Neutrosophic nil radicals of neutrosophic ideals in rings

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Abstract

The notion of neutrosophic nil radicals of neutrosophic ideals in rings is introduced, and related properties are investigated. In addition, we study some relations between the nil radical of the neutrosophic polynomial ideal of the polynomial ring $R[x]$ induced by a neutrosophic ideal of a ring R and the nil radical of a neutrosophic ideal of R. Finally, we find the result of neutrosophic nil radicals of neutrosophic ideals from the epimorphism of rings.

Keywords: ring; neutrosophic polynomial ideal; neutrosophic ideal; neutrosophic nil radical; epimorphism.

1 Introduction

The concept of fuzzy sets was proposed by Zadeh.[13] The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in.[1,3,4] The idea of intuitionistic fuzzy sets suggested by Atanassov[2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision-making.[6–8] The notion of neutrosophic sets was introduced by Smarandache[11] in 1999, which is a more general platform that extends the notions of classic sets (intuitionistic) fuzzy sets and interval-valued (intuitionistic) fuzzy sets. Neutrosophic set theory is applied to various parts, which is referred to on the site `http://fs.unm.edu/neutrosophy.htm`. Smarandache[12] generalized the intuitionistic fuzzy set, paraconsistent set, and intuitionistic set to the neutrosophic set. Cetkin and Aygun[5] introduced an approach to single-valued neutrosophic ideals over a ring. Hameed et al.[9] introduced the notions of γ-single valued neutrosophic subrings, γ-single valued neutrosophic ideals, and the sum and product of γ-single valued neutrosophic ideals.

This article introduces the concept of neutrosophic nil radicals of neutrosophic ideals in rings and explores features that are connected to it. Furthermore, we examine some connections between the nil radical of the neutrosophic polynomial ideal of the polynomial ring $R[x]$ induced by a neutrosophic ideal of a ring R and the nil radical of a neutrosophic ideal of R. Finally, using the epimorphism of rings, we determine the outcome of neutrosophic nil radicals of neutrosophic ideals.

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2 Preliminaries

Let's start by reviewing the definition of a neutrosophic set as it was defined by Smarandache[11] in 1999.

Let R be a nonempty set. A neutrosophic set A on R is defined to be a structure

$$A := \{\langle x, \mu(x), \gamma(x), \psi(x)\rangle \mid x \in R\}, \tag{1}$$

where $\mu : R \to [0,1]$ is a truth membership function, $\gamma : R \to [0,1]$ is an indeterminate membership function, and $\psi : R \to [0,1]$ is a false membership function. The neutrosophic set in (1) is simply denoted by $A = (\mu_A, \gamma_A, \psi_A)$.

In this section, we review some definitions which will be used in the later section. Throughout this paper unless stated otherwise all rings are commutative rings with identity.

Definition 2.1. A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ of a ring R is said to be a neutrosophic subring of R if

$$(\forall x, y \in R) \begin{pmatrix} \mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(x-y) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(x-y) \leq \max\{\psi_A(x), \psi_A(y)\} \end{pmatrix}, \tag{2}$$

$$(\forall x, y \in R) \begin{pmatrix} \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(xy) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(xy) \leq \max\{\psi_A(x), \psi_A(y)\} \end{pmatrix}. \tag{3}$$

Definition 2.2. A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ of a ring R is said to be a neutrosophic ideal of R if it satisfies (2) and

$$(\forall x, y \in R) \begin{pmatrix} \mu_A(xy) \geq \max\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(xy) \geq \max\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(xy) \leq \min\{\psi_A(x), \psi_A(y)\} \end{pmatrix}. \tag{4}$$

Sometimes the notation $\min\{a, b\}$ is denoted by $a \wedge b$, and $\max\{a, b\}$ is denoted by $a \vee b$.

Definition 2.3. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic sets in a ring R (not necessarily commutative). The neutrosophic sum $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ is defined to be the neutrosophic set $A + B = (\mu_{A+B}, \gamma_{A+B}, \psi_{A+B})$ in R given by

$$\mu_{A+B}(x) = \begin{cases} \bigvee\limits_{x=y+z} (\mu_A(y) \wedge \mu_B(z)) & \text{if } x = y+z \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_{A+B}(x) = \begin{cases} \bigvee\limits_{x=y+z} (\gamma_A(y) \wedge \gamma_B(z)) & \text{if } x = y+z \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{A+B}(x) = \begin{cases} \bigwedge\limits_{x=y+z} (\psi_A(y) \vee \psi_b(z)) & \text{if } x = y+z \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic sets in a ring R (not necessarily commutative). The neutrosophic product of $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ is defined to be the neutrosophic set $A * B = (\mu_{AB}, \gamma_{AB}, \psi_{AB})$ in R given by

$$\mu_{AB}(x) = \bigvee \left\{ \bigwedge_{1 \leq i \leq k} \mu_A(a_i) \wedge \mu_B(b_i) : \sum_{i=1}^{n} a_i b_i = x, k \in \mathbb{N} \right\},$$

$$\gamma_{AB}(x) = \bigvee \left\{ \bigwedge_{1 \leq i \leq k} \gamma_A(a_i) \wedge \gamma_B(b_i) : \sum_{i=1}^{n} a_i b_i = x, k \in \mathbb{N} \right\},$$

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$$\psi_{AB}(x) = \bigwedge \left\{ \bigvee_{1 \le i \le k} \psi_A(a_i) \vee \psi_B(b_i) : \sum_{i=1}^{n} a_i b_i = x, k \in \mathbb{N} \right\},$$

if we can express $x = \sum\limits_{i=1}^{n} a_i b_i$ for some $a_i, b_i \in \mathbb{R}$, where each $a_i b_i \neq 0$ and $k \in \mathbb{N}$. Otherwise, we define $AB = 0$, that is, $\mu_{AB}(x) = 0$, $\gamma_{AB}(x) = 0$, and $\psi_{AB}(x) = 1$.

Theorem 2.5. [10] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic ideal of a ring R and let $f(x) = \sum\limits_{i=0}^{m} a_i x_i \in \mathbb{R}[x]$. Define the neutrosophic set $A_x = (\mu_{A_x}, \gamma_{A_x}, \psi_{A_x})$ on $\mathbb{R}[x]$ by $\mu_{A_x}(f(x)) = \min_i\{\mu_A(a_i)\}$, $\gamma_{A_x}(f(x)) = \min_i\{\gamma_A(a_i)\}$, and $\psi_{A_x}(f(x)) = \max_i\{\psi_A(a_i)\}$. Then A_x is a neutrosophic ideal of $\mathbb{R}[x]$.*

3 Radical of the neutrosophic polynomial induced by a neutrosophic ideal

In this section, we study some relations between the nil radical of the neutrosophic polynomial ideal of $R[x]$ induced by a neutrosophic ideal of a ring R and the nil radical of a neutrosophic ideal of the ring.

Definition 3.1. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic ideal of a ring R. Then the neutrosophic nil radical of A is defined to be a neutrosophic set $\sqrt{A} = (\mu_{\sqrt{A}}, \gamma_{\sqrt{A}}, \psi_{\sqrt{A}})$ by $\mu_{\sqrt{A}}(x) = \bigvee\limits_{n \ge 1} \mu_A(x^n)$, $\gamma_{\sqrt{A}}(x) = \bigvee\limits_{n \ge 1} \gamma_A(x^n)$, and $\psi_{\sqrt{A}}(x) = \bigwedge\limits_{n \ge 1} \psi_A(x^n)$ for all $x \in R$ and for $n \in \mathbb{N}$.

Proposition 3.2. *If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of a ring R, then*

(1) $A \subseteq \sqrt{A}$,

(2) if $A \subseteq B$, then $\sqrt{A} \subseteq \sqrt{B}$,

(3) $\sqrt{\sqrt{A}} = \sqrt{A}$.

Proof. (1) Let $x \in R$. Then

$$\begin{aligned} \mu_{\sqrt{A}}(x) &= \bigvee_{m \ge 1} \mu_A(x^m) \ge \mu_A(x^m) \ge \mu_A(x), \\ \gamma_{\sqrt{A}}(x) &= \bigvee_{m \ge 1} \gamma_A(x^m) \ge \gamma_A(x^m) \ge \gamma_A(x), \\ \psi_{\sqrt{A}}(x) &= \bigwedge_{m \ge 1} \psi_A(x^m) \le \psi_A(x^m) \le \psi_A(x). \end{aligned}$$

Hence, $A \subseteq \sqrt{A}$.
(2) Let $x \in R$ and $A \subseteq B$. Then

$$\begin{aligned} \mu_{\sqrt{A}}(x) &= \bigvee_{m \ge 1} \mu_A(x^m) \le \bigvee_{m \ge 1} \mu_B(x^m) \le \mu_B(x), \\ \gamma_{\sqrt{A}}(x) &= \bigvee_{m \ge 1} \gamma_A(x^m) \le \bigvee_{m \ge 1} \gamma_B(x^m) \le \gamma_B(x), \\ \psi_{\sqrt{A}}(x) &= \bigwedge_{m \ge 1} \psi_A(x^m) \ge \bigwedge_{m \ge 1} \psi_B(x^m) \ge \psi_B(x). \end{aligned}$$

Hence, $\sqrt{A} \subseteq \sqrt{B}$.
(3) Let $x \in R$. Then

$$\begin{aligned} \mu_{\sqrt{\sqrt{A}}}(x) &= \bigvee_{m \ge 1} \mu_{\sqrt{A}}(x^m) = \bigvee_{m \ge 1} \bigvee_{n \ge 1} \mu_A((x^m))^n = \bigvee_{k \ge 1} \mu_A(x^k) = \mu_{\sqrt{A}}(x), \\ \gamma_{\sqrt{\sqrt{A}}}(x) &= \bigvee_{m \ge 1} \gamma_{\sqrt{A}}(x^m) = \bigvee_{m \ge 1} \bigvee_{n \ge 1} \gamma_A((x^m))^n = \bigvee_{k \ge 1} \gamma_A(x^k) = \gamma_{\sqrt{A}}(x), \\ \psi_{\sqrt{\sqrt{A}}}(x) &= \bigwedge_{m \ge 1} \psi_{\sqrt{A}}(x^m) = \bigwedge_{m \ge 1} \bigwedge_{n \ge 1} \psi_A((x^m))^n = \bigwedge_{k \ge 1} \psi_A(x^k) = \psi_{\sqrt{A}}(x). \end{aligned}$$

Hence, $\sqrt{\sqrt{A}} = \sqrt{A}$. □

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Theorem 3.3. *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic ideal of a ring R, then $\sqrt{A}$ is a neutrosophic ideal of R.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned}
\mu_{\sqrt{A}}(x) \wedge \mu_{\sqrt{A}}(y) &= (\bigvee_{m\geq 1} \mu_A(x^m)) \wedge (\bigvee_{n\geq 1} \mu_A(y^n)) \\
&= \bigvee_{n\geq 1} (\bigvee_{m\geq 1} \mu_A(x^m) \wedge \mu_A(y^n)), \\
\gamma_{\sqrt{A}}(x) \wedge \gamma_{\sqrt{A}}(y) &= (\bigvee_{m\geq 1} \gamma_A(x^m)) \wedge (\bigvee_{n\geq 1} \gamma_A(y^n)) \\
&= \bigvee_{n\geq 1} (\bigvee_{m\geq 1} \gamma_A(x^m) \wedge \gamma_A(y^n)), \\
\psi_{\sqrt{A}}(x) \vee \psi_{\sqrt{A}}(y) &= (\bigwedge_{m\geq 1} \psi_A(x^m)) \vee (\bigwedge_{n\geq 1} \psi_A(y^n)) \\
&= \bigwedge_{n\geq 1} (\bigwedge_{m\geq 1} \psi_A(x^m) \vee \psi_A(y^n)).
\end{aligned}$$

Let $m, n \in \mathbb{N}$. Since R is commutative, each term in the binomial expansion of $(x+y)^{m+n}$ contains either x^m or y^n as a factor. Hence, there exist $r, t \in R$ such that $(x+y)^{m+n} = rx^m + ty^n$. Thus

$$\begin{aligned}
\mu_A(x^m) \wedge \mu_A(y^n) &\leq (\mu_A(x^m) \vee \mu_A(r)) \wedge (\mu_A(y^n) \vee \mu_A(t)) \\
&\leq \mu_A(rx^m) \wedge \mu_A(ty^n) \\
&\leq \mu_A(rx^m + ty^n) \\
&= \mu_A((x+y)^{m+n}) \\
&\leq \bigvee_{k\geq 1} \mu_A((x+y)^k) \\
&= \mu_{\sqrt{A}}(x+y), \\
\gamma_A(x^m) \wedge \gamma_A(y^n) &\leq (\gamma_A(x^m) \vee \gamma_A(r)) \wedge (\gamma_A(y^n) \vee \gamma_A(t)) \\
&\leq \gamma_A(rx^m) \wedge \gamma_A(ty^n) \\
&\leq \gamma_A(rx^m + ty^n) \\
&= \gamma_A((x+y)^{m+n}) \\
&\leq \bigvee_{k\geq 1} \gamma_A((x+y)^k) \\
&= \gamma_{\sqrt{A}}(x+y), \\
\psi_A(x^m) \vee \psi_A(y^n) &\geq (\psi_A(x^m) \wedge \psi_A(r)) \vee (\psi_A(y^n) \wedge \psi_A(t)) \\
&\geq \psi_A(rx^m) \vee \psi_A(ty^n) \\
&\geq \psi_A(rx^m + ty^n) \\
&= \psi_A((x+y)^{m+n}) \\
&\geq \bigwedge_{k\geq 1} \psi_A((x+y)^k) \\
&= \psi_{\sqrt{A}}(x+y).
\end{aligned}$$

Notice that

$$\begin{aligned}
\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y) &\Leftrightarrow \mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y), \\
\gamma_A(x-y) \geq \gamma_A(x) \wedge \gamma_A(y) &\Leftrightarrow \gamma_A(x+y) \geq \gamma_A(x) \wedge \gamma_A(y), \\
\psi_A(x-y) \leq \psi_A(x) \vee \psi_A(y) &\Leftrightarrow \psi_A(x+y) \leq \psi_A(x) \vee \psi_A(y),
\end{aligned}$$

respectively. Next, we have

$$\begin{aligned}
\mu_{\sqrt{A}}(x) \vee \mu_{\sqrt{A}}(y) &= (\bigvee_{n\geq 1} \mu_A(x^n)) \vee (\bigvee_{n\geq 1} \mu_A(y^n)) \\
&= \bigvee_{n\geq 1} (\bigvee_{n\geq 1} \mu_A(x^n) \vee \mu_A(y^n)), \qquad (5)
\end{aligned}$$

$$\begin{aligned}
\gamma_{\sqrt{A}}(x) \vee \gamma_{\sqrt{A}}(y) &= (\bigvee_{n\geq 1} \gamma_A(x^n)) \vee (\bigvee_{n\geq 1} \gamma_A(y^n)) \\
&= \bigvee_{n\geq 1} (\bigvee_{n\geq 1} \gamma_A(x^n) \vee \gamma_A(y^n)), \qquad (6)
\end{aligned}$$

$$\begin{aligned}
\psi_{\sqrt{A}}(x) \wedge \psi_{\sqrt{A}}(y) &= (\bigwedge_{n\geq 1} \psi_A(x^n)) \wedge (\bigwedge_{n\geq 1} \psi_A(y^n)) \\
&= \bigwedge_{n\geq 1} (\bigwedge_{n\geq 1} \psi_A(x^n) \wedge \psi_A(y^n)). \qquad (7)
\end{aligned}$$

Also

$$\begin{array}{rcl} \mu_A(x^n)\wedge\mu_A(y^n) & \leq & \mu_A(x^ny^n) \\ & = & \mu_A((xy)^n) \\ & \leq & \bigvee\limits_{k\geq 1}\mu_A((xy)^k) \\ & = & \mu_{\sqrt{A}}(xy), \\ \gamma_A(x^n)\wedge\gamma_A(y^n) & \leq & \gamma_A(x^ny^n) \\ & = & \gamma_A((xy)^n) \\ & \leq & \bigvee\limits_{k\geq 1}\gamma_A((xy)^k) \\ & = & \gamma_{\sqrt{A}}(xy), \\ \psi_A(x^n)\vee\psi_A(y^n) & \geq & \psi_A(x^ny^n) \\ & = & \psi_A((xy)^n) \\ & \geq & \bigwedge\limits_{k\geq 1}\psi_A((xy)^k) \\ & = & \psi_{\sqrt{A}}(xy). \end{array}$$

It follows that $\mu_{\sqrt{A}}(xy)\geq\mu_{\sqrt{A}}(x)\wedge\mu_{\sqrt{A}}(y)$, $\gamma_{\sqrt{A}}(xy)\geq\gamma_{\sqrt{A}}(x)\wedge\gamma_{\sqrt{A}}(y)$, and $\psi_{\sqrt{A}}(xy)\leq\psi_{\sqrt{A}}(x)\vee\psi_{\sqrt{A}}(y)$. Hence, $\sqrt{A}$ is a neutrosophic ideal of R. □

Corollary 3.4. *If $A=(\mu_A,\gamma_A,\psi_A)$ is a neutrosophic ideal of a ring R, then $\sqrt{A}$ is a neutrosophic subring of R.*

Proof. Let $x,y\in R$. Then by Theorem 3.3, $\sqrt{A}$ is a neutrosophic ideal of R. Thus

$$(\forall x,y\in R)\left(\begin{array}{c}\mu_{\sqrt{A}}(x-y)\geq\min\{\mu_{\sqrt{A}}(x),\mu_{\sqrt{A}}(y)\} \\ \gamma_{\sqrt{A}}(x-y)\geq\min\{\gamma_{\sqrt{A}}(x),\gamma_{\sqrt{A}}(y)\} \\ \psi_{\sqrt{A}}(x-y)\leq\max\{\psi_{\sqrt{A}}(x),\psi_{\sqrt{A}}(y)\}\end{array}\right),$$

$$(\forall x,y\in R)\left(\begin{array}{c}\mu_{\sqrt{A}}(xy)\geq\max\{\mu_{\sqrt{A}}(x),\mu_{\sqrt{A}}(y)\} \\ \gamma_{\sqrt{A}}(xy)\geq\max\{\gamma_{\sqrt{A}}(x),\gamma_{\sqrt{A}}(y)\} \\ \psi_{\sqrt{A}}(xy)\leq\min\{\psi_{\sqrt{A}}(x),\psi_{\sqrt{A}}(y)\}\end{array}\right).$$

On the other hand, we have

$$\mu_{\sqrt{A}}(xy)\geq\max\{\mu_{\sqrt{A}}(x),\mu_{\sqrt{A}}(y)\}\geq\min\{\mu_{\sqrt{A}}(x),\mu_{\sqrt{A}}(y)\},$$

$$\gamma_{\sqrt{A}}(xy)\geq\max\{\gamma_{\sqrt{A}}(x),\gamma_{\sqrt{A}}(y)\}\geq\min\{\gamma_{\sqrt{A}}(x),\gamma_{\sqrt{A}}(y)\},$$

$$\psi_{\sqrt{A}}(xy)\leq\min\{\psi_{\sqrt{A}}(x),\psi_{\sqrt{A}}(y)\}\leq\max\{\psi_{\sqrt{A}}(x),\psi_{\sqrt{A}}(y)\}.$$

Hence, $\sqrt{A}$ is a neutrosophic subring of R. □

Theorem 3.5. *If $A=(\mu_A,\gamma_A,\psi_A)$ and $B=(\mu_B,\gamma_B,\psi_B)$ are neutrosophic ideals of a ring R, then*

(1) $\sqrt{A\cap B}=\sqrt{A}\cap\sqrt{B}$,

(2) $\sqrt{A\cup B}=\sqrt{A}\cup\sqrt{B}$,

(3) $\sqrt{A}+\sqrt{B}\subseteq\sqrt{A+B}$,

(4) $\sqrt{A}\sqrt{B}\subseteq\sqrt{AB}$.

Proof. (1) Since $A\cap B\subseteq A$ and $A\cap B\subseteq B$, it follows from Proposition 3.2 (1) that $\sqrt{A\cap B}\subseteq\sqrt{A}$ and $\sqrt{A\cap B}\subseteq\sqrt{B}$ and so $\sqrt{A\cap B}\subseteq\sqrt{A}\cap\sqrt{B}$. For another inclusion, let $x\in R$. Then

$$\begin{array}{rcl} \mu_{\sqrt{A}\cap\sqrt{B}}(x) & = & \mu_{\sqrt{A}}(x)\wedge\mu_{\sqrt{A}}(y) \\ & = & (\bigvee\limits_{m\geq 1}\mu_A(x^m))\wedge(\bigvee\limits_{n\geq 1}\mu_A(y^n)) \\ & = & \bigvee\limits_{m\geq 1}\bigvee\limits_{n\geq 1}(\mu_A(x^m)\wedge\mu_A(y^n)), \end{array}$$

$$\begin{array}{rcl}\gamma_{\sqrt{A}\cap\sqrt{B}}(x) & = & \gamma_{\sqrt{A}}(x)\wedge\gamma_{\sqrt{A}}(y)\\ & = & (\bigvee\limits_{m\geq 1}\gamma_A(x^m))\wedge(\bigvee\limits_{n\geq 1}\gamma_A(y^n))\\ & = & \bigvee\limits_{m\geq 1}\bigvee\limits_{n\geq 1}(\gamma_A(x^m)\wedge\gamma_A(y^n)),\end{array}$$

$$\begin{array}{rcl}\psi_{\sqrt{A}\cap\sqrt{B}}(x) & = & \psi_{\sqrt{A}}(x)\vee\psi_{\sqrt{A}}(y)\\ & = & (\bigwedge\limits_{m\geq 1}\psi_A(x^m))\vee(\bigwedge\limits_{n\geq 1}\psi_A(y^n))\\ & = & \bigwedge\limits_{m\geq 1}\bigwedge\limits_{n\geq 1}(\psi_A(x^m)\vee\psi_A(y^n)).\end{array}$$

Now, let $m, n \in \mathbb{N}$. Then

$$\begin{array}{rcl}\mu_A(x^m)\wedge\mu_B(y^n) & \leq & \mu_A(x^{mn})\wedge\mu_B(y^{mn})\\ & = & \mu_{A\cap B}(x^{mn})\\ & \leq & \bigvee\limits_{k\geq 1}\mu_{A\cap B}(x^k)\\ & = & \mu_{\sqrt{A\cap B}}(x),\end{array}$$

$$\begin{array}{rcl}\gamma_A(x^m)\wedge\gamma_B(y^n) & \leq & \gamma_A(x^{mn})\wedge\gamma_B(y^{mn})\\ & = & \gamma_{A\cap B}(x^{mn})\\ & \leq & \bigvee\limits_{k\geq 1}\gamma_{A\cap B}(x^k)\\ & = & \gamma_{\sqrt{A\cap B}}(x),\end{array}$$

$$\begin{array}{rcl}\psi_A(x^m)\vee\psi_B(y^n) & \geq & \psi_A(x^{mn})\vee\psi_B(y^{mn})\\ & = & \psi_{A\cap B}(x^{mn})\\ & \geq & \bigwedge\limits_{k\geq 1}\psi_{A\cap B}(x^k)\\ & = & \psi_{\sqrt{A\cap B}}(x).\end{array}$$

Thus $\mu_{\sqrt{A}\cap\sqrt{B}}(x) \leq \mu_{\sqrt{A\cap B}}(x)$, $\gamma_{\sqrt{A}\cap\sqrt{B}}(x) \leq \gamma_{\sqrt{A\cap B}}(x)$, and $\psi_{\sqrt{A}\cap\sqrt{B}}(x) \geq \psi_{\sqrt{A\cap B}}(x)$. Hence, $\sqrt{A\cap B} = \sqrt{A}\cap\sqrt{B}$.
(2) The proof follows similar to the proof of (1).
(3) The proof follows from the Definition 2.3.
(4) The proof follows from the Definition 2.4. □

If A is a neutrosophic ideal of a ring R, then A_x is a neutrosophic ideal of a polynomial ring $R[x]$ by Theorem 2.5 and the neutrosophic set $\sqrt{A_x}$ is the neutrosophic nil radical of A_x.

The following theorem gives that the two neutrosophic nil radicals have the same value.

Theorem 3.6. *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic ideal of a ring R, then $(\sqrt{A_x})_x = (\sqrt{A})_x$.*

Proof. Let $f(x) = \sum\limits_{i=0}^{m} a_i x^i \in R[x]$. By Theorem 2.5, we have $A_x = (\mu_{A_x}, \gamma_{A_x}, \psi_{A_x})$, where

$$\begin{array}{rcl}\mu_{A_x}(a_j^n) & = & \mu_{A_x}(a_j^n + 0x + 0x^2 + \cdots + 0x^m)\\ & = & \min\{\mu_A(a_j^n), \mu_A(0), \ldots, \mu_A(0)\}\\ & = & \mu_A(a_j^n),\end{array}$$

$$\begin{array}{rcl}\gamma_{A_x}(a_j^n) & = & \gamma_{A_x}(a_j^n + 0x + 0x^2 + \cdots + 0x^m)\\ & = & \min\{\gamma_A(a_j^n), \gamma_A(0), \ldots, \gamma_A(0)\}\\ & = & \gamma_A(a_j^n),\end{array}$$

$$\begin{array}{rcl}\psi_{A_x}(a_j^n) & = & \psi_{A_x}(a_j^n + 0x + 0x^2 + \cdots + 0x^m)\\ & = & \max\{A(a_j^n), \psi_A(0), \ldots, \psi_A(0)\}\\ & = & \psi_A(a_j^n).\end{array}$$

Since $\sqrt{A_x}$ is a neutrosophic ideal of $R[x]$, we have $(\sqrt{A_x})_x = (\mu_{(\sqrt{A_x})_x}, \gamma_{(\sqrt{A_x})_x}, \psi_{(\sqrt{A_x})_x})$, where

$$\begin{aligned}
\mu_{(\sqrt{A_x})_x}(f(x)) &= \min_{i=0}^{m}\{\mu_{(\sqrt{A_x})_x}(a_i)\} \\
&= \min_{i=0}^{m}\{\bigvee_{n\geq 1} \mu_{A_x}(a_i^n)\} \\
&= \min_{i=0}^{m}\{\bigvee_{n\geq 1} \mu_{A}(a_i^n)\} \\
&= \min_{i=0}^{m}\{\mu_{\sqrt{A}}(a_i)\} \\
&= \mu_{(\sqrt{A})_x}(f(x)),
\end{aligned}$$

$$\begin{aligned}
\gamma_{(\sqrt{A_x})_x}(f(x)) &= \min_{i=0}^{m}\{\gamma_{(\sqrt{A_x})_x}(a_i)\} \\
&= \min_{i=0}^{m}\{\bigvee_{n\geq 1} \gamma_{A_x}(a_i^n)\} \\
&= \min_{i=0}^{m}\{\bigvee_{n\geq 1} \gamma_{A}(a_i^n)\} \\
&= \min_{i=0}^{m}\{\gamma_{\sqrt{A}}(a_i)\} \\
&= \gamma_{(\sqrt{A})_x}(f(x)),
\end{aligned}$$

$$\begin{aligned}
\psi_{(\sqrt{A_x})_x}(f(x)) &= \max_{i=0}^{m}\{\psi_{(\sqrt{A_x})_x}(a_i)\} \\
&= \max_{i=0}^{m}\{\bigwedge_{n\geq 1} \psi_{A_x}(a_i^n)\} \\
&= \max_{i=0}^{m}\{\bigwedge_{n\geq 1} \psi_{A}(a_i^n)\} \\
&= \max_{i=0}^{m}\{\psi_{\sqrt{A}}(a_i)\} \\
&= \psi_{(\sqrt{A})_x}(f(x)).
\end{aligned}$$

Hence, $(\sqrt{A_x})_x = (\sqrt{A})_x$. □

Theorem 3.7. *If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of a ring R, then*

(1) $(\sqrt{A \cap B})_x = (\sqrt{A})_x \cap (\sqrt{B})_x$,

(2) $(\sqrt{A})_x \cup (\sqrt{B})_x \subseteq (\sqrt{A \cup B})_x$,

(3) $(\sqrt{A})_x + (\sqrt{B})_x = (\sqrt{A + B})_x$,

(4) $(\sqrt{AB})_x \subseteq (\sqrt{A})_x(\sqrt{B})_x$.

Proof. Let A and B be neutrosophic ideals of R. Then A_x and B_x are neutrosophic polynomial ideals of $R[x]$ by Theorem 2.5.

(1)

$$\begin{aligned}
(\sqrt{A \cap B})_x &= (\sqrt{(A \cap B)_x})_x \\
&= (\sqrt{A_x \cap B_x})_x \\
&= (\sqrt{A_x} \cap \sqrt{B_x})_x \\
&= (\sqrt{A_x})_x \cap (\sqrt{B_x})_x \\
&= (\sqrt{A})_x \cap (\sqrt{B})_x.
\end{aligned}$$

(2)

$$\begin{aligned}
(\sqrt{A \cup B})_x &\supseteq (\sqrt{(A \cup B)_x})_x \\
&= (\sqrt{A_x \cup B_x})_x \\
&\supseteq (\sqrt{A_x} \cup \sqrt{B_x})_x \\
&= (\sqrt{A_x})_x \cup (\sqrt{B_x})_x \\
&= (\sqrt{A})_x \cup (\sqrt{B})_x.
\end{aligned}$$

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(3)

$$\begin{aligned}(\sqrt{A})_x + (\sqrt{B})_x &= (\sqrt{A_x})_x + (\sqrt{B_x})_x \\ &\subseteq (\sqrt{A_x} + \sqrt{B_x})_x \\ &= (\sqrt{A_x + B_x})_x \\ &\subseteq (\sqrt{(A+B)_x})_x \\ &= (\sqrt{A+B})_x.\end{aligned}$$

(4)

$$\begin{aligned}(\sqrt{A})_x(\sqrt{B})_x &= (\sqrt{A})_x(\sqrt{B})_x \\ &\supseteq (\sqrt{A_x}\sqrt{B_x})_x \\ &= (\sqrt{A_x B_x})_x \\ &\supseteq (\sqrt{(AB)_x})_x \\ &= (\sqrt{AB})_x.\end{aligned}$$

□

Theorem 3.8. *Let $f : R \to R'$ be a homomorphism of rings and let B be a neutrosophic ideal of R'. If f_x is the induced homomorphism of f, that is, $\mu_x(\sum\limits_{i=0}^{n} a_i x_i) = \sum\limits_{i=0}^{n} f(a_i)x_i$, then $f^{-1}(\sqrt{B})_x = \sqrt{f^{-1}(B))}_x$.*

Proof. Let $g(x) = \sum\limits_{i=0}^{n} b_i x_i \in R[x]$. Then $\sqrt{f^{-1}(B)}_x = (\mu_{\sqrt{f^{-1}(B)}_x}, \mu_{\sqrt{f^{-1}(B)}_x}, \psi_{\sqrt{f^{-1}(B)}_x})$, where

$$\begin{aligned}\mu_{\sqrt{f^{-1}(B)}_x}(g(x)) &= \min_{i}^{m}\{\mu_{\sqrt{f^{-1}(B)}}(b_i)\} \\ &= \min_{i}^{m}\{\bigvee_{n\geq 1} \mu_{f^{-1}(B)}(b_i^n)\} \\ &= \bigvee_{n\geq 1} \min_{i}^{m}\{\mu_{f^{-1}(B)}(b_i^n)\} \\ &= \bigvee_{n\geq 1} \min_{i}^{m}\{\mu_B(f(b_i^n))\} \\ &= \bigvee_{n\geq 1} \min_{i}^{m}\{\mu_B(f(b_i)^n)\} \\ &= \min_{i}^{m}\{\bigvee_{n\geq 1} (f(b_i)^n)\} \\ &= \min_{i}^{m}\{\mu_{\sqrt{B}}(f(b_i))\} \\ &= \mu_{(\sqrt{B})_x}(\mu_x(g(x))) \\ &= \mu_{\mu_x^{-1}(\sqrt{B})_x}(g(x)),\end{aligned}$$

$$\begin{aligned}\gamma_{\sqrt{f^{-1}(B)}_x}(g(x)) &= \min_{i}^{m}\{\gamma_{\sqrt{f^{-1}(B)}}(b_i)\} \\ &= \min_{i}^{m}\{\bigvee_{n\geq 1} \gamma_{f^{-1}(B)}(b_i^n)\} \\ &= \bigvee_{n\geq 1} \min_{i}^{m}\{\gamma_{f^{-1}(B)}(b_i^n)\} \\ &= \bigvee_{n\geq 1} \min_{i}^{m}\{\gamma_B(f(b_i^n))\} \\ &= \bigvee_{n\geq 1} \min_{i}^{m}\{\gamma_B(f(b_i)^n)\} \\ &= \min_{i}^{m}\{\bigvee_{n\geq 1} (f(b_i)^n)\} \\ &= \min_{i}^{m}\{\gamma_{\sqrt{B}}(f(b_i))\} \\ &= \gamma_{(\sqrt{B})_x}(\gamma_x(g(x))) \\ &= \gamma_{\gamma_x^{-1}(\sqrt{B})_x}(g(x)),\end{aligned}$$

Received: May 12, 2023 Revised: July 12, 2023 Accepted: November 01, 2023

$$\begin{aligned}
\psi_{\sqrt{f^{-1}(B)}_x}(g(x)) &= \max_i^m\{\psi_{\sqrt{f^{-1}(B)}}(b_i)\}\\
&= \max_i^m\{\bigvee_{n\geq 1} \psi_{f^{-1}(B)}(b_i^n)\}\\
&= \bigwedge_{n\geq 1} \max_i^m\{\psi_{f^{-1}(B)}(b_i^n)\}\\
&= \bigwedge_{n\geq 1} \max_i^m\{\psi_B(f(b_i^n))\}\\
&= \bigwedge_{n\geq 1} \max_i^m\{\psi_B(f(b_i)^n)\}\\
&= \max_i^m\{\bigwedge_{n\geq 1} (g(b_i)^n)\}\\
&= \max_i^m\{\psi_{\sqrt{B}}(g(b_i))\}\\
&= \psi_{(\sqrt{B})_x}(\psi_x(g(x)))\\
&= \psi_{\mu_x^{-1}(\sqrt{B})_x}(g(x)).
\end{aligned}$$

Hence, $f^{-1}(\sqrt{B})_x = \sqrt{f^{-1}(B))}_x$. □

Proposition 3.9. *Let $f : R \to R'$ be an epimorphism of rings. If A is a neutrosophic ideal of R, then $f(\sqrt{A}) \subseteq \sqrt{f(A)}$. Further, if A is constant on* $\operatorname{Ker} f$*, then $f(\sqrt{A}) = \sqrt{f(A)}$.*

Proof. Clearly, $f(A)$ and $f(\sqrt{A})$ are neutrosophic ideals of R'. If $y \in R'$ and $f(x) = y$ for some $x \in R$, then $f(x^n) = y^n$ for all $n = 1, 2, \ldots$. Thus

$$\begin{aligned}
\mu_{\sqrt{A}}(y) &= \sup\{\mu_{\sqrt{A}}(x) : x \in f^{-1}(y)\}\\
&= \sup\{\bigvee_{n\geq 1} \mu_A(x^n) : x \in f^{-1}(y)\}\\
&= \bigvee_{n\geq 1} \sup\{\mu_A(x^n) : x \in f^{-1}(y)\}\\
&= \bigvee_{n\geq 1} \sup\{\mu_A(x^n) : x^n \in f^{-1}(y^n)\}\\
&= \bigvee_{n\geq 1} \sup\{\mu_A(z^n) : z \in f^{-1}(y^n)\}\\
&= \bigvee_{n\geq 1} \mu_{f(A)}(y^n)\\
&= \mu_{\sqrt{f(A)}}(y),
\end{aligned}$$

$$\begin{aligned}
\gamma_{\sqrt{A}}(y) &= \sup\{\gamma_{\sqrt{A}}(x) : x \in f^{-1}(y)\}\\
&= \sup\{\bigvee_{n\geq 1} \gamma_A(x^n) : x \in f^{-1}(y)\}\\
&= \bigvee_{n\geq 1} \sup\{\gamma_A(x^n) : x \in f^{-1}(y)\}\\
&= \bigvee_{n\geq 1} \sup\{\gamma_A(x^n) : x^n \in f^{-1}(y^n)\}\\
&= \bigvee_{n\geq 1} \sup\{\gamma_A(z^n) : z \in f^{-1}(y^n)\}\\
&= \bigvee_{n\geq 1} \gamma_{f(A)}(y^n)\\
&= \gamma_{\sqrt{f(A)}}(y),
\end{aligned}$$

$$\begin{aligned}
\psi_{\sqrt{A}}(y) &= \inf\{\psi_{\sqrt{A}}(x) : x \in f^{-1}(y)\}\\
&= \inf\{\bigwedge_{n\geq 1} \psi_A(x^n) : x \in f^{-1}(y)\}\\
&= \bigwedge_{n\geq 1} \inf\{\psi_A(x^n) : x \in f^{-1}(y)\}\\
&= \bigwedge_{n\geq 1} \inf\{\psi_A(x^n) : x^n \in f^{-1}(y^n)\}\\
&= \bigwedge_{n\geq 1} \inf\{\psi_A(z^n) : z \in f^{-1}(y^n)\}\\
&= \bigwedge_{n\geq 1} \psi_{f(A)}(y^n)\\
&= \psi_{\sqrt{f(A)}}(y).
\end{aligned}$$

Hence, $f(\sqrt{A}) \subseteq \sqrt{f(A)}$. Further, if A is constant on $\operatorname{Ker} f$ and $x_0 \in f^{-1}(y)$ is a fixed element, then $\mu_A(x^n) = \mu_A(x_0^n)$, $\gamma_A(x^n) = \gamma_A(x_0^n)$, and $\psi_A(x^n) = \psi_A(x_0^n)$ for all $x \in f^{-1}(y)$, and $\mu_A(x) = \mu_A(x_0^n)$,

Received: May 12, 2023 Revised: July 12, 2023 Accepted: November 01, 2023

$\gamma_A(x) = \gamma_A(x_0^n)$, and $\psi_A(x) = \psi_A(x_0^n)$ for all $x \in f^{-1}(y^n)$. Hence,

$$\begin{aligned}
\mu_{f(\sqrt{A})}(y) &= \sup\{\mu_{\sqrt{A}}(x) : x \in f^{-1}(y)\} \\
&= \sup\{\bigvee_{n\geq 1} \mu_A(x^n) : x \in f^{-1}(y)\} \\
&= \bigvee_{n\geq 1} \sup\{\mu_A(x^n) : x \in f^{-1}(y)\} \\
&= \bigvee_{n\geq 1} \sup\{\mu_A(x^n) : x^n \in f^{-1}(y^n)\} \\
&= \bigvee_{n\geq 1} \sup\{\mu_A(z^n) : z \in f^{-1}(y^n)\} \\
&= \bigvee_{n\geq 1} \mu_{f(A)}(y^n) \\
&= \mu_{\sqrt{f(A)}}(y), \\
\gamma_{f(\sqrt{A})}(y) &= \sup\{\gamma_{\sqrt{A}}(x) : x \in f^{-1}(y)\} \\
&= \sup\{\bigvee_{n\geq 1} \gamma_A(x^n) : x \in f^{-1}(y)\} \\
&= \bigvee_{n\geq 1} \sup\{\gamma_A(x^n) : x \in f^{-1}(y)\} \\
&= \bigvee_{n\geq 1} \sup\{\gamma_A(x^n) : x^n \in f^{-1}(y^n)\} \\
&= \bigvee_{n\geq 1} \sup\{\gamma_A(z^n) : z \in f^{-1}(y^n)\} \\
&= \bigvee_{n\geq 1} \gamma_{f(A)}(y^n) \\
&= \gamma_{\sqrt{f(A)}}(y), \\
\psi_{f(\sqrt{A})}(y) &= \inf\{\psi_{\sqrt{A}}(x) : x \in f^{-1}(y)\} \\
&= \inf\{\bigvee_{n\geq 1} \psi_A(x^n) : x \in f^{-1}(y)\} \\
&= \bigwedge_{n\geq 1} \inf\{\psi_A(x^n) : x \in f^{-1}(y)\} \\
&= \bigwedge_{n\geq 1} \inf\{\psi_A(x^n) : x^n \in f^{-1}(y^n)\} \\
&= \bigwedge_{n\geq 1} \inf\{\psi_A(z^n) : z \in f^{-1}(y^n)\} \\
&= \bigwedge_{n\geq 1} \psi_{f(A)}(y^n) \\
&= \gamma_{\sqrt{f(A)}}(y).
\end{aligned}$$

Hence, $f(\sqrt{A}) = \sqrt{f(A)}$. ☐

Theorem 3.10. *Let $f : R \to R'$ be an epimorphism of rings and let f_x be the induced homomorphism of f. If a neutrosophic ideal A of R is constant on* $\operatorname{Ker} f$*, then the neutrosophic polynomial ideal A_x is constant on* $\operatorname{Ker} f_x$.

Proof. Let $\mu_A(x) = \alpha_0$, $\gamma_A(x) = \beta_0$, and $\psi_A(x) = \delta_0$ for all $x \in \operatorname{Ker} f$, where $\alpha_0, \beta_0, \delta_0 \in [0, 1]$ are constant such that $\alpha_0 + \beta_0 + \delta_0 \leq 1$. Let $g(x) = \sum_{i=0}^{m} b_i x_i \in \operatorname{Ker} f_x$. Then $0 = f_x(g(x)) = \sum_{i=0}^{m} f(b_i)x^i$, so $f(b_i) = 0$ for all $i = 1, 2, \ldots, m$. Hence, $b_i \in \operatorname{Ker} f$ for all $i = 1, 2, \ldots, m$, that is, $\mu_A(b_i) = \alpha_0$, $\gamma_A(b_i) = \beta_0$, and $\psi_A(b_i) = \delta_0$ for all $i = 1, 2, \ldots, m$. Thus $\mu_{A_x}(g(x)) = \min_{i=0}^{m}\{\mu_A(b_i)\} = \alpha_0$, $\gamma_{A_x}(g(x)) = \min_{i=0}^{m}\{\mu_A(b_i)\} = \beta_0$, and $\psi_{A_x}(g(x)) = \max_{i=0}^{m}\{\mu_A(b_i)\} = \delta_0$. Hence, A_x is constant on $\operatorname{Ker} f_x$. ☐

Corollary 3.11. *Let $f : R \to R'$ be an epimorphism of rings and let f_x be the induced homomorphism of f. If an f-invariant neutrosophic ideal A of R is constant on* $\operatorname{Ker} f$*, then $f_x(\sqrt{A_x}) = \sqrt{(f(A))_x}$.*

Proof. It follows from Proposition 3.9 and Theorem 3.6 that $f_x(\sqrt{A_x}) = \sqrt{f_x(A_x)} = \sqrt{(f(A))_x}$. ☐

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