respectively as:

$$Q(\omega|\theta,y)=\omega^{n+b-1}\,e^{-\omega\left[b-\sum_{i=1}^{n}ln\left(1-e^{-\theta y_i}\right)\right]}\quad(15)$$

$$Q(\theta|\omega,y)=\theta^{n+c-1}\,e^{-\theta\left(\sum_{i=1}^{n}y_i+h\right)}\prod_{i=2}^{n}\left(1-e^{-\theta y_i}\right)^{\omega-1}\quad(16)$$

Note that when $\omega\ge1$ we can generate a random sample directly from the log-concave density $Q(\omega|\theta,y)$, and when $\omega<1$ we can depend on the order statistics to perform generation using the (GED), so we can re-write (16) as follow:

$$Q(\theta|\omega,y)=e^{-\theta y_{(1)}}\left(1-e^{-\theta y_i}\right)^{\omega-1}\left[\theta^{n+c-1}\,e^{-\theta\left(\sum_{i=2}^{n}y_i+h\right)}\prod_{i=2}^{n}\left(1-e^{-\theta y_i}\right)^{\omega-1}\right]\quad(17)$$

where $y_{(1)},\dots,y_{(n)}$ are the orders of $y's$, $y_{(1)}$ is a continuous function goes to finite constant as $\theta\to0$ and goes to 0 as $\theta\to\infty$. That leads to the function having a finite maximum.

We can summarized (GS) that applied to find the Bayes estimators for ω and θ as follows:

1. Assume initial values for ω and θ say $\left(\omega_0=\hat\omega_{ML},\theta_0=\hat\theta_{ML}\right)$
2. Generate parameters ω^j and θ^j from the posterior $Q(\omega_0,\theta_0|y)$.
3. Let $j=j+1$
4. Generate new values for parameters ω,θ say ω_{j+1} and θ_{j+1} from the posteriors $Q(\omega_j|\theta_j,y)$ and $Q(\theta_j|\omega_j,y)$ respectively.
5. Repeat step 4 such that $j=1,2,3,\dots,N$.
6. The final Bayes estimator for $g(\omega,\theta)$ under squared loss function according to the following formula:

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$$g(\widehat{\omega},\hat{\theta})_{BGS} = E[Q(\omega,\theta|y)] = \frac{1}{N-M}\sum_{j=M+1}^{N} g(\omega_j,\theta_j) \quad (18)$$

where M is the number of observations that are burned to remove the influence of the initial sampling values.

Importance Sampling: [1], [11], [12]

To solve (12) we suggest employing an Importance Sampling (IS) method, the (IS) mechanism assumes that the integral which is formulated as follows:

$$G = E\{g(\varphi)\} = \frac{\int_{\forall\varphi} g(\varphi)p(\varphi)q(\varphi)d\varphi}{\int_{\forall\varphi} p(\varphi)q(\varphi)d\varphi} \quad (19)$$

where:

$q(\varphi)$: a function of (φ).

$p(\varphi)$: a generator function from known probability density function

$g(\theta)$: any function of (φ) to be estimated

Then (G) have an approximate solution equal to

$$\hat{G} \approx \frac{1}{N}\sum_{i=1}^{N} g_i(\varphi_i)w_i(\varphi_i) \quad (20)$$

Where $w(\varphi_i)$ is the (IS) weighted function

$$w(\varphi_i) = \frac{q(\varphi_i)}{\sum_{j=1}^{N} q(\varphi_j)} \quad , \quad i \neq j \quad (21)$$

Now to apply the (IS) let's re-write the Likelihood function as:

$$L(y_i,\omega,\theta) \propto \omega^n\theta^n\, e^{\left[-n\theta\bar{y}-(\omega-1)\sum_{i=1}^{n}\ln\left(1-e^{-\theta y_i}\right)\right]} \quad (22)$$

then from (8) and (22), the joint posterior density becomes:

$$Q(\omega,\theta|y) \propto \omega^{n+b}\, e^{-\omega\left[d-\sum_{i=1}^{n}\ln\left(1-e^{-\theta y_i}\right)\right]}\, \theta^{n+c}e^{-n\theta\bar{y}+h}\, e^{-\sum_{i=1}^{n}\ln\left(1-e^{-\theta y_i}\right)}$$

that is:

$$Q(\omega,\theta|y) \propto gamma\{n+b\,,d-S_\theta\}\; gamma\{n+c\,,n\bar{y}+h\}\; e^{S_\theta}$$

where $S_\theta = -\sum_{i=1}^{n}\ln\left(1-e^{-\theta y_i}\right)$

Thus the marginal posterior density of θ is given by:

$$Q(\theta|\omega,y) \propto \theta^{n+c}e^{-n\theta\bar{y}+h}\left[(S_\theta+d)^{-(n+c)}\, e^{S_\theta}\right] \quad (23)$$

that is:

$$Q(\theta|\omega,y) \propto gamma\{n+c\,,n\bar{y}+h\}\; Z(\theta)$$

where $Z(\theta) = \left[(S_\theta+d)^{-(n+c)}e^{S_\theta}\right]$

From (23) we can express the Bayes estimate of θ say $\hat{\theta}_{BIS}$ by using importance sampling as:

$$\hat{\theta}_{BIS} = \frac{E[\theta\, Z(\theta)]}{E[Z(\theta)]} \quad (24)$$

Now the marginal posterior density for ω given θ and y is:

$$Q(\omega|\theta,y) \propto gamma\{n+b\,,d-S_\theta\}$$

then the consequently marginal posterior density can be obtained from the following conditional formula:

$$Q(\omega|\theta,y) = \frac{E[Z(\theta)\; gamma\{n+b\,,d-S_\theta\}]}{E[gamma\{n+c\,,h-S_\theta\}]} \quad (25)$$

The Bayes estimator of ω say ω_{BIS} depends on the fact that $E(\omega|\theta,y) = (n+b)/(S_\theta+d)$ and obtained as:

$$\widehat{\omega}_{BIS} = \frac{E[Z(\theta)\,(n+b)/(S_\theta+d)]}{E[Z(\theta)]} \quad (26)$$

We can summarized the (IS) steps to find the Bayes estimators for ω and θ as follows:

1. Generate θ_1 from $gamma\{n+c\,,n\bar{y}+h\}$.
2. Calculate $S_{\theta 1}$, $Z_1(\theta_1)$ and $Q_1(\theta_1|\omega,y)$ from (28).
3. Generate $(\omega_1|\theta_1)$ from $gamma\{n+b\,,d-S_\theta\}$
4. Calculate $Q_1(\omega_1|\theta_1,y)$.
5. Repeat steps above to obtain $(\theta_1,\theta_2,\dots,\theta_N)$ and $(\omega_1,\omega_2,\dots,\omega_N)$.
6. For each sample in the above step calculate $[Q_1(\theta_1|\omega,y),\dots,Q_1(\theta_n|\omega,y)]$ and $[Q_1(\omega_1|\theta_1,y),\dots,Q_n(\omega_n|\theta_n,y)]$.
7. The (IS) Bayes estimators for θ and ω under (SLF) can be obtained respectively as the following formulas:

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$$\hat{\theta}_{BIS} = \frac{\sum_{i=1}^{N} \theta_i \ Z_i(\theta_i)}{\sum_{i=1}^{N} Z_i(\theta_i)} \quad (27)$$

$$\hat{\theta}_{BIS} = \frac{\sum_{i=1}^{N} [(n+b)/(S_{\theta i}+d)] \ Z_i(\theta_i)}{\sum_{i=1}^{N} Z_i(\theta_i)} \quad (28)$$

Neutrosophic Bayes Method:

In this section, we will use the Bayes method to estimate the unknown neutrosophic parameters, assuming that $\omega = w_0 + w_1 I$ and $\theta = \theta_0 + \theta_1 I$ have the following gamma prior distributions:

$\pi(\omega) \propto \omega^{b-1} e^{-d\omega} \qquad , w_0, w_0 + w_1, b, d > 0$

$\pi(\theta) \propto \theta^{c-1} e^{-h\theta} \qquad , \theta_0, \theta_0 + \theta_1, c, h > 0$

thus the joint prior distribution for ω and θ:

$$\pi(\omega, \theta) \propto \ \omega^{b-1} \, \theta^{c-1} e^{-d\omega} \, e^{-h\theta} \qquad , \theta_0, \theta_0 + \theta_1, w_0, w_0 + w_1, b, d, c, h > 0$$

where the parameters b, d, c, and h are non-negative and known.

Suppose that $(y_1, y, \dots, y_n)$ is a random sample from $GE(\omega, \theta)$, then based on (2) and (10) we can obtain the joint posterior density function of the parameters ω and θ as follows:

$$Q(\omega, \theta | y) = \frac{L(y|\omega, \theta) \, \pi(\omega, \theta)}{\int \int_{\forall\omega\forall\theta} L(y|\omega, \theta) \, \pi(\omega, \theta) \, d\omega \, d\theta}$$

$$Q(\omega, \theta | y) = \frac{\omega^{n+b-1} \, e^{-\omega\left[d - \sum_{i=1}^{n} \ln\left(1 - e^{-\theta y_i}\right)\right]} \, \theta^{n+c-1} e^{-\theta\left(\sum_{i=1}^{n} y_i + h\right)}}{\iint_{\forall\omega\forall\theta} \omega^{n+b-1} \, e^{-\omega\left[d - \sum_{i=1}^{n} \ln\left(1 - e^{-\theta y_i}\right)\right]} \, \theta^{n+c-1} e^{-\theta\left(\sum_{i=1}^{n} y_i + h\right)} \, d\omega \, d\theta} \quad (11)$$

Where $y = y_0 + y_1 I$

Then, the Bayes estimator under the squared error loss function of the $g(\omega, \theta)$ which is a function for the ω and θ can be defined as:

$$g(\hat{\omega}, \hat{\theta})_B = \frac{\int \int_{\forall\omega\forall\theta} g(\omega, \theta) L(y|\omega, \theta) \, \pi(\omega, \theta) \, d\omega \, d\theta}{\int \int_{\forall\omega\forall\theta} L(y|\omega, \theta) \, \pi(\omega, \theta) \, d\omega \, d\theta}$$

$$g(\hat{\omega}, \hat{\theta})_B = \frac{\iint_{\forall\omega\forall\theta} \omega^{n+b} \, e^{-\omega\left[d - \sum_{i=1}^{n} \ln\left(1 - e^{-\theta y_i}\right)\right]} \, \theta^{n+c} e^{-\theta\left(\sum_{i=1}^{n} y_i + h\right)} \, d\omega \, d\theta}{\iint_{\forall\omega\forall\theta} \omega^{n+b-1} \, e^{-\omega\left[d - \sum_{i=1}^{n} \ln\left(1 - e^{-\theta y_i}\right)\right]} \, \theta^{n+c-1} e^{-\theta\left(\sum_{i=1}^{n} y_i + h\right)} \, d\omega \, d\theta}$$

Clearly from (12), it is not easy to find an exact solution to get the estimators of ω and θ, so we must use approximation methods.

Neutrosophic Lindley Approximation:

The 3rd method to solve (12) is Lindley approximation (LA), in the case of two parameters, Lindley's approximation for $\hat{g} = g(\hat{\omega}, \hat{\theta})$ can be written as follows:

$$\hat{g} = g(\hat{\omega}, \hat{\theta}) + \frac{1}{2}[U + l_{30} V_{12} + l_{03} V_{21} + l_{21} W_{21} + l_{12} W_{12}] + P_1 U_{12} + P_2 U_{21}$$

where:

$$U = \sum_{i=1}^{2} \sum_{j=1}^{2} \mu_{ij} \, \delta_{ij} \ , \quad l_{ij} = \frac{\partial_{i+j} ln L(y_i, \omega, \theta)}{\partial\omega_i \, \partial\theta_j} \qquad , i, j = 0,1,2,3 \ \ , i + j = 3$$

$$P_1 = \frac{\partial ln\{\pi(\omega, \theta)\}}{\partial\omega} \ , \quad P_2 = \frac{\partial ln\{\pi(\omega, \theta)\}}{\partial\theta} \ , \quad \mu_{ij} = \frac{\partial^2 g}{\partial\omega_i \, \partial\theta_j} \ , \quad \mu_i = \frac{\partial g}{\partial\omega_i} \ , \quad \mu_j = \frac{\partial g}{\partial\theta_j}$$

$$U_{ij} = \mu_i \delta_{ii} + \mu_j \delta_{ji} \ , \ V_{ij} = (\mu_i \delta_{ii} + \mu_j \delta_{ij}) \delta_{ii} \ , \ W_{ij} = 3\mu_i \delta_{ii} \delta_{ij} + \mu_j (\delta_{ii} \delta_{jj} + 2\delta_{ij}^2)$$

And $\mu_i = \mu_{i_0} + \mu_{i_1} I, \delta_{ii} = \delta_{ii_0} + \delta_{ii_1} I$

here:

$\hat{\omega}$ and $\hat{\theta}$ the MLEs of ω and θ respectively.

δ_{ij} the $(i,j)th$ element of the inverse of the Fisher information matrix.

therefore, we have:

$$l_{30} = \frac{2n}{\hat{\omega}^3}, l_{03} = \frac{2n}{\hat{\theta}^3} + (\hat{\omega} - 1) \sum_{i=1}^{n} \frac{y_i^3 e^{-\hat{\theta} y_i} \left(1 + e^{-\hat{\theta} y_i}\right)}{\left(1 - e^{-\hat{\theta} y_i}\right)^3}, l_{12} = -\sum_{i=1}^{n} \frac{y_i^3 e^{-\hat{\theta} y_i}}{\left(1 - e^{-\hat{\theta} y_i}\right)^2}, l_{12} = 0$$

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$$\delta_{11}=\frac{\left[\frac{n}{\hat\theta^2}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right]}{\left[\left\{\frac{n}{\hat\omega^2}\right\}\left\{\frac{n}{\hat\theta^2}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\right]-\left[\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right]^2}$$

$$\delta_{12}=\delta_{21}=\frac{\left[\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right]}{\left[\left\{\frac{n}{\hat\omega^2}\right\}\left\{\frac{n}{\hat\theta^2}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\right]-\left[\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right]^2}$$

$$\delta_{22}=\frac{\left[\frac{n}{\hat\omega^2}\right]}{\left[\left\{\frac{n}{\hat\omega^2}\right\}\left\{\frac{n}{\hat\theta^2}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\right]-\left[\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right]^2}$$

Now when $g(\omega,\theta)=\omega$ we have $\mu_1=1,\ \mu_2=0,\ \mu_{ij}=0,\ i,j=1,2$, therefore:

$U=0, U_{12}=\delta_{11}, U_{21}=\delta_{12}, V_{12}=\delta_{11}^2, V_{21}=\delta_{21}\delta_{22}$

$W_{12}=3\delta_{11}\delta_{12}, W_{21}=\delta_{11}\delta_{22}+2\delta_{21}^2$

$P_1=\left(\frac{h-1}{\hat\omega}-c\right), P_2=\left(\frac{d-1}{\hat\omega}-b\right)$

and when $g(\omega,\theta)=\theta$ we have $\mu_1=0,\ \mu_2=1,\ \mu_{ij}=0,\ i,j=1,2$, therefor:

$U=0, U_{12}=\delta_{21}, U_{21}=\delta_{22}, V_{12}=\delta_{12}\delta_{11}, V_{21}=\delta_{22}^2$

$W_{12}=\delta_{11}\delta_{22}+2\delta_{12}^2, W_{21}=3\delta_{22}\delta_{21}$

From the above the Lindley approximate for the Bayes estimators of ω and θ under the squared errors loss function are respectively:

$$\hat\omega_{BLA}=\frac{1}{2}\left[\frac{2n}{\hat\omega^3}\delta_{11}^2\left\{\frac{2n}{\hat\theta^3}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\delta_{21}\delta_{22}-\left\{\sum_{i=1}^{n}\frac{y_i^2e^{-\hat\theta y_i}}{\left(1-e^{-\hat\theta y_i}\right)^2}\right\}(\delta_{11}\delta_{22}+2\delta_{21}^2)\right]+\left(\frac{h-1}{\hat\omega}-c\right)\delta_{11}+\left(\frac{d-1}{\hat\omega}-b\right)\delta_{12}\quad(13)$$

$$\hat\theta_{BLA}=\frac{1}{2}\left[\frac{2n}{\hat\omega^3}\delta_{11}\delta_{12}\left\{\frac{2n}{\hat\theta^3}+(\hat\omega-1)\sum_{i=1}^{n}\frac{y_i^3e^{-\hat\theta y_i}\left(1+e^{-\hat\theta y_i}\right)}{\left(1-e^{-\hat\theta y_i}\right)^3}\right\}\delta_{22}^2-\left\{\sum_{i=1}^{n}\frac{y_i^2e^{-\hat\theta y_i}}{\left(1-e^{-\hat\theta y_i}\right)^2}\right\}\delta_{22}\delta_{21}\right]+\left(\frac{h-1}{\hat\omega}-c\right)\delta_{11}+\left(\frac{d-1}{\hat\omega}-b\right)\delta_{22}\quad(14)$$

Neutrosophic Gibbs Sampling:

The first method that solves the ratio of the two integrals in posterior distribution is MCMC specifically Gibbs sampling (GS), since the joint posterior of ω and θ is defined in (11), then the conditional posterior densities $(\omega|\theta)$ and $(\theta|\omega)$ can be written respectively as:

$$Q(\omega|\theta,y)=\omega^{n+b-1}\,e^{-\omega\left[b-\sum_{i=1}^{n}ln\left(1-e^{-\theta y_i}\right)\right]}$$

$$Q(\theta|\omega,y)=\theta^{n+c-1}\,e^{-\theta\left(\sum_{i=1}^{n}y_i+h\right)}\prod_{i=2}^{n}\left(1-e^{-\theta y_i}\right)^{\omega-1}$$

Where $n=n_0+n_1I, b=b_0+b_1I, h=h_0+h_1I$

Note that when $\omega\geq 1$ we can generate a random sample directly from the log-concave density $Q(\omega|\theta,y)$, and when $\omega<1$ we can depend on the order statistics to perform generation using the (GED), so we can re-write (16) as follow:

$$Q(\theta|\omega,y)=e^{-\theta y_{(1)}}\left(1-e^{-\theta y_i}\right)^{\omega-1}\left[\theta^{n+c-1}\,e^{-\theta\left(\sum_{i=2}^{n}y_i+h\right)}\prod_{i=2}^{n}\left(1-e^{-\theta y_i}\right)^{\omega-1}\right]$$

where $y_{(1)},\dots,y_{(n)}$ are the orders of $y{\cdot}s$, $y_{(1)}$ is a continuous function goes to finite constant as $\theta\to 0$ and goes to 0 as $\theta\to\infty$. That leads to the function having a finite maximum.

We can summarized (GS) that applied to find the Bayes estimators for ω and θ as follows:

7. Assume initial values for ω and θ say $\left(\omega_0=\hat\omega_{ML},\theta_0=\hat\theta_{ML}\right)$

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8. Generate parameters ω^j and θ^j from the posterior $Q(\omega_0, \theta_0|y)$.
9. Let $j = j + 1$
10. Generate new values for parameters ω, θ say ω_{j+1} and θ_{j+1} from the posteriors $Q(\omega_j|\theta_j, y)$ and $Q(\theta_j|\omega_j, y)$ respectively.
11. Repeat step 4 such that $j = 1,2,3, \dots, N$.
12. The final Bayes estimator for $g(\omega, \theta)$ under squared loss function according to the following formula:

$$g(\hat{\omega}, \hat{\theta})_{BGS} = E[Q(\omega, \theta|y)] = \frac{1}{N - M} \sum_{j=M+1}^{N} g(\omega_j, \theta_j)$$

where M is the number of observations that are burned to remove the influence of the initial sampling values.

Simulation Study:

Simulation experiments include the following stages and steps:

first stage: selection of the sample sizes as $n = 25, 50, 100$, while the different default values for the parameters (ω) and (θ) are shown in Table (1) below:

Table 1: Simulation Experiments

Experiment I		Experiment II		Experiment III		Experiment I		Experiment II		Experiment III	
ω	θ	ω	θ	ω	θ	θ	ω	θ	ω	θ	ω
	0.5		0.5		0.5		1		1		1
1	1	2	1	4	1	0.5	2	1	2	2	2
	2		2		2		4		4		4

Second stage: in this stage, data for the random variable (t) that follows the generalized exponential distribution with the two parameters (ω) and (θ) will be generated. This will be implemented according to the parameter values and sample sizes assumed in the first stage and according to the Inverse Transformation Method the following these steps:

1. Generating the random variable $U \sim uniform(0,1)$, that is:
   $U_i \sim Uniform(0,1) \quad , \quad i = 1,2, \dots, n$
2. By using the following transformation we can obtained the (GED) variable (y_i):
   $$y_i = -\frac{1}{\theta} Ln\left(1 - U_i^{\frac{1}{\omega}}\right) \quad , \quad i = 1,2, \dots, n$$

Note that we chose parameters values $c = b = \alpha, d = h = \varphi$ as parameters for Gamma prior distribution.

Third stage: this stage included the (GED) parameters estimation and finding the comparison criterion between approximation methods, as given on the theoretical side. The previous stages and steps are repeated for each experiment (k=1, 2, …, K, K=5000), and the comparison between estimators depending up on the mean squares error (MSE), which evaluate according to the formulas below:

$$MSE(\hat{\omega}) = \frac{1}{K} \sum_{k=1}^{K} (\omega - \hat{\omega}_k)^2 \ , \qquad MSE(\theta) = \frac{1}{K} \sum_{k=1}^{K} (\theta - \hat{\theta}_k)^2$$

Results and discussions:

The results of the simulation experiments were obtained based on a program written in (R) language. The results have been placed in Tables (2), (3), (4) and Figure (1) below, it is clear that the estimated values for parameters (ω) and (θ) in all experiments and sample sizes are close to the real (assumed) values.

From Tables (2-4) below, we notice that when $(\omega = 2,4)$ and $(\varphi = 1,2)$ the bias of the estimators are increases, and the parameters estimated values are closer to the assumed values when $(\omega = \theta = 1, \omega = \theta = 2)$. Also from (MSE), we can say that the values of (MSE) when $(\omega = 4)$ were higher than the values of (MSE) when $(\omega = 2)$, which in turn was higher than the values of (MSE) when $(\omega = 1)$. The same behavior applies to the values of (MSE) when estimating the parameter, while the two experiments $(\omega = \theta = 1, \omega = \theta = 2)$

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recorded the lowest (MSE) values. Furthermore, the (BIS) method gives the lowest (MSE) values followed by the (BLA) method and then the (BGS) method which gave the highest values for (MSE).

Table 2: Estimated values and Mean Squared Error for the parameters ω, θ for Experiment I

ω, θ	Method	n	$\widehat{\omega}$	MSE	θ, ω	Method	n	$\hat{\theta}$	MSE
1,0.5	BL	25	0.97054	0.01172	0.5,1	BL	25	0.48973	0.00515
		50	1.02435	0.01040			50	0.49151	0.00461
		100	0.98061	0.00922			100	0.50676	0.00413
	BGS	25	0.93781	0.02366		BGS	25	0.47831	0.00811
		50	1.05723	0.02070			50	0.48004	0.00739
		100	0.94847	0.01717			100	0.48203	0.00630
	BIS	25	0.97561	0.00990		BIS	25	0.50851	0.00439
		50	1.02195	0.00887			50	0.49234	0.00394
		100	1.01976	0.00794			100	0.49311	0.00354
1,1	BL	25	0.96667	0.02444	0.5,2	BL	25	0.48673	0.00713
		50	1.02755	0.02179			50	0.51097	0.00638
		100	0.97806	0.01937			100	0.50874	0.00571
	BGS	25	1.07037	0.03406		BGS	25	0.47198	0.01569
		50	1.06475	0.03020			50	0.52579	0.01418
		100	0.94170	0.02533			100	0.47679	0.01204
	BIS	25	0.97240	0.02073		BIS	25	0.51099	0.00607
		50	1.02484	0.01861			50	0.49011	0.00545
		100	1.02235	0.01669			100	0.49110	0.00489
1,2	BL	25	0.96899	0.01403	0.5,4	BL	25	0.51557	0.00952
		50	1.02563	0.01245			50	0.48713	0.00852
		100	0.97959	0.01106			100	0.48975	0.00762
	BGS	25	0.67267	0.02663		BGS	25	0.53288	0.02135
		50	1.30121	0.02334			50	0.46975	0.01928
		100	1.27119	0.01940			100	0.52724	0.01635
	BIS	25	0.97432	0.01187		BIS	25	0.51289	0.00810
		50	1.02311	0.01063			50	0.48839	0.00728
		100	1.02080	0.00952			100	0.48956	0.00653

Table 3 : Estimated values and Mean Squared Error for the parameters ω, θ for Experiment II

α, φ	Method	n	$\hat{\alpha}$	MSE	φ, α	n	Method	$\hat{\varphi}$	MSE
2,0.5	BL	25	1.96181	0.01878	1,1	25	BL	1.01386	0.00695
		50	1.96843	0.01663		50		0.98854	0.00622
		100	2.02514	0.01474		100		0.99088	0.00555
	BGS	25	2.08062	0.03939		25	BGS	1.02926	0.01312

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		50	1.92581	0.03442		50		0.97308	0.01181
		100	2.06679	0.02853		100		1.02424	0.01000
	BIS	25	1.96838	0.01584		25	BIS	1.01147	0.00591
		50	2.02846	0.01419		50		0.98967	0.00531
		100	2.02561	0.01270		100		0.99071	0.00477
2,1	BL	25	2.04105	0.02214	1,2	25	BL	1.01551	0.00864
		50	2.03394	0.01962		50		0.98718	0.00773
		100	1.97298	0.01739		100		0.98979	0.00691
	BGS	25	1.91333	0.04570		25	BGS	1.03274	0.02050
		50	2.07975	0.03994		50		0.96987	0.01845
		100	1.92820	0.03313		100		1.02713	0.01561
	BIS	25	1.96601	0.01868		25	BIS	1.01284	0.00735
		50	2.03059	0.01673		50		0.98844	0.00660
		100	2.02753	0.01499		100		0.98960	0.00593
2,2	BL	25	2.03979	0.02462	1,4	25	BL	1.01822	0.01119
		50	2.03181	0.02181		50		0.98494	0.01000
		100	2.02533	0.01940		100		0.98801	0.00894
	BGS	25	1.53843	0.05225		25	BGS	1.03846	0.02765
		50	2.42475	0.04572		50		0.96461	0.02482
		100	2.38241	0.03794		100		1.03186	0.02096
	BIS	25	1.96379	0.02207		25	BIS	1.01508	0.00951
		50	2.03259	0.01975		50		0.98643	0.00854
		100	2.02933	0.01770		100		0.98778	0.00767

Table 4: Estimated values and Mean Squared Error for the parameters ω, θ for Experiment III

α, φ	Method	n	$\hat{\alpha}$	MSE	φ, α	n	Method	$\hat{\varphi}$	MSE
4,0.5	BL	25	3.95069	0.02644	2,1	25	BL	2.02270	0.01528
		50	3.95924	0.02334		50		1.98123	0.01364
		100	3.96754	0.02064		100		1.98506	0.01218
	BGS	25	4.10410	0.06377		25	BGS	2.04793	0.03294
		50	3.90421	0.05547		50		1.95590	0.02942
		100	3.91376	0.04584		100		2.03971	0.02476
	BIS	25	3.95917	0.02225		25	BIS	2.01880	0.01298
		50	4.03675	0.01990		50		1.98308	0.01164
		100	4.03307	0.01781		100		1.98477	0.01045
4,1	BL	25	3.94850	0.02966	2,2	25	BL	2.01926	0.01081
		50	3.95743	0.02621		50		1.98408	0.00964
		100	3.96610	0.02318		100		1.98732	0.00861
	BGS	25	4.10871	0.06987		25	BGS	2.04067	0.02949
		50	3.89996	0.06082		50		1.96258	0.02632

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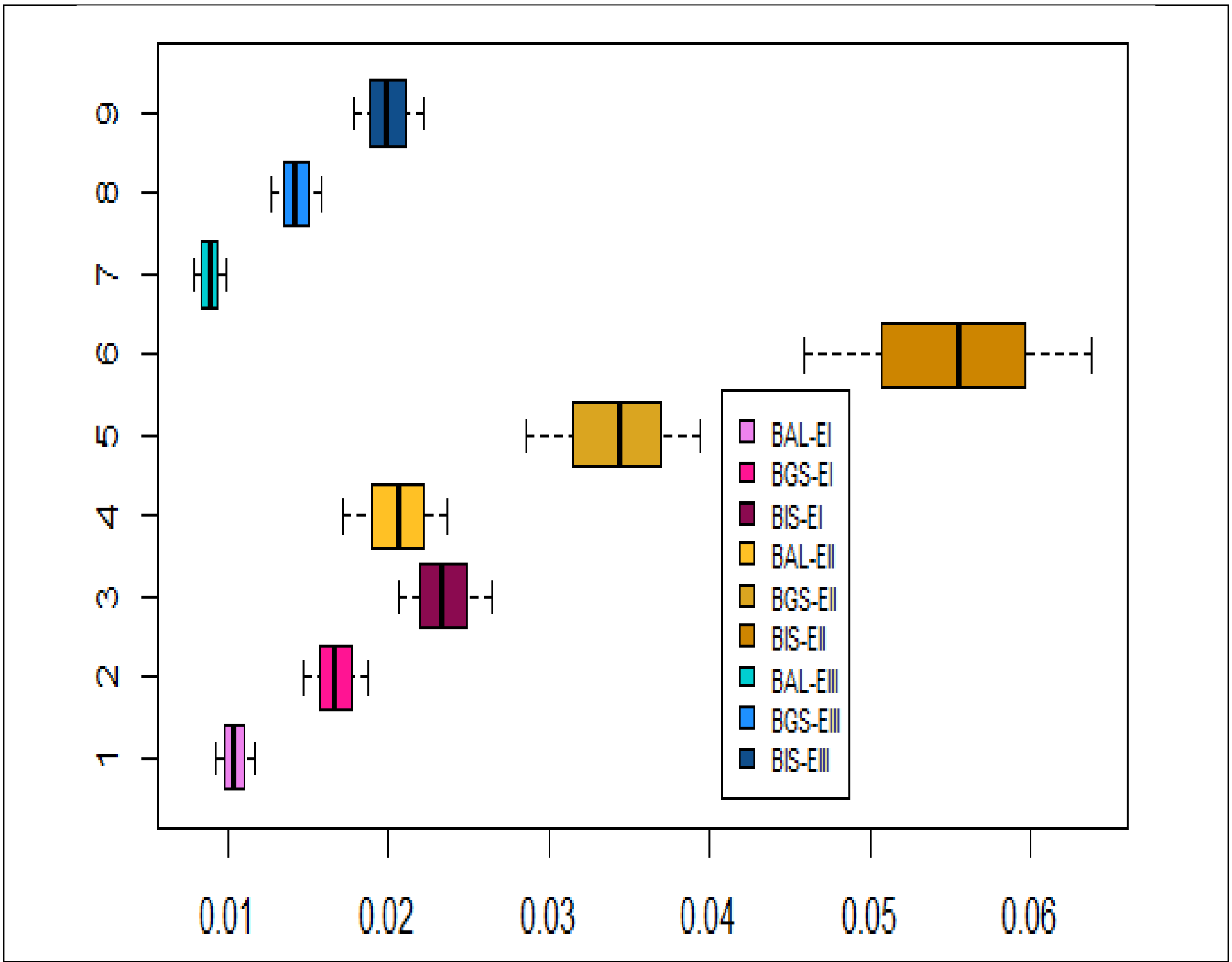
		100	3.90993	0.05029		100		2.03369	0.02214
	BIS	25	3.95736	0.02498		25	BIS	2.01595	0.00918
		50	4.03837	0.02234		50		1.98565	0.00824
		100	4.03454	0.01999		100		1.98708	0.00740
4,2	BL	25	3.94305	0.03523	2,4	25	BL	2.02468	0.01799
		50	3.95292	0.03111		50		1.97960	0.01606
		100	3.96251	0.02750		100		1.98375	0.01435
	BGS	25	4.60116	0.08505		25	BGS	2.05210	0.04861
		50	4.55320	0.07398		50		1.95205	0.04342
		100	3.50195	0.06114		100		2.04317	0.03654
	BIS	25	3.95284	0.02966		25	BIS	2.02044	0.01529
		50	4.04244	0.02652		50		1.98161	0.01371
		100	4.03820	0.02373		100		1.98345	0.01232



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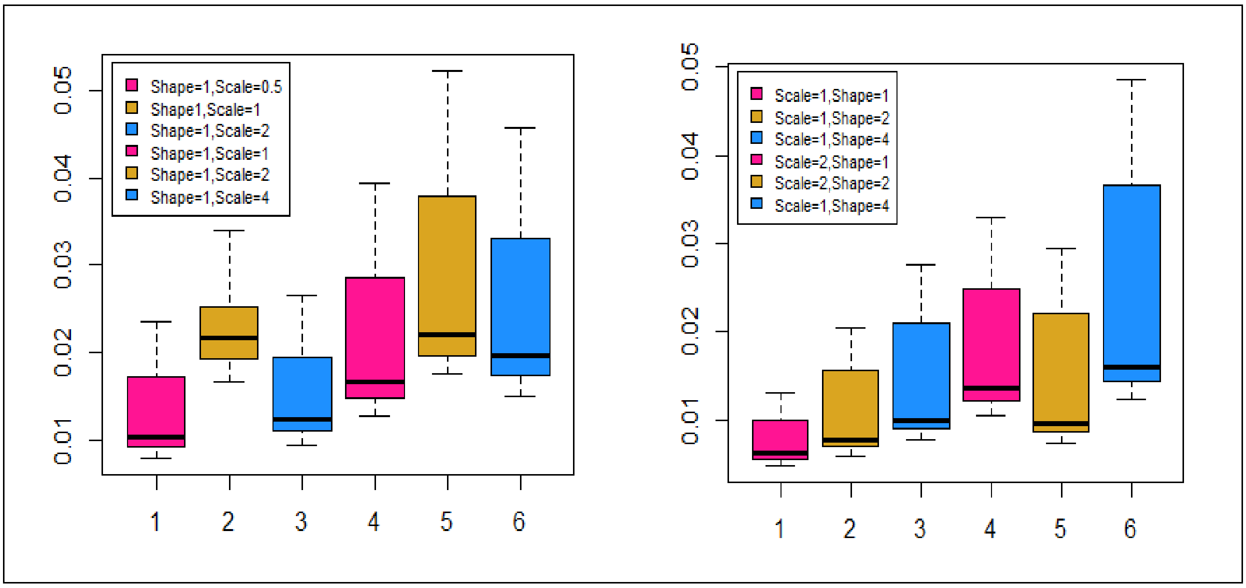

Figure (1) MSE for different simulation experiments

Real data:

The real data represents the number of operating hours for a machine from the printing, publishing and translation house in Baghdad, these data has been recorded during three months in (2022), specifically the printing machine, as well as the availability of data for the same period in (2021), these data have been placed in Table (5) below.

Table 5: Real data

	2021	2022		2021	2022		2021	2022		2021	2022
1	1.84	4.16	13	4.91	0.17	25	1.89	2.68	37	1.23	1.1
2	0.67	3.54	14	0.29	1.88	26	1.86	1.05	38	1.45	0.85
3	5.76	1.59	15	0.1	1	27	6.73	0.35	39	1.71	0.24
4	2.63	0.84	16	2.27	0.61	28	0.13	1.56	40	1.35	1.68
5	0.1	1.48	17	0.13	1.68	29	2.47	6.09	41	0.47	1.92
6	0.17	0.78	18	3.1	2.37	30	4.67	3.95	42	3.8	6.48
7	0.65	0.88	19	1.16	0.32	31	1.17	2.32	43	0.26	1.48
8	0.32	1.1	20	0.25	1.15	32	0.78	4	44	0.48	1.77
9	3.11	1.05	21	3.1	0.94	33	0.88	6.95	45	5.86	1.2
10	0.6	2	22	3.83	4.35	34	1.42	2.36	46	0.51	4.98
11	0.95	7.73	23	1.23	0.3	35	1.80	1.85	47	2.33	1.14
12	1.20	5.83	24	0.63	0.2	36	0.56	2.39	48	4.69	0.59

Table 6: some statistical measures

Year	Min	Max	Mean	Median	Skewness	Kurtosis	S.D.
2022	0.17	7.73	2.185	1.575	1.391	1.144	1.923
2021	0.1	6.73	1.82	1.23	1.259	0.823	1.716

Table (6) above includes some statistical measures of the operation data of the printing machine for the years (2021) and (2022), and it is clear from it that the operation recorded in (2021), while the highest operating time in (2022), the skewness and kurtosis coefficients for all variables are positive.

The Kolmogorov Smirnov (KS) test at (0.05) significance level was conducted to purpose of knowing the distribution of the two machines, the hypotheses were as follows:

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H_0: The operating times for the printing machine in (2022) follows GE distribution.

H_0; The operating times for the printing machine in (2021) follows Gamma distribution.

Table 7: Kolmogorov Smirnov tests for real data

Year	K-S	p-value	Year	K-S	p-value
2021	0.17042	0.1231	2022	0.13443	0.3509

The results in table (7) below refers to, the p-values indicate to accept the above null hypotheses as being greater than the significance level, thus the operation data of the two machines are follow a (GE) and Gamma distributions respectively. The box plots in Figure (2) below represents the operating data of the printing machine, it is clear from it that these data suffering from of outliers, especially the (GE) data in the left side.

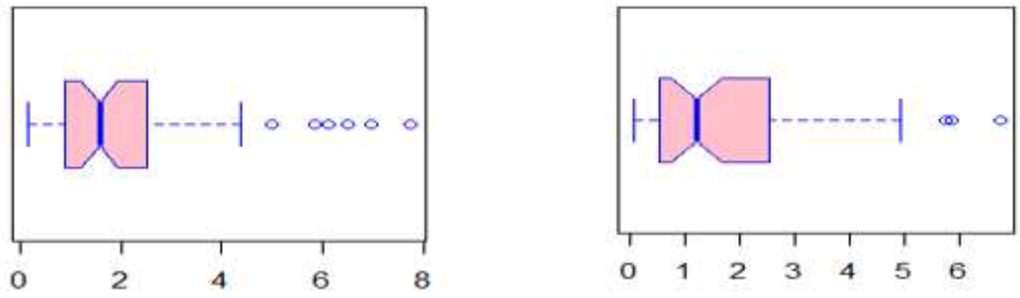

Figure 2: Box plots for real data (2022 in the left)

Table (8) below contain the estimated values of (α) and (φ) for the printing machine by using the (BLA) and (BIS) methods, also the (SE) values for these estimators, Furthermore, we calculating accuracy criteria for these methods that is: (Log-L), (AIC) and (BIC).

Table (8) accuracy criteria for BLA and BIS methods

Method	Parameter	Estimate	SE	Log-L	AIC	BIC
BLA	$\hat{\alpha}$	1.42676	0.13092	117.36	-230.72	-226.98
	$\hat{\varphi}$	0.55369	0.07141			
BIS	$\hat{\alpha}$	1.33196	0.07619	112.19	-222.38	-216.64
	$\hat{\varphi}$	0.53532	0.04926			

Clearly form table (8) above, the (SE) values for estimating (α) are smaller than the (SE) values for estimating (φ), and the (SE) values for the estimators (α) and (φ) by using (BLA) method are larger than these values for (BIS) method, also the (BIS) method gives the larger values of (Log-L), while (BIS) method gives recorded the smallest values of (AIC) and (BIC).

3. Conclusions:

It seems that the proposal to employ the Importance Sampling method was successful, as this method provided the best estimations for the shape and scale parameters for the Generalized Exponential Distribution. Also, the Lindley approximation gave acceptable results in finding solutions to the integrals of the posterior distribution for the Bayes method, where the estimating the shape and scale parameters of the generalized exponential distribution was superior to the Gibbs Sampling method. Thus using the MCMC method based on the importance function in generating the posterior distribution to estimate the shape and scale parameters for the generalized exponential distribution gives better results than the dependence of the order statistics in generating the posterior distribution. Also, a usage of neutrosophic simulation methods was applied on the data to check the validity of the classical simulation, and the results are approximately similar.

For all approximate methods, the MSE values when estimating the scale parameter of the generalized exponential distribution, are less than the values of those criteria when estimating the shape parameter. Furthermore, when estimating the shape and the scale parameters for the generalized exponential distribution for all estimation methods, the MSE values increase by increasing the values of the two parameters, exception of the

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case in which the values of the two parameters are equal, where the MSE values decrease when estimating the scale parameter and increase when estimating the shape parameter.

As for the real data, the outliers in 2021 are little than outliers in 2022, i.e., the (GED) variables have more than outliers for gamma distribution variables. Also by applying the Importance Sampling methods and Lindley approximate to the real data to find the shape and scale parameters of the generalized exponential distribution using the Bayes method, the results reached in the simulation experiments were confirmed, that is the values of the three comparison criteria Log-L, AIC and BIC indicated the superiority of the Importance Sampling method over the neutrosophic Lindley approximation method.

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