



# Properties of Fuzzy Semi-Open Sets with Regards to Fuzzy Ideals

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## Abstract

This paper introduces a broadened concept of fuzzy semi-open sets by framing them in the context of fuzzy ideals. Where many of their elementary properties will be presented in terms of theorems and lemmas. Also, many related examples about the validity of semi-open sets and their relationships with fuzzy ideals will be provided and discussed.

**Keywords:** Fuzzy Semi-open sets; fuzzy Semi-closed sets; fuzzy Generalized closed sets; fuzzy Ideals

## 1. Introduction

By Zadeh [2], the fuzzy set concept was first presented. This was done to measure the degree of belonging of an element to a set, and then this concept played a big role in the study of many concepts in Applied Mathematics, specifically those related to intelligent systems [9] and Computer Science [10-12] in general. Chang also introduced the fuzzy topological space [1]. Several fuzzy topologists have endeavored to investigate fuzzy closed sets and the related principles of fuzzy topology from diverse perspectives. This becomes apparent in the initial sections of various studies, where exploration is conducted on subjects like “fuzzy semi-open sets” and “generalized fuzzy closed sets”. For further details, you may refer to the cited works [1-6]. The idea of utilizing fuzzy topological ideals to describe fuzzy topological ideas has been a fascinating area of research for a while now [see some of the pioneering papers in 7, 8, 9]. This concept is pertinent to our present study. It is essential to recall that a non-empty collection represents a “fuzzy ideal” on a “fuzzy topological space”  $(\tilde{M}, \tilde{\tau})$ . This fuzzy ideal pertains to fuzzy subsets of  $\tilde{M}$  exhibiting the fuzzy heredity property, meaning that if  $\mu\tilde{M} \leq \mu\tilde{N}$  and  $\mu\tilde{P} < \mu\tilde{M}$ , then  $\mu\tilde{P} \leq \mu\tilde{N}$ . Additionally, the fuzzy ideal adheres to fuzzy finite additivity, which implies that if  $\mu\tilde{M}$  and  $\mu\tilde{N}$  are both less than or equal to  $\mu\tilde{P}$ , then the maximum of  $\mu\tilde{M}$  and  $\mu\tilde{N}$  is also less than or equal to  $\mu\tilde{P}$ .

In this study, we delve into various aspects of “fuzzy semi-open sets” and characterize them in relation to a “fuzzy ideal”. Specifically, our concept of fuzzy semi-openness encompasses both the traditional notion of fuzzy ‘semi-openness’ and the concept of fuzzy semi-openness considered in previous research. We focus on a fuzzy topological space  $(\tilde{M}, \tilde{\tau})$  (or simply  $\tilde{M}$ ) throughout, without assuming any separation axioms. The standard notation  $(\tilde{k})(\tilde{Q})$  will be used to denote the closure, and  $\text{int}$  represents the interior of a fuzzy subset  $\tilde{Q}$  within the fuzzy topological space  $(\tilde{M}, \tilde{\tau})$ .

### Basic concepts of fuzzy set

**Definition (1-1) [2] is as follows:**

Consider  $\tilde{w}$  as a non-empty set representing a vague set. The characteristic of  $\tilde{B}$  in  $\tilde{Y}$  is determined by a membership function  $\mu\tilde{B}: \tilde{w} \rightarrow [0,1]$ . This fuzzy collection can be represented as:

$$C^{\wedge} = \{(y, \mu\tilde{B}(y)): y \in \Sigma, \mu\tilde{B}(y) \leq 1\}$$

When  $J^{\wedge}y = \{\tilde{C}: \tilde{C} \text{ is a fuzzy-set in } y\}$ , the set comprising all fuzzy-sets in  $\tilde{\Sigma}$  will be represented as  $J^{\wedge}y$ .

**Definition (1-2) [2] :-** The backing of a fuzzy-set  $\tilde{C}$  is the set of all  $y \in Y$  such that  $\mu\tilde{C}(y) > 0$  and is represented by  $S(\tilde{C})$ .

**Definition (1-3) :-** The fuzzy-set in  $d$  corresponding to the degree of membership  $\mu$  is a fuzzy element  $\tilde{Q}_c^a$ :

$$\tilde{Q}_c^a(d) = \begin{cases} a, & \text{if } c = d \\ 0, & \text{if } c \neq d \end{cases}$$

When  $0 < a \leq 1$ ,  $d$  is the support of  $\tilde{Q}_c^a(c)$ .

**Definition (1-4) [2]:-** A ‘fuzzy set’  $\tilde{D}$  is termed finite if its support, denoted as  $S(\tilde{D})$ , comprises a finite set of elements.

**Remark (1-5):-**

1) An inhabited set “a” is a ‘fuzzy set’ with degree of membership  $\mu_{\tilde{a}}(a) = 1, \forall a \in A$  and “a” is referred to as a crisp set.

2) For any element “a” in set A, membership-function  $\mu_{\tilde{\emptyset}}(a) = 0, \forall a \in A$  is characterizing an empty set and expressed by  $\tilde{\emptyset}$ .

**Definition (1-6) [2]:-** Let  $\tilde{M}_c^r$  be a ‘fuzzy point’ and  $\tilde{D}$  be a fuzzy set in non- empty set  $\tilde{C}$ . Subsequently,  $\tilde{M}_c^r$  is in  $\tilde{D}$  or  $\tilde{D}$  contains  $\tilde{M}_c^r$  if  $\mu_{\tilde{M}_c^r} \leq \mu_{\tilde{D}}(c)$  for all  $c \in C$  and denoted by  $c \in S(\tilde{D})$ .

**Definition (1-7) [2] :-** Let  $\tilde{M}$  and  $\tilde{N}$  by fuzzy sets of a universal set  $\tilde{R}$  then

1)  $\tilde{M} \subseteq \tilde{N}$  if and only if  $\mu_{\tilde{M}}(r) \leq \mu_{\tilde{N}}(r), \forall r \in R$

2)  $\tilde{M} = \tilde{N}$  if and only if  $\mu_{\tilde{M}}(r) = \mu_{\tilde{N}}(r)$  for all  $r \in R$

3)  $\tilde{M}^c$  is the complement of a fuzzy set  $\tilde{M}$  with membership function  $\mu_{\tilde{M}^c} = 1 - \mu_{\tilde{M}}(r)$

4)  $\tilde{C} = \tilde{M} \cup \tilde{N}$  necessary and sufficient condition if  $\mu_{\tilde{C}}(r) = \max \{ \mu_{\tilde{M}}(r), \mu_{\tilde{N}}(r) \}, \forall r \in R$

5)  $\tilde{D} = \tilde{M} \cap \tilde{N}$  necessary and sufficient condition  $\mu_{\tilde{D}}(r) = \min \{ \mu_{\tilde{M}}(r), \mu_{\tilde{N}}(r) \}, \forall r \in R$

6) More generally, for a family of fuzzy sets  $\{ \tilde{M}_m : m \in \Lambda \text{ where } \Lambda \text{ is the any index set} \}$

the union  $\tilde{C} = \bigcup_{m \in \Lambda} Mm$  and the intersection  $\tilde{D} = \bigcap_{m \in \Lambda} Mm$  and defined respectively by

$$\mu_{\tilde{C}}(r) = \sup_{m \in \Lambda} \{ \mu_{\tilde{M}}(r) ; r \in R \}$$

$$\mu_{\tilde{D}}(r) = \inf_{m \in \Lambda} \{ \mu_{\tilde{M}}(r) ; r \in R \}$$

## 2. Fuzzy Semi-openness With Respect To an fuzzy Ideal

Assume that the topological space  $\tilde{M}$  is fuzzy. Remember that if there is a fuzzy open-set  $\tilde{R}$  such that  $\mu_{\tilde{R}} \leq \mu_{\tilde{G}} < \mu(\tilde{k})(\tilde{R})$ , then a fuzzy-subset  $\tilde{G}$  of  $\tilde{M}$  is considered fuzzy ‘semi-open’. This inspires our initial definition.

**Definition 2-1:-** If there is an open set  $\tilde{R}$  that is fuzzy and such that  $\mu_{\tilde{R}} - \mu_{\tilde{G}} \leq r$  and  $\mu_{\tilde{G}} - \mu(\tilde{k})(\tilde{R}) \leq \mu$ , then a fuzzy-subset  $\tilde{G}$  of  $\tilde{M}$  is considered fuzzy ‘semi-open’ with regard to a fuzzy ideal  $\tilde{g}$  (written as fuzzy  $\tilde{g}$ -semi-open). It is evident that  $\tilde{M}$  is fuzzy  $\tilde{g}$ -‘semi-open’ if  $\mu_{\tilde{M}} \leq \mu_{\tilde{g}}$ . Furthermore, for each fuzzy ideal  $\{ > \text{ on } \tilde{X}$ , every fuzzy open set  $\tilde{M}$  is fuzzy ‘semi-open’, and every fuzzy ‘semi-open’ set  $(N \tilde{)} is fuzzy \{ -‘semi-open’ for any fuzzy ideal \tilde{g} on \tilde{M}.$

**Example 2-2:-** Let  $M = \{(\delta, 0.1), (\epsilon, 0.1), (\zeta, 0.1)\}$  and  $\tilde{\tau} = \{ \tilde{\emptyset}, \{(\delta, 0.1), (\epsilon, 0.0), (\zeta, 0.0)\}, \{(\delta, 0.1), (\epsilon, 0.0), (\zeta, 0.1)\}, \tilde{M} \}$  be a fuzzy topological space. Select  $\tilde{U} = \{ \tilde{\emptyset}, \{(\delta, 0.0), (\epsilon, 0.1), (\epsilon, 0.0)\}, \{(\delta, 0.0), (\epsilon, 0.1), (\zeta, 0.1)\}, \{(\delta, 0.0), (\epsilon, 0.1), (\zeta, 0.1)\}, \text{ and note that } \{(\delta, 0.0), (\epsilon, 0.1), (\zeta, 0.0)\} \text{ is a fuzzy ‘semi-open’ } \tilde{U}; \text{ on the other hand, } \{(\delta, 0.0), (\epsilon, 0.1), (\zeta, 0.0)\} \text{ is not fuzzy ‘semi-open’ in the sense that there is no fuzzy open set } \tilde{R} \text{ such that } \mu_{\tilde{R}} < \{(\delta, 0.0), (\epsilon, 0.1), (\zeta, 0.0)\} < \mu(\tilde{g})(\tilde{R}). \text{ Therefore, a fuzzy set may not be fuzzy semi open in the conventional sense even if it is fuzzy semi open.}$

The following scenario illustrates how fuzzy semi-openness and fuzzy  $\tilde{g}$  semi-openness coincide for a fuzzy ideal  $\tilde{g}$  that is not countably additive.

**Theorem 2-3:-** On a fuzzy topological space  $\tilde{M}$ , the subsequently are the same as a fuzzy ideal $\tilde{I}$ :

1.  $\tilde{g}$  is the fuzzy minimal ideal on  $\tilde{M}$ ; that is,  $\tilde{I} = \{\emptyset\}$ ; 2. There is no distinction between fuzzy semi-openness and fuzzy  $\tilde{g}$ -semi-openness.

**Proof.** Assume first that  $\tilde{I} = \{\emptyset\}$ . It is sufficient to demonstrate that a fuzzy-set  $\tilde{G}$  is always fuzzy ‘semi-open’ in the conventional sense if it is fuzzy  $\tilde{g}$ -semi open. In fact, if  $\tilde{G}$  is fuzzy  $\tilde{g}$ -‘semi-open’, then  $\mu_{\tilde{R}} - \mu_{\tilde{G}}, \mu_{\tilde{G}} - \mu(k)(\tilde{R}) \leq \mu_{\tilde{I}} = \{\emptyset\}$ , and so  $\mu_{\tilde{R}} < \mu_{\tilde{G}} < \mu(k)(\tilde{R})$ , demonstrating that  $\tilde{G}$  is fuzzy ‘semi-open’. On the other hand, assume that a fuzzy set  $\tilde{G}$  is fuzzy ‘semi-open’ whenever it is fuzzy  $\tilde{g}$ -‘semi-open’. Assume  $\mu_{\tilde{H}} \leq \mu_{\tilde{I}}$ .

Hence,  $\tilde{H}$  is assumed to be fuzzy ‘semi-open’ and fuzzy  $\tilde{g}$ -‘semi-open’. Thus,  $\mu(V1) < \mu_{\tilde{H}} < \mu(\text{cl})((V1))$  implies the existence of a fuzzy open set  $(V1)$ .  $\text{Max}\{\mu_{\tilde{H}}, \mu(V1)\} \leq \mu_{\tilde{g}}$  since  $\mu_{\tilde{H}} \leq \mu_{\tilde{g}}$  and  $\mu(V1) < \mu_{\tilde{H}}$  implies that  $\mu(V1) \leq \mu_{\tilde{g}}$ . This means that there is a fuzzy open set  $(V2)$  for which  $\mu(V2) < \text{max}\{\mu_{\tilde{H}}, \mu(V1)\} < \mu(\text{cl})((V2))$  because  $\text{max}\{\mu_{\tilde{H}}, \mu(V1)\}$  is fuzzy  $\tilde{g}$ -‘semi-open’. The fuzzy open set  $(V3)$  is similarly defined so that  $\mu(V3) < \text{max}\{\mu_{\tilde{H}}, \mu(V1), \mu(V2)\} < \mu(\text{cl})((V3))$ . This means that if we keep going, we will have an endless number of fuzzy open sets  $(V1), (V2), (V3), \dots$ , such that  $\text{max}\{\mu_{\tilde{H}}, \mu(V1), \mu(V2), (V3) \dots\} \leq \mu_{\tilde{I}}$ , which is not feasible since the fuzzy ideal  $\tilde{I}$  is not fuzzy closed under countable additivity. Hence,  $\mu(V1) = \emptyset$  (in a similar manner for the other  $V_i$ ’s); consequently,  $\mu(\text{cl})((V1)) = \emptyset$ , and the relations  $\mu(V1) < \mu_{\tilde{H}} < \mu(\text{cl})((V1))$  subsequently yield  $\mu_{\tilde{H}} = \emptyset$ , demonstrating that  $\mu_{\tilde{I}} = \{\emptyset\}$ .

**Proposition 2-4:-** Let  $\tilde{M}$  be a fuzzy topological space, and let  $\tilde{g}$  and  $\tilde{g}'$  and  $\tilde{I}$  be two fuzzy ideals on it.

1. If  $\mu_{\tilde{I}} < \mu_{\tilde{I}'}$ ; every fuzzy  $\tilde{G}$ -‘semi-open’ set  $\tilde{G}$  is fuzzy  $\mu_{\tilde{I}}$ -‘semi-open’
2. If  $\tilde{G}$  is concurrently ‘semi-open’ with respect to fuzzy  $\mu_{\tilde{I}}$ -‘semi-open’ and fuzzy  $\mu_{\tilde{I}'}$ -‘semi-open’, then it is  $\text{min}\{\mu_{\tilde{I}}, \mu_{\tilde{I}'}\}$ -‘semi-open’.

**Corollary 2-5:-** Remember that  $\mu_{\tilde{I}} \tilde{G} = \{\text{min}\{\mu_{\tilde{G}}, \mu_{\tilde{I}}\} | \mu_{\tilde{S}} \leq \mu_{\tilde{I}}\}$  is also a fuzzy ideal on  $\tilde{M}$  for a fuzzy subset  $\tilde{G}$  of  $\tilde{M}$  and a fuzzy ideal  $\tilde{I}$  on  $\tilde{M}$ .

1. A fuzzy-set  $\tilde{H}$  is considered fuzzy  $\tilde{I}$ -‘semi-open’ if and only if it qualifies as fuzzy  $\mu_{\tilde{I}}$ -‘semi-open’.
2. The smallest fuzzy ideal is  $\mu_{\tilde{I}} \tilde{G} = \mu_{\tilde{I}} \{\emptyset\} = \emptyset$  if  $\mu_{\tilde{G}} = \emptyset$ . If a fuzzy set  $\tilde{C}$  fuzzy  $\mu_{\tilde{I}} \tilde{G}$ -‘semi-open’, then it is also fuzzy  $\mu_{\tilde{I}}$ -‘semi-open’.

**Proposition 2-6:-** If their union, expressed as  $\text{max}\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\}$ , is fuzzy  $\tilde{g}$ -‘semi-open’, then both sets  $\tilde{G}$  and  $\tilde{H}$  are deemed ‘semi-open’ with respect to the fuzzy ideal  $\tilde{g}$ .

**Proof.** Assuming the specified conditions are satisfied. To demonstrate  $\text{max}\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\}$  is fuzzy  $\tilde{g}$ -‘semi-open’, We are required to generate. an fuzzy open set  $\tilde{R}$  such that  $\mu_{\tilde{R}} - \text{max}\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\} \leq \mu_{\tilde{g}}$  and  $\text{max}\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\} - \mu_{\tilde{k}}(\tilde{R}) \leq \mu_{\tilde{g}}$ . Since  $\tilde{G}$  and  $\tilde{H}$  are both fuzzy  $\tilde{g}$ -‘semi-open’, there exist fuzzy open sets  $\tilde{R}1$  and  $\tilde{R}2$  such that

$$\mu_{\tilde{R}1} - \mu_{\tilde{G}} \leq \mu_{\tilde{g}}, \mu_{\tilde{G}} - \mu_{\tilde{k}}(\tilde{R}1) \leq \mu_{\tilde{g}}, \mu_{\tilde{R}2} - \mu_{\tilde{H}} \leq \mu_{\tilde{g}}, \mu_{\tilde{H}} - \mu_{\tilde{k}}(\tilde{R}2) \leq \mu_{\tilde{g}}.$$

Choose  $\mu_{\tilde{R}} = \text{max}\{\mu_{\tilde{R}1}, \mu_{\tilde{R}2}\}$  and observe that

$$(\text{max}\{\mu_{\tilde{R}1}, \mu_{\tilde{R}2}\}) - (\text{max}\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\}) = \text{max}\{((\mu_{\tilde{R}1} - \mu_{\tilde{G}}) - \mu_{\tilde{H}}), ((\mu_{\tilde{R}2} - \mu_{\tilde{H}}) - \mu_{\tilde{G}})\} \leq \mu_{\tilde{g}}.$$

Also,

$$\text{max}\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\} - \mu_{\tilde{k}}(\text{max}\{\mu_{\tilde{R}1}, \mu_{\tilde{R}2}\}) = \text{max}\{((\mu_{\tilde{G}} - \mu_{\tilde{k}}(\tilde{R}1)) - \mu_{\tilde{k}}(\tilde{R}2)), ((\mu_{\tilde{G}} - \mu_{\tilde{k}}(\tilde{R}2)) - \mu_{\tilde{k}}(\tilde{R}1))\} \leq \mu_{\tilde{g}}.$$

Therefore, by definition,  $\text{max}\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\}$  is fuzzy  $\tilde{g}$ -‘semi-open’.

**Proposition 2-7:-** Under the conditions of a fuzzy topological space  $\tilde{M}$ , where there exists an open single subset  $\{\tilde{h}\}$  such that the degree of membership of its closure  $\mu(k)(\{\tilde{h}\})$  equals  $\mu_{\tilde{M}}$ , and for any fuzzy ideal  $\tilde{g}$  on  $\tilde{M}$  with  $\mu_{\{\tilde{h}\}} \leq \mu_{\tilde{g}}$ , the following conditions hold:

1. Every individual fuzzy subset  $\tilde{M}$  within exhibits ‘semi-open’ properties in relation to  $\tilde{g}$ .
2. Each finite fuzzy subset of  $(\tilde{M})$  is characterized by being ‘semi-open’ with respect to  $\tilde{g}$ .

**Proof.** Under the given conditions, let's assume  $\{\tilde{s}\}$  is a fuzzy single subset of  $\tilde{M}$ . Considering  $(\{\tilde{h}\})$  is fuzzy open and  $\mu_{\{\tilde{h}\}} - \mu_{\{\tilde{s}\}} = \mu_{\{\tilde{h}\}} \leq \mu_{\tilde{g}}$ , and  $\mu_{\{\tilde{s}\}} - \mu_{\text{cl}}(\{\tilde{a}\}) = \mu_{\{\tilde{s}\}} - \mu_{\tilde{M}} = \emptyset \leq \mu_{\tilde{g}}$ , it follows that  $\{\tilde{s}\}$  is fuzzy  $\tilde{g}$ -

'semi-open'; thereby establishing (1). To confirm (2), holds, let  $\mu\tilde{G} = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \dots, \tilde{s}_n\}$  be a fuzzy finite subset of  $\tilde{M}$ .

Since  $\mu\tilde{G} = \max\{\{\tilde{s}_1\}, \{\tilde{s}_2\}, \{\tilde{s}_3\}, \dots, \{\tilde{s}_n\}\}$ , the outcome can be derived from recognizing that each individual fuzzy singleton subset  $\{\tilde{s}_i\}$  (where  $i = 1, 2, 3, \dots, n$ ) is 'semi-open' with respect to the fuzzy ideal  $\tilde{g}$ , combined with multiple applications of Proposition 2 stated earlier.

Proposition 3 lacks validity for any chosen fuzzy ideal.

**Example 2-8:-** Contemplate  $\tilde{M} = \{(\delta, 0.6), (\epsilon, 0.6), (\zeta, 0.6)\}$ ,  $(\tau) = \{\emptyset, \{(\delta, 0.6), (\epsilon, 0.0), (\zeta, 0.0)\}, \{(\delta, 0.6), (\epsilon, 0.0), (\zeta, 0.6)\}, \tilde{M}\}$ , and observe that  $\mu\tilde{g}(\{(\delta, 0.6), (\epsilon, 0.0), (\zeta, 0.0)\}) = \mu\tilde{M}$ . If we opt for the smallest possible fuzzy ideal.

If we consider  $\mu\tilde{\zeta} = \{\emptyset\}$  on  $\tilde{M}$ , then the fuzzy singleton subset  $\{(\delta, 0.0), (\epsilon, 0.6), (\zeta, 0.0)\}$  does not qualify as fuzzy  $\tilde{\zeta}$ - 'semi-open'. This is due to the absence of a fuzzy open set  $\tilde{R}$  meeting  $\mu\tilde{R} - \mu\{(\delta, 0.0)\} \leq \mu\tilde{g}$  and  $\mu\{(\delta, 0.0)\} - \mu\tilde{g}(\tilde{R}) \leq \mu\tilde{g}$  concurrently.

**Proposition 2-9:-** Consider fuzzy subsets  $\tilde{G}$  and  $\tilde{H}$  within a fuzzy topological field  $\tilde{M}$ . Suppose  $\tilde{G}$  is fuzzy open,  $\mu\tilde{G} < \mu\tilde{H}$  and  $\tilde{G}$  is fuzzy dense in  $\tilde{H}$  (that is,  $\mu\tilde{G} < \mu\tilde{k}(\tilde{G})$ ). In that case,  $\tilde{H}$  is 'semi-open' with respect to any fuzzy ideal  $\tilde{g}$  on  $\tilde{M}$ . This conclusion remains true, especially when the degree of membership of  $\tilde{B}$  equals the closure of  $\tilde{G}$ .

**Remark 2-10:** Their intersection  $\min\{\mu\tilde{G}, \mu\tilde{H}\}$  may not necessarily be 'semi-open' with respect to  $\tilde{\zeta}$ , even if two fuzzy sets  $\tilde{G}$  and  $\tilde{H}$  are 'semi-open' with respect to the fuzzy ideal  $\tilde{g}$ . For instance, consider the fuzzy topological space  $\tilde{M} = \{(\delta, 0.8), (\epsilon, 0.8), (\zeta, 0.8)\}$  with a specific fuzzy topology  $\tilde{\sigma} = \{\emptyset, \{(\delta, 0.8), (\epsilon, 0.0), (\zeta, 0.0)\}, \{(\delta, 0.0), (\epsilon, 0.0), (\zeta, 0.8)\}, \{(\delta, 0.8), (\epsilon, 0.0), (\zeta, 0.8)\}, \tilde{M}\}$ . Note that  $\mu\tilde{g}(\{(\delta, 0.8), (\epsilon, 0.0), (\zeta, 0.0)\}) = \{(\delta, 0.8), (\epsilon, 0.8), (\zeta, 0.0)\}$  and  $\mu\tilde{\zeta}(\{(\delta, 0.0), (\epsilon, 0.0), (\zeta, 0.8)\}) = \{(\delta, 0.0), (\epsilon, 0.8), (\zeta, 0.8)\}$ ; moreover, the fuzzy subsets  $\{(\delta, 0.8), (\epsilon, 0.8), (\zeta, 0.0)\}$  and  $\{(\delta, 0.0), (\epsilon, 0.8), (\zeta, 0.8)\}$  exhibit 'semi-open' characteristics in relation to the minimal fuzzy ideal  $\mu\tilde{g} = \{\emptyset\}$ , as demonstrated in Proposition 4. Nevertheless, the specific fuzzy singleton subset  $\mu\{\tilde{g}\} = \min\{(\delta, 0.8), (\epsilon, 0.8), (\zeta, 0.0)\}, \{(\delta, 0.0), (\epsilon, 0.8), (\zeta, 0.8)\}$  does not adhere to the criteria for being fuzzy 'semi-open' under the fuzzy minimal ideal  $\mu\tilde{g} = \{\emptyset\}$ .

Certainly, if both  $\tilde{G}$  and  $\mu\tilde{H}$  are less than or equal to  $\mu\tilde{g}$ , then  $\min\{\mu\tilde{G}, \mu\tilde{H}\}$  will exhibit fuzzy 'semi-open' properties under the fuzzy ideal  $\tilde{g}$ . However, for fuzzy subsets not belonging to the fuzzy ideal  $\tilde{g}$ , a stringent condition ensuring their intersection is fuzzy 'semi-open' under the fuzzy ideal  $\tilde{g}$  is outlined below.

**Proposition 2-11:-** In a fuzzy topological space  $\tilde{M}$ , where every non-empty fuzzy open-subset  $\tilde{M}$  is dense, and the fuzzy open-subsets  $\tilde{M}$  satisfy the property of finite intersections, let be considered as a fuzzy ideal.

1. If  $\tilde{H}$  is also 'semi-open' with respect to the same fuzzy ideal  $\tilde{g}$ , given that  $\mu\tilde{G} < \mu\tilde{H}$ , then  $\tilde{M}$  is half-open relate to the fuzzy ideal  $\tilde{g}$ .
2. If  $\tilde{G}$  is half-open relate to the fuzzy ideal  $\tilde{g}$ , then the maximum of  $\mu\tilde{G}$  and  $\mu\tilde{H}$ , denoted as  $\max\{\mu\tilde{G}, \mu\tilde{H}\}$ , is also 'semi-open' for any fuzzy subset  $\tilde{G}$  of  $\tilde{M}$ .
3. If their intersection  $\min\{\mu\tilde{G}, \mu\tilde{H}\}$  is also 'semi-open' under the same fuzzy ideal, then both  $\tilde{G}$  and  $\tilde{H}$  are half-open relate to the fuzzy ideal  $\tilde{g}$ .

**Proof.** (1) Assume  $\tilde{G}$  is 'semi-open' under the fuzzy ideal  $\tilde{g}$ , and  $\mu\tilde{G} < \mu\tilde{H}$ . There exists an open fuzzy set  $\tilde{R}$  satisfying  $\mu\tilde{R} - \mu\tilde{G} \leq \mu\tilde{g}$  and  $\mu\tilde{G} - \mu\tilde{k}(\tilde{R}) \leq \mu\tilde{g}$ . Given that we are considering subsets of  $\tilde{M}$  not belonging to the fuzzy ideal  $\tilde{g}$ , it's imperative to note that such an open set  $\tilde{R}$  must be non-empty. Because  $\mu\tilde{G} < \mu\tilde{H}$ , the  $\mu\tilde{R} - \mu\tilde{H} < \mu\tilde{R} - \mu\tilde{G} \leq \mu\tilde{g}$ ; moreover,  $\mu\tilde{H} - \mu\tilde{k}(\tilde{R}) = \mu\tilde{G} - \mu\tilde{M} = \mu\tilde{\emptyset} \leq \mu\tilde{g}$ . Thus,  $\tilde{H}$  is fuzzy  $\tilde{g}$ - 'semi-open'.

(2) Since  $\mu\tilde{G} < \mu\tilde{H} \Leftrightarrow \max\{\mu\tilde{G}, \mu\tilde{H}\} = \mu\tilde{H}$ , (2) is a direct consequence of (1).

(3) Assume both  $\tilde{G}$  and  $\tilde{H}$  are 'semi-open' under the fuzzy ideal  $\tilde{g}$ . Without loss of generality, consider  $\min\{\mu\tilde{G}, \mu\tilde{H}\} \neq \emptyset$ ; if it were empty,  $\min\{\mu\tilde{G}, \mu\tilde{H}\}$  would be trivially 'semi-open' under  $\tilde{g}$ . According to the given conditions, there exist open fuzzy sets  $\tilde{R}$  and  $\tilde{S}$  such that  $\mu\tilde{R} - \mu\tilde{G}, \mu\tilde{G} - \mu\tilde{k}(\tilde{R}) \leq \mu\tilde{g}$ , and  $\mu\tilde{V} - \mu\tilde{H}, \mu\tilde{H} - \mu\tilde{k}(\tilde{S}) \leq \mu\tilde{g}$ . Consider the fuzzy open set obtained from the intersection of  $\mu\tilde{R}$  and  $\mu\tilde{S}$ , affirming its non-emptiness owing to the fuzzy finite intersection property. Since the difference between this set and  $\min\{\mu\tilde{G}, \mu\tilde{H}\}$  is bounded by  $\mu\tilde{g}$ , and the difference between  $\min\{\mu\tilde{G}, \mu\tilde{H}\}$  and its closure in  $\min\{\mu\tilde{R}, \mu\tilde{S}\}$  is also within  $\mu\tilde{g}$ , it can be concluded that  $\min\{\mu\tilde{G}, \mu\tilde{H}\}$  is 'semi-open' under the fuzzy ideal  $\tilde{g}$ .

**Remark 2-12:-** The fuzzy singleton subset, composed of elements such as  $\{(\delta,0.0),(\varepsilon,0.6),(\zeta,0.0)\}$ , revealed specific characteristics in Example 2 was not ‘semi-open’ under the minimal fuzzy ideal  $\mu_{\tilde{g}} = \{\tilde{\emptyset}\}$ . However, it's noteworthy that the fuzzy set  $\{(\delta,0.6),(\varepsilon,0.6),(\zeta,0.0)\} = \max\{ \{(\delta,0.6),(\varepsilon,0.0),(\zeta,0.0)\}, \{(\delta,0.0),(\varepsilon,0.6),(\zeta,0.0)\} \}$  is ‘semi-open’ under  $\mu_{\tilde{g}} = \{\tilde{\emptyset}\}$ . This is because the non-empty open singleton subset  $\{(\delta,0.6),(\varepsilon,0.0),(\zeta,0.0)\}$  is densely distributed in  $\tilde{M}$ , which aligns with Proposition 5(2) mentioned earlier.

**Proposition 2-12:-** In the context described in Proposition 5,  $\tilde{G}$  is ‘semi-open’ under the fuzzy ideal  $\tilde{g}$  if and only if the closure of  $\tilde{G}$ , denoted as  $(\tilde{k})(\tilde{G})$ , is also ‘semi-open’ under the same fuzzy ideal  $\tilde{g}$ .

**Proof.** If  $\tilde{G}$  is ‘semi-open’ under the fuzzy ideal  $\tilde{g}$ , then, because  $\mu_{\tilde{G}} < \mu_{(\tilde{k})(\tilde{G})}$ , its closure  $(\tilde{k})(\tilde{G})$  is also ‘semi-open’, as stated in Proposition 5(2). On the other hand, if  $(\tilde{k})(\tilde{G})$  is ‘semi-open’ under the fuzzy ideal  $\tilde{g}$ , an open fuzzy set exists  $\tilde{R}$  such that  $\mu_{\tilde{R}} - (\tilde{k})(\tilde{G}) \leq \mu_{\tilde{g}}$  and  $(\tilde{k})(\tilde{G}) - (\tilde{k})(\tilde{R}) \leq \mu_{\tilde{g}}$ . It's important to note that  $\tilde{U}$  must be non-empty; otherwise,  $(\tilde{k})(\tilde{R}) = \mu_{\tilde{\emptyset}}$ , which implies  $\mu_{\tilde{G}} \leq \mu_{\tilde{g}}$  - a situation we aim to steer clear of situations involving subsets that do not fall within the fuzzy ideal  $\tilde{g}$ . To establish  $\tilde{G}$  as ‘semi-open’ under the fuzzy ideal  $\tilde{g}$ , consider the fuzzy open set  $\mu_{\tilde{S}} = \mu_{\tilde{R}} - \mu_{(\tilde{k})(\tilde{G})}$ , obtained from the assumption that  $\mu_{\tilde{R}}$  and  $\mu_{((\tilde{k})(\tilde{G}))} \wedge c \leq \mu_{\tilde{g}}$ . It is evident that  $\mu_{\tilde{S}} - \mu_{\tilde{G}} = \min\{\mu_{\tilde{R}}, \mu_{((\tilde{k})(\tilde{G}))} \wedge c, \mu_{\tilde{G}} \wedge c\} \leq \mu_{\tilde{g}}$  due to the heredity property. Additionally,  $\mu_{\tilde{G}} - \mu_{(\tilde{k})(\tilde{S})} = \mu_{\tilde{G}} - \mu_{(\tilde{k})(\min\{\mu_{\tilde{R}}, \mu_{((\tilde{k})(\tilde{G}))} \wedge c\})} = \mu_{\tilde{G}} - \mu_{\tilde{M}} = \mu_{\tilde{\emptyset}} \leq \mu_{\tilde{g}}$ . This demonstrates that  $\tilde{G}$  qualifies as ‘semi-open’ under the fuzzy ideal  $\tilde{g}$ .

**Proposition 2-14:-** For a fuzzy subset  $\tilde{G}$  of  $\tilde{M}$ , the conditions listed below are interchangeable:

1. The set  $\mu_{\tilde{M}} - \mu_{\tilde{G}}$  exhibits ‘semi-open’ properties under the fuzzy ideal  $\tilde{g}$ .
2. A closed fuzzy set exists  $(\tilde{W})$  such that  $\mu_{\text{int}(\tilde{W})} - \mu_{\tilde{G}} \leq \mu_{\tilde{g}}$  and  $\mu_{\tilde{G}} - \mu_{\tilde{W}} \leq \mu_{\tilde{g}}$ .

**Proof.** Let's begin by assuming that  $\mu_{\tilde{M}} - \mu_{\tilde{G}}$  is ‘semi-open’ under the fuzzy ideal  $\tilde{g}$ . This implies the existence of an open fuzzy set  $\tilde{R}$  such that  $\mu_{\tilde{R}} - (\mu_{\tilde{M}} - \mu_{\tilde{G}}) \leq \mu_{\tilde{g}}$  and  $(\mu_{\tilde{M}} - \mu_{\tilde{G}}) - \mu_{(\tilde{k})(\tilde{R})} \leq \mu_{\tilde{g}}$ . Since  $\mu_{\tilde{R}} - (\mu_{\tilde{R}} - \mu_{\tilde{G}})$  is equivalent to  $\mu_{\tilde{G}} - (\mu_{\tilde{M}} - \mu_{\tilde{R}})$  and  $(\mu_{\tilde{M}} - \mu_{\tilde{G}}) - \mu_{(\tilde{k})(\tilde{R})}$  corresponds to  $\text{int}(\mu_{\tilde{M}} - \mu_{\tilde{R}}) - \mu_{\tilde{G}}$ , we can satisfy condition (2) by selecting the fuzzy closed set  $\mu_{\tilde{M}} - \mu_{\tilde{R}}$  as  $\tilde{W}$ . Conversely, assuming condition (2) holds, choosing the fuzzy open set  $\mu_{\tilde{R}} = \mu_{\tilde{M}} - \mu_{\tilde{W}}$  demonstrates that  $\mu_{\tilde{M}} - \mu_{\tilde{G}}$  is ‘semi-open’ under the fuzzy ideal  $\tilde{g}$ .

**Definition 2-15:-** A ‘fuzzy subset’  $\tilde{G}$  of  $\tilde{M}$  is deemed semi-closed under the fuzzy ideal  $\tilde{g}$  (denoted as fuzzy  $\tilde{g}$ -semi-closed) whenever  $\mu_{\tilde{M}} - \mu_{\tilde{G}}$  exhibits ‘semi-open’ properties under the same fuzzy ideal  $\tilde{g}$ .

**Proposition 2-16:-** If  $\tilde{G}$  and  $\tilde{H}$  are both semi-closed under the fuzzy ideal  $\tilde{g}$ , their intersection  $\min\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\}$  also qualifies as semi-closed under the same fuzzy ideal  $\tilde{g}$ .

**Proof.** Assuming the specified conditions are satisfied. Fuzzy closed sets  $\tilde{W}1$  and  $\tilde{W}2$  are exist subject to the condition that  $\text{int}(\tilde{W}1) - \mu_{\tilde{G}}, \mu_{\tilde{G}} - \mu_{\tilde{W}1} \leq \mu_{\tilde{g}}$  and  $\text{int}(\tilde{W}2) - \mu_{\tilde{H}}, \mu_{\tilde{H}} - \mu_{\tilde{W}2} \leq \mu_{\tilde{g}}$ . With  $\mu_{\tilde{W}} = \min\{\mu_{\tilde{W}1}, \mu_{\tilde{W}2}\}$ , we have that  $\text{Int}(\min\{\mu_{\tilde{W}1}, \mu_{\tilde{W}2}\}) - (\min\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\}) = \max\{\min\{\text{int}(\mu_{\tilde{W}1}) - \mu_{\tilde{W}}, \text{int}(\mu_{\tilde{W}2})\}, \min\{\text{int}(\mu_{\tilde{W}1}), \text{int}(\mu_{\tilde{W}2}) - \mu_{\tilde{H}}\}\} \leq \mu_{\tilde{g}}$ ,

and

$$(\min\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\}) - (\min\{\mu_{\tilde{W}1}, \mu_{\tilde{W}2}\}) = \max\{\min\{\mu_{\tilde{G}} - \mu_{\tilde{W}1}, \mu_{\tilde{H}}\}, \min\{\mu_{\tilde{G}}, \mu_{\tilde{H}} - \mu_{\tilde{W}2}\}\} \leq \mu_{\tilde{g}};$$

therefore,  $\min\{\mu_{\tilde{G}}, \mu_{\tilde{H}}\}$  is fuzzy  $\tilde{g}$ -semi-closed.

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