



On Symbolic 7-Plithogenic and 8-Plithogenic Number Theoretical Concepts

Luis Albarracín Zambrano¹, Fabricio Lozada Torres¹, Bolívar Villalta Jadan¹, Nabil Khuder Salman²

¹Universidad Regional Autónoma de los Andes (UNIANDES), Ecuador

²College Of Pharmacy, AL-Farahidi University, Baghdad, Iraq

Emails: uq.luisalbarracin@uniandes.edu.ec; ua.fabriciolozada@uniandes.edu.ec;
us.bolivarvillalta@uniandes.edu.ec; nsalman@uoalfarahidi.edu.iq

Abstract

This paper is dedicated to studying the foundations of 7-plithogenic and 8-plithogenic number theory, where the central concepts about symbolic 7-plithogenic/8-plithogenic integers will be discussed such as symbolic 7-plithogenic/8-plithogenic Pythagoras triples and quadruples, symbolic 7-plithogenic/8-plithogenic linear Diophantine equations, and the divisors. On the other hand, we prove that Euler's theorem is still true in the case of the symbolic 7-plithogenic/8-plithogenic number theory.

Keywords: symbolic 7-plithogenic integer; symbolic 7-plithogenic congruencies; symbolic 7-plithogenic phi-Euler's theorem; symbolic 8-plithogenic phi-Euler's theorem.

1. Introduction

Numerical systems that expand integers play an important role in the study of pure mathematics, and also through many applications that relate to the theory of cryptography and the construction of algorithms related to it [12].

The traditional number theory [24] was initially generalized through the system of neutrosophic integer numbers, where these numbers were used in the generalization of the El-Gamal, and RSA algorithms [10-11].

Many researchers around the world have studied plithogenic structures, where the plithogenic sets introduced by Smarandache [4,18] were used to generalize matrices, rings, special functions, and also vector spaces [1-3,5-6, 13-17, 19-21].

The study of the theory of plithogenic numbers began in [7,16, 22-23], where they were studied for special cases of n values between 2 and 6.

In this research paper, we followed up the tireless efforts made by the researchers, where we studied in details the foundations of symbolic 7-plithogenic and 8-plithogenic number theory, and presented many theorems with proofs to explain the novelty of our work.

2. Main discussion

Definition:

The ring of symbolic 7-plithogenic integers is defined as follows:

$$7 - SP_Z = \{t_0 + \sum_{i=1}^7 t_i P_i; t_i \in Z\}, \text{ where } P_i \times P_j = p_{\max(i,j)}, P_i^2 = P_i.$$

Definition.

Let $T = t_0 + \sum_{i=1}^7 t_i P_i, C = c_0 + \sum_{i=1}^7 c_i P_i, D = d_0 + \sum_{i=1}^7 d_i P_i \in 7 - SP_Z$, we say that:

- 1). $T \setminus C$ if there exists $D \in 7 - SP_Z$ such that $T \cdot D = C$.
- 2). $T \equiv C \pmod{D}$ if $D \setminus T - C$.
- 3). $D = \gcd(T, C)$ if $D \setminus X, D \setminus Y$ and if $Y \setminus C, Y \setminus T$, then $Y \setminus D$.
- 4). T, C are relatively prime if $\gcd(T, C) = 1$.

Theorem1.

Let $T = t_0 + \sum_{i=1}^7 t_i P_i, C = c_0 + \sum_{i=1}^7 c_i P_i, D = d_0 + \sum_{i=1}^7 d_i P_i \in 7 - SP_Z$, then:

- 1). $Z = \gcd(T, C)$ if and only if:

$$\begin{cases} d_0 = \gcd(c_0, t_0) \\ \sum_{i=0}^j d_i = \gcd\left(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i\right); 1 \leq j \leq 7 \end{cases}$$

- 2). $T \equiv C \pmod{D}$ if and only if $\sum_{i=0}^j t_i \equiv \sum_{i=0}^j c_i \pmod{\sum_{i=0}^j d_i}$, where $0 \leq j \leq 7$.
- 3). If $T \setminus C$ then $\sum_{i=0}^j t_i \setminus \sum_{i=0}^j c_i; 0 \leq j \leq 7$.

Theorem2.

Let $T = t_0 + \sum_{i=1}^7 t_i P_i, C = c_0 + \sum_{i=1}^7 c_i P_i, D = d_0 + \sum_{i=1}^7 d_i P_i, L = l_0 + \sum_{i=1}^7 l_i P_i, K = k_0 + \sum_{i=1}^7 k_i P_i, G = g_0 + \sum_{i=1}^7 g_i P_i \in 7 - SP_Z$, then:

- 1). If $D \setminus T, D \setminus C$, then $D \setminus LT + KC$.
- 2). If $D = \gcd(T, C)$, then there exists $L, K \in 7 - SP_Z$ such that $LT + KC = D$.
- 3). If $T \equiv C \pmod{D}$, then:

$$\begin{cases} T + G = C + G \pmod{D} & (I) \\ T - G = C - G \pmod{D} & (II) \\ T \cdot G = CG \pmod{D} & (III) \end{cases}$$

Theorem3.

Let $TX + CY = D$ be symbolic 7-plithogenic Diophantine equation in two variables, $T, C, D, X, Y \in 7 - SP_Z$, hence it is solvable if and only if:

$\sum_{i=0}^j t_i \sum_{i=0}^j x_i + \sum_{i=0}^j c_i \sum_{i=0}^j y_i = \sum_{i=0}^j d_i; 0 \leq j \leq 7$ are solvable, i.e. $\gcd(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i) \setminus \sum_{i=0}^j d_i; 0 \leq j \leq 7$.

Theorem4.

(T, C, D) is a symbolic 7-plithogenic Pythagoras triple i.e. it is a solution of the non linear Diophantine equation $T^2 + C^2 = D^2$, if and only if $(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i, \sum_{i=0}^j d_i); 0 \leq j \leq 7$ is a Pythagoras triple in Z .

Theorem5.

(T, C, D, L) is a symbolic 7-plithogenic Pythagoras quadruple i.e. it is a solution of the non linear Diophantine equation $T^2 + C^2 + D^2 = L^2$, if and only if $(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i, \sum_{i=0}^j d_i, \sum_{i=0}^j l_i); 0 \leq j \leq 7$ is a Pythagoras quadruple in Z .

Proof of theorem1.

- 1). We put

$$\begin{aligned} D = d_0 + \sum_{i=1}^7 d_i P_i, d_0 = \gcd(c_0, t_0), \sum_{i=1}^1 d_i = \gcd\left(\sum_{i=1}^1 c_i, \sum_{i=1}^1 t_i\right), \sum_{i=1}^2 d_i = \gcd\left(\sum_{i=1}^2 c_i, \sum_{i=1}^2 t_i\right) \\ \sum_{i=1}^3 d_i = \gcd\left(\sum_{i=1}^3 c_i, \sum_{i=1}^3 t_i\right), \sum_{i=1}^4 d_i = \gcd\left(\sum_{i=1}^4 c_i, \sum_{i=1}^4 t_i\right), \sum_{i=1}^5 d_i = \gcd\left(\sum_{i=1}^5 c_i, \sum_{i=1}^5 t_i\right), \sum_{i=1}^6 d_i \\ = \gcd\left(\sum_{i=1}^6 c_i, \sum_{i=1}^6 t_i\right), \sum_{i=1}^7 d_i = \gcd\left(\sum_{i=1}^7 c_i, \sum_{i=1}^7 t_i\right) \end{aligned}$$

Assume that $K = k_0 + \sum_{i=1}^7 k_i P_i$ with $K \setminus C, K \setminus T$, hence:

$$\begin{cases} \sum_{i=0}^j d_i \setminus \sum_{i=0}^j t_i, \sum_{i=0}^j d_i \setminus \sum_{i=0}^j c_i; 0 \leq j \leq 7 \\ \sum_{i=1}^j k_i \setminus \sum_{i=0}^j t_i, \sum_{i=1}^j k_i \setminus \sum_{i=0}^j c_i; 0 \leq j \leq 7 \end{cases}$$

So that $\sum_{i=1}^j k_i \setminus \sum_{i=0}^j d_i; 0 \leq j \leq 7$, hence $K \setminus D$ and $D = \gcd(C, T)$.

- 2). $C \equiv T \pmod{D}$ if and only if $D \setminus T - C$, which is equivalent to $\sum_{i=0}^j d_i \setminus \sum_{i=0}^j (c_i - t_i); 0 \leq j \leq 7$, hence $\sum_{i=0}^j t_i \equiv \sum_{i=0}^j c_i \pmod{\sum_{i=0}^j d_i}; 0 \leq j \leq 7$.
- 3). Assume that $T \setminus C$, hence:

$$\left\{ \begin{aligned} t_0 d_0 &= c_0 \quad (1) \\ t_0 d_1 + t_1 d_0 + t_1 d_1 &= c_1 \quad (2) \\ t_0 d_2 + t_1 d_2 + t_2 d_2 + t_2 d_0 + t_2 d_1 &= c_2 \quad (3) \\ t_0 d_3 + t_1 d_3 + t_2 d_3 + t_3 d_3 + t_3 d_0 + t_3 d_1 + t_3 d_2 &= c_3 \quad (4) \\ t_0 d_4 + t_1 d_4 + t_2 d_4 + t_3 d_4 + t_4 d_4 + t_4 d_0 + t_4 d_1 + t_4 d_2 + t_4 d_3 &= c_4 \quad (5) \\ t_0 d_5 + t_1 d_5 + t_2 d_5 + t_3 d_5 + t_4 d_5 + t_5 d_5 + t_5 d_0 + t_5 d_1 + t_5 d_2 + t_5 d_3 + t_5 d_4 &= c_5 \quad (6) \\ t_0 d_6 + t_1 d_6 + t_2 d_6 + t_3 d_6 + t_4 d_6 + t_5 d_6 + t_6 d_6 + t_6 d_0 + t_6 d_1 + t_6 d_2 + t_6 d_3 + t_6 d_4 + t_6 d_5 &= c_6 \quad (7) \\ t_0 d_7 + t_1 d_7 + t_2 d_7 + t_3 d_7 + t_4 d_7 + t_5 d_7 + t_6 d_7 + t_7 d_7 + t_7 d_0 + t_7 d_1 + t_7 d_2 + t_7 d_3 + t_7 d_4 + t_7 d_5 + t_7 d_6 &= c_7 \quad (8) \end{aligned} \right.$$

By adding (1) + (2), (1) + (2) + (3), (1) + (2) + (3) + (4), (1) + (2) + (3) + (4) + (5), (1) + (2) + (3) + (4) + (5) + (6), (1) + (2) + (3) + (4) + (5) + (6) + (7), (1) + (2) + (3) + (4) + (5) + (6) + (7) + (8) we get:

$$\left\{ \begin{aligned} t_0 d_0 &= c_0 \\ \sum_{i=1}^1 t_i \sum_{i=1}^1 d_i &= \sum_{i=1}^1 c_i \\ \sum_{i=1}^2 t_i \sum_{i=1}^2 d_i &= \sum_{i=1}^2 c_i \\ \sum_{i=1}^3 t_i \sum_{i=1}^3 d_i &= \sum_{i=1}^3 c_i \\ \sum_{i=1}^4 t_i \sum_{i=1}^4 d_i &= \sum_{i=1}^4 c_i \\ \sum_{i=1}^5 t_i \sum_{i=1}^5 d_i &= \sum_{i=1}^5 c_i \\ \sum_{i=1}^6 t_i \sum_{i=1}^6 d_i &= \sum_{i=1}^6 c_i \\ \sum_{i=1}^7 t_i \sum_{i=1}^7 d_i &= \sum_{i=1}^7 c_i \end{aligned} \right.$$

Which means that $\sum_{i=0}^j t_i \setminus \sum_{i=0}^j c_i ; 0 \leq j \leq 7$

Proof of Theorem 2.

1). Assume that $D \setminus T, D \setminus C$, then we get:

$$\sum_{i=0}^j d_i \setminus \sum_{i=0}^j t_i, \text{ and } \sum_{i=0}^j d_i \setminus \sum_{i=0}^j c_i ; 0 \leq j \leq 7.$$

So that $d \setminus (\sum_{i=0}^j l_i \sum_{i=0}^j t_i + \sum_{i=0}^j k_i \sum_{i=0}^j c_i)$ for $0 \leq j \leq 7$ and $D \setminus LT + KC$.

2). Assume that $D = \gcd(T, C)$, then $\sum_{i=0}^j d_i = \gcd(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i)$ for all $0 \leq j \leq 7$.

According to Bezout's theorem, we can write:

$$\text{There exists } l_j, k_j \in Z \text{ such that } \sum_{i=0}^j d_i = l_j \sum_{i=0}^j t_i + k_j \sum_{i=0}^j c_i$$

by putting

$$L = l_0 + (l_1 - l_0)P_1 + (l_2 - l_1)P_2 + (l_3 - l_2)P_3 + (l_4 - l_3)P_4 + (l_5 - l_4)P_5 + (l_6 - l_5)P_6 + (l_7 - l_6)P_7,$$

$$K = k_0 + (k_1 - k_0)P_1 + (k_2 - k_1)P_2 + (k_3 - k_2)P_3 + (k_4 - k_3)P_4 + (k_5 - k_4)P_5 + (k_6 - k_5)P_6 +$$

$$(k_7 - k_6)P_7, \text{ we get:}$$

$$D = LT + KC.$$

3). Assume that $T \equiv C \pmod{D}$, then:

$$\sum_{i=0}^j d_i \setminus \sum_{i=0}^j (t_i - c_i) \text{ for all } 0 \leq j \leq 7, \text{ hence:}$$

$$\left\{ \begin{aligned} \sum_{i=0}^j d_i \setminus \sum_{i=0}^j (t_i - g_i + g_i - c_i) \\ \sum_{i=0}^j d_i \setminus \sum_{i=0}^j (t_i + g_i - g_i + c_i) \end{aligned} \right.$$

Hence $T \pm G = C \pm G \pmod{D}$, also:

$$\sum_{i=0}^j d_i \setminus \sum_{i=0}^j (t_i - c_i) \sum_{i=0}^j g_i \text{ i.e. } \sum_{i=0}^j d_i \setminus \sum_{i=0}^j t_i \sum_{i=0}^j g_i - \sum_{i=0}^j g_i \sum_{i=0}^j c_i$$

$$\text{Hence } TG \equiv CG \pmod{D}.$$

Proof of theorem3.

It is easy to check that $TX + CY = D$ is equivalent to:

$$\sum_{i=0}^j t_i \sum_{i=0}^j x_i + \sum_{i=0}^j c_i \sum_{i=0}^j y_i = \sum_{i=0}^j d_i; 0 \leq j \leq 7$$

The previous eight Diophantine equations are solvable if and only if:

$$\gcd\left(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i\right) \mid \sum_{i=0}^j d_i; 0 \leq j \leq 7$$

And the proof holds.

Proof of theorem4.

$T^2 + C^2 = D^2$ implies that:

$$\left\{ \begin{array}{l} t_0^2 + c_0^2 = d_0^2 \\ \left(\sum_{i=0}^1 t_i\right)^2 + \left(\sum_{i=0}^1 c_i\right)^2 = \left(\sum_{i=0}^1 d_i\right)^2 \\ \left(\sum_{i=0}^2 t_i\right)^2 + \left(\sum_{i=0}^2 c_i\right)^2 = \left(\sum_{i=0}^2 d_i\right)^2 \\ \left(\sum_{i=0}^3 t_i\right)^2 + \left(\sum_{i=0}^3 c_i\right)^2 = \left(\sum_{i=0}^3 d_i\right)^2 \\ \left(\sum_{i=0}^4 t_i\right)^2 + \left(\sum_{i=0}^4 c_i\right)^2 = \left(\sum_{i=0}^4 d_i\right)^2 \\ \left(\sum_{i=0}^5 t_i\right)^2 + \left(\sum_{i=0}^5 c_i\right)^2 = \left(\sum_{i=0}^5 d_i\right)^2 \\ \left(\sum_{i=0}^6 t_i\right)^2 + \left(\sum_{i=0}^6 c_i\right)^2 = \left(\sum_{i=0}^6 d_i\right)^2 \\ \left(\sum_{i=0}^7 t_i\right)^2 + \left(\sum_{i=0}^7 c_i\right)^2 = \left(\sum_{i=0}^7 d_i\right)^2 \end{array} \right.$$

Which implies the proof.

Proof of Theorem 5

$T^2 + C^2 + D^2 = L^2$ implies that:

$$\left\{ \begin{aligned} & t_0^2 + c_0^2 + d_0^2 = l_0^2 \\ & \left(\sum_{i=0}^1 t_i \right)^2 + \left(\sum_{i=0}^1 c_i \right)^2 + \left(\sum_{i=0}^1 d_i \right)^2 = \left(\sum_{i=0}^1 l_i \right)^2 \\ & \left(\sum_{i=0}^2 t_i \right)^2 + \left(\sum_{i=0}^2 c_i \right)^2 + \left(\sum_{i=0}^2 d_i \right)^2 = \left(\sum_{i=0}^2 l_i \right)^2 \\ & \left(\sum_{i=0}^3 t_i \right)^2 + \left(\sum_{i=0}^3 c_i \right)^2 + \left(\sum_{i=0}^3 d_i \right)^2 = \left(\sum_{i=0}^3 l_i \right)^2 \\ & \left(\sum_{i=0}^4 t_i \right)^2 + \left(\sum_{i=0}^4 c_i \right)^2 + \left(\sum_{i=0}^4 d_i \right)^2 = \left(\sum_{i=0}^4 l_i \right)^2 \\ & \left(\sum_{i=0}^5 t_i \right)^2 + \left(\sum_{i=0}^5 c_i \right)^2 + \left(\sum_{i=0}^5 d_i \right)^2 = \left(\sum_{i=0}^5 l_i \right)^2 \\ & \left(\sum_{i=0}^6 t_i \right)^2 + \left(\sum_{i=0}^6 c_i \right)^2 + \left(\sum_{i=0}^6 d_i \right)^2 = \left(\sum_{i=0}^6 l_i \right)^2 \\ & \left(\sum_{i=0}^7 t_i \right)^2 + \left(\sum_{i=0}^7 c_i \right)^2 + \left(\sum_{i=0}^7 d_i \right)^2 = \left(\sum_{i=0}^7 l_i \right)^2 \end{aligned} \right.$$

Which implies the proof.

Definition.

Let $S = s_0 + \sum_{i=0}^7 s_i P_i \in 7 - SP_Z$, hence we say that $S > 0$ if and only if $s_0 > 0, \sum_{i=0}^k s_i > 0 ; 1 \leq k \leq 7$

For example: $S = 3 + P_1 + P_2 + 5P_3 - 2P_4 + P_7 > 0$.

If $J = j_0 + \sum_{i=1}^7 j_i P_i \in 7 - SP_Z$, we say that $S \geq J$ if and only if $s_0 \geq j_0, \sum_{i=0}^k s_i \geq \sum_{i=0}^k j_i ; 1 \leq k \leq 7$.

Definition.

Let $T = t_0 + \sum_{i=1}^7 t_i P_i, C = c_0 + \sum_{i=1}^7 c_i P_i \geq 0$, hence:

$$\begin{aligned} T^C = t_0^{c_0} + P_1 & \left[\left(\sum_{i=0}^1 t_i \right)^{\sum_{i=0}^1 c_i} - t_0^{c_0} \right] + P_2 \left[\left(\sum_{i=0}^2 t_i \right)^{\sum_{i=0}^2 c_i} - \left(\sum_{i=0}^1 t_i \right)^{\sum_{i=0}^1 c_i} \right] \\ & + P_3 \left[\left(\sum_{i=0}^3 t_i \right)^{\sum_{i=0}^3 c_i} - \left(\sum_{i=0}^2 t_i \right)^{\sum_{i=0}^2 c_i} \right] + P_4 \left[\left(\sum_{i=0}^4 t_i \right)^{\sum_{i=0}^4 c_i} - \left(\sum_{i=0}^3 t_i \right)^{\sum_{i=0}^3 c_i} \right] \\ & + P_5 \left[\left(\sum_{i=0}^5 t_i \right)^{\sum_{i=0}^5 c_i} - \left(\sum_{i=0}^4 t_i \right)^{\sum_{i=0}^4 c_i} \right] + P_6 \left[\left(\sum_{i=0}^6 t_i \right)^{\sum_{i=0}^6 c_i} - \left(\sum_{i=0}^5 t_i \right)^{\sum_{i=0}^5 c_i} \right] \\ & + P_7 \left[\left(\sum_{i=0}^7 t_i \right)^{\sum_{i=0}^7 c_i} - \left(\sum_{i=0}^6 t_i \right)^{\sum_{i=0}^6 c_i} \right] \end{aligned}$$

Definition.

Let $T = t_0 + \sum_{i=1}^7 t_i P_i > 0$, then:

$$\begin{aligned} \varphi(T) = \varphi(t_0) + P_1 & \left[\varphi \left(\sum_{i=0}^1 t_i \right) - \varphi(t_0) \right] + P_2 \left[\varphi \left(\sum_{i=0}^2 t_i \right) - \varphi \left(\sum_{i=0}^1 t_i \right) \right] + P_3 \left[\varphi \left(\sum_{i=0}^3 t_i \right) - \varphi \left(\sum_{i=0}^2 t_i \right) \right] \\ & + P_4 \left[\varphi \left(\sum_{i=0}^4 t_i \right) - \varphi \left(\sum_{i=0}^3 t_i \right) \right] + P_5 \left[\varphi \left(\sum_{i=0}^5 t_i \right) - \varphi \left(\sum_{i=0}^4 t_i \right) \right] \\ & + P_6 \left[\varphi \left(\sum_{i=0}^6 t_i \right) - \varphi \left(\sum_{i=0}^5 t_i \right) \right] + P_7 \left[\varphi \left(\sum_{i=0}^7 t_i \right) - \varphi \left(\sum_{i=0}^6 t_i \right) \right] \end{aligned}$$

Where φ is Euler's function on Z .

Theorem 6.

Let $XT = t_0 + \sum_{i=1}^7 t_i P_i, C = c_0 + \sum_{i=1}^7 c_i P_i, 0 \in 7 - SP_Z, gcd(T, C) = 1$ and $T, C > 0$, hence:
 $T^{\varphi(C)} \equiv 1(mod C)$

Proof.

$gcd(t_0, c_0) = 1$, hence $t_0^{\varphi(c_0)} \equiv 1(mod c_0)$.

$gcd(\sum_{i=0}^1 t_i, \sum_{i=0}^1 c_i) = 1$, hence $(\sum_{i=0}^1 t_i)^{\varphi(\sum_{i=0}^1 c_i)} \equiv 1(mod \sum_{i=0}^1 c_i)$

By a similar argument, we get:

$$\begin{aligned} \left(\sum_{i=0}^2 t_i\right)^{\varphi(\sum_{i=0}^2 c_i)} &\equiv 1 \left(mod \sum_{i=0}^2 c_i\right), \left(\sum_{i=0}^3 t_i\right)^{\varphi(\sum_{i=0}^3 c_i)} \equiv 1 \left(mod \sum_{i=0}^3 c_i\right) \\ \left(\sum_{i=0}^4 t_i\right)^{\varphi(\sum_{i=0}^4 c_i)} &\equiv 1 \left(mod \sum_{i=0}^4 c_i\right), \left(\sum_{i=0}^5 t_i\right)^{\varphi(\sum_{i=0}^5 c_i)} \equiv 1 \left(mod \sum_{i=0}^5 c_i\right), \left(\sum_{i=0}^6 t_i\right)^{\varphi(\sum_{i=0}^6 c_i)} \\ &\equiv 1 \left(mod \sum_{i=0}^6 c_i\right), \left(\sum_{i=0}^7 t_i\right)^{\varphi(\sum_{i=0}^7 c_i)} \equiv 1 \left(mod \sum_{i=0}^7 c_i\right) \end{aligned}$$

This implies

$$T^{\varphi(C)} \equiv 1(mod Y).$$

Definition:

The ring of symbolic 8-plithogenic integers is defined as follows:

$$8 - SP_Z = \{t_0 + \sum_{i=1}^8 t_i P_i; t_i \in Z\}, \text{ where } P_i \times P_j = p_{max(i,j)}, P_i^2 = P_i.$$

Definition.

Let $T = t_0 + \sum_{i=1}^8 t_i P_i, C = c_0 + \sum_{i=1}^8 c_i P_i, D = d_0 + \sum_{i=1}^8 d_i P_i \in 8 - SP_Z$, we say that:

- 1). $T \setminus C$ if there exists $D \in 7 - SP_Z$ such that $T.D = C$.
- 2). $T \equiv C(mod D)$ if $D \setminus T - C$.
- 3). $D = gcd(T, C)$ if $D \setminus X, D \setminus Y$ and if $Y \setminus C, Y \setminus T$, then $Y \setminus D$.
- 4). T, C are relatively prime if $gcd(T, C) = 1$.

Theorem7.

Let $T = t_0 + \sum_{i=1}^8 t_i P_i, C = c_0 + \sum_{i=1}^8 c_i P_i, D = d_0 + \sum_{i=1}^8 d_i P_i \in 8 - SP_Z$, then:

- 1). $Z = gcd(T, C)$ if and only if:

$$\begin{cases} d_0 = gcd(c_0, t_0) \\ \sum_{i=0}^j d_i = gcd\left(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i\right); 1 \leq j \leq 8 \end{cases}$$

- 2). $T \equiv C(mod D)$ if and only if $\sum_{i=0}^j t_i \equiv \sum_{i=0}^j c_i (mod \sum_{i=0}^j d_i)$, where $0 \leq j \leq 8$.

- 3). If $T \setminus C$ then $\sum_{i=0}^j t_i \setminus \sum_{i=0}^j c_i; 0 \leq j \leq 8$.

Theorem8.

Let $T = t_0 + \sum_{i=1}^8 t_i P_i, C = c_0 + \sum_{i=1}^8 c_i P_i, D = d_0 + \sum_{i=1}^8 d_i P_i, L = l_0 + \sum_{i=1}^8 l_i P_i, K = k_0 + \sum_{i=1}^8 k_i P_i, G = g_0 + \sum_{i=1}^8 g_i P_i \in 8 - SP_Z$, then:

- 1). If $D \setminus T, D \setminus C$, then $D \setminus LT + KC$.
- 2). If $D = gcd(T, C)$, then there exists $L, K \in 7 - SP_Z$ such that $LT + KC = D$.
- 3). If $T \equiv C(mod D)$, then:

$$\begin{cases} T + G = C + G (mod D) \quad (I) \\ T - G = C - G (mod D) \quad (II) \\ T.G = CG (mod D) \quad (III) \end{cases}$$

Theorem9.

Let $TX + CY = D$ be symbolic 8-plithogenic Diophantine equation in two variables, $T, C, D, X, Y \in 8 - SP_Z$, hence it is solvable if and only if:

$$\sum_{i=0}^j t_i \sum_{i=0}^j x_i + \sum_{i=0}^j c_i \sum_{i=0}^j y_i = \sum_{i=0}^j d_i; 0 \leq j \leq 8 \text{ are solvable, i.e. } gcd(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i) \setminus \sum_{i=0}^j d_i; 0 \leq j \leq 8.$$

Theorem 10.

(T, C, D) is a symbolic 8-plithogenic Pythagoras triple i.e. it is a solution of the non linear Diophantine equation $T^2 + C^2 = D^2$, if and only if $(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i, \sum_{i=0}^j d_i); 0 \leq j \leq 8$ is a Pythagoras triple in Z .

Theorem 11.

(T, C, D, L) is a symbolic 8-plithogenic Pythagoras quadruple i.e. it is a solution of the non linear Diophantine equation $T^2 + C^2 + D^2 = L^2$, if and only if $(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i, \sum_{i=0}^j d_i, \sum_{i=0}^j l_i); 0 \leq j \leq 8$ is a Pythagoras quadruple in Z .

Proof of theorem7.

1). We put

$$D = d_0 + \sum_{i=1}^8 d_i P_i, d_0 = gcd(c_0, t_0), \sum_{i=1}^1 d_i = gcd\left(\sum_{i=1}^1 c_i, \sum_{i=1}^1 t_i\right), \sum_{i=1}^2 d_i = gcd\left(\sum_{i=1}^2 c_i, \sum_{i=1}^2 t_i\right)$$

$$\sum_{i=1}^3 d_i = gcd\left(\sum_{i=1}^3 c_i, \sum_{i=1}^3 t_i\right), \sum_{i=1}^4 d_i = gcd\left(\sum_{i=1}^4 c_i, \sum_{i=1}^4 t_i\right), \sum_{i=1}^5 d_i = gcd\left(\sum_{i=1}^5 c_i, \sum_{i=1}^5 t_i\right), \sum_{i=1}^6 d_i = gcd\left(\sum_{i=1}^6 c_i, \sum_{i=1}^6 t_i\right)$$

$$= gcd\left(\sum_{i=1}^6 c_i, \sum_{i=1}^6 t_i\right), \sum_{i=1}^7 d_i = gcd\left(\sum_{i=1}^7 c_i, \sum_{i=1}^7 t_i\right), \sum_{i=1}^8 d_i = gcd\left(\sum_{i=1}^8 c_i, \sum_{i=1}^8 t_i\right)$$

Assume that $K = k_0 + \sum_{i=1}^8 k_i P_i$ with $K \setminus C, K \setminus T$, hence:

$$\left\{ \begin{array}{l} \sum_{i=0}^j d_i \setminus \sum_{i=0}^j t_i, \sum_{i=0}^j d_i \setminus \sum_{i=0}^j c_i; 0 \leq j \leq 8 \\ \sum_{i=1}^j k_i \setminus \sum_{i=0}^j t_i, \sum_{i=1}^j k_i \setminus \sum_{i=0}^j c_i; 0 \leq j \leq 8 \end{array} \right.$$

So that $\sum_{i=1}^j k_i \setminus \sum_{i=0}^j d_i; 0 \leq j \leq 8$, hence $K \setminus D$ and $D = gcd(C, T)$.

2). $C \equiv T \pmod{D}$ if and only if $D \setminus T - C$, which is equivalent to

$$\sum_{i=0}^j d_i \setminus \sum_{i=0}^j (c_i - t_i); 0 \leq j \leq 8, \text{ hence } \sum_{i=0}^j t_i \equiv \sum_{i=0}^j c_i \pmod{\sum_{i=0}^j d_i}; 0 \leq j \leq 8.$$

3). Assume that $T \setminus C$, hence:

$$\left\{ \begin{array}{l} t_0 d_0 = c_0 \quad (1) \\ t_0 d_1 + t_1 d_0 + t_1 d_1 = c_1 \quad (2) \\ t_0 d_2 + t_1 d_2 + t_2 d_2 + t_2 d_0 + t_2 d_1 = c_2 \quad (3) \\ t_0 d_3 + t_1 d_3 + t_2 d_3 + t_3 d_3 + t_3 d_0 + t_3 d_1 + t_3 d_2 = c_3 \quad (4) \\ t_0 d_4 + t_1 d_4 + t_2 d_4 + t_3 d_4 + t_4 d_4 + t_4 d_0 + t_4 d_1 + t_4 d_2 + t_4 d_3 = c_4 \quad (5) \\ t_0 d_5 + t_1 d_5 + t_2 d_5 + t_3 d_5 + t_4 d_5 + t_5 d_5 + t_5 d_0 + t_5 d_1 + t_5 d_2 + t_5 d_3 + t_5 d_4 = c_5 \quad (6) \\ t_0 d_6 + t_1 d_6 + t_2 d_6 + t_3 d_6 + t_4 d_6 + t_5 d_6 + t_6 d_6 + t_6 d_0 + t_6 d_1 + t_6 d_2 + t_6 d_3 + t_6 d_4 + t_6 d_5 = c_6 \quad (7) \\ t_0 d_7 + t_1 d_7 + t_2 d_7 + t_3 d_7 + t_4 d_7 + t_5 d_7 + t_6 d_7 + t_7 d_7 + t_7 d_0 + t_7 d_1 + t_7 d_2 + t_7 d_3 + t_7 d_4 + t_7 d_5 + t_7 d_6 = c_7 \quad (8) \\ t_0 d_8 + t_1 d_8 + \dots + t_8 d_8 + t_8 d_0 + t_8 d_1 + \dots + t_8 d_7 = c_8 \quad (9) \end{array} \right.$$

By adding (1) + (2), (1) + (2) + (3), (1) + (2) + (3) + (4), (1) + (2) + (3) + (4) + (5), (1) + (2) + (3) + (4) + (5) + (6), (1) + (2) + (3) + (4) + (5) + (6) + (7), (1) + (2) + (3) + (4) + (5) + (6) + (7) + (8) + (9) we get:

$$\left\{ \begin{array}{l} t_0 d_0 = c_0 \\ \sum_{i=1}^1 t_i \sum_{i=1}^1 d_i = \sum_{i=1}^1 c_i \\ \sum_{i=1}^2 t_i \sum_{i=1}^2 d_i = \sum_{i=1}^2 c_i \\ \sum_{i=1}^3 t_i \sum_{i=1}^3 d_i = \sum_{i=1}^3 c_i \\ \sum_{i=1}^4 t_i \sum_{i=1}^4 d_i = \sum_{i=1}^4 c_i \\ \sum_{i=1}^5 t_i \sum_{i=1}^5 d_i = \sum_{i=1}^5 c_i \\ \sum_{i=1}^6 t_i \sum_{i=1}^6 d_i = \sum_{i=1}^6 c_i \\ \sum_{i=1}^7 t_i \sum_{i=1}^7 d_i = \sum_{i=1}^7 c_i \\ \sum_{i=1}^8 t_i \sum_{i=1}^8 d_i = \sum_{i=1}^8 c_i \end{array} \right.$$

Which means that $\sum_{i=0}^j t_i \setminus \sum_{i=0}^j c_i ; 0 \leq j \leq 8$

Proof of Theorem 8.

1). Assume that $D \setminus T, D \setminus C$, then we get:

$$\sum_{i=0}^j d_i \setminus \sum_{i=0}^j t_i, \text{ and } \sum_{i=0}^j d_i \setminus \sum_{i=0}^j c_i ; 0 \leq j \leq 8.$$

So that $d \setminus (\sum_{i=0}^j l_i \sum_{i=0}^j t_i + \sum_{i=0}^j k_i \sum_{i=0}^j c_i)$ for $0 \leq j \leq 8$ and $D \setminus LT + KC$.

2). Assume that $D = gcd(T, C)$, then $\sum_{i=0}^j d_i = gcd(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i)$ for all $0 \leq j \leq 8$.

According to Bezout's theorem, we can write:

$$\text{There exists } l_j, k_j \in Z \text{ such that } \sum_{i=0}^j d_i = l_j \sum_{i=0}^j t_i + k_j \sum_{i=0}^j c_i$$

by putting

$$L = l_0 + (l_1 - l_0)P_1 + (l_2 - l_1)P_2 + (l_3 - l_2)P_3 + (l_4 - l_3)P_4 + (l_5 - l_4)P_5 + (l_6 - l_5)P_6 + (l_7 - l_6)P_7, \\ K = k_0 + (k_1 - k_0)P_1 + (k_2 - k_1)P_2 + (k_3 - k_2)P_3 + (k_4 - k_3)P_4 + (k_5 - k_4)P_5 + (k_6 - k_5)P_6 + \\ (k_7 - k_6)P_7 + (k_8 - k_7)P_8, \text{ we get:}$$

$$D = LT + KC.$$

3). Assume that $T \equiv C(mod D)$, then:

$$\sum_{i=0}^j d_i \setminus \sum_{i=0}^j (t_i - c_i) \text{ for all } 0 \leq j \leq 8, \text{ hence:}$$

$$\left\{ \begin{array}{l} \sum_{i=0}^j d_i \setminus \sum_{i=0}^j (t_i - g_i + g_i - c_i) \\ \sum_{i=0}^j d_i \setminus \sum_{i=0}^j (t_i + g_i - g_i + c_i) \end{array} \right.$$

Hence $T \pm G = C \pm G(mod D)$, also:

$$\sum_{i=0}^j d_i \setminus \sum_{i=0}^j (t_i - c_i) \sum_{i=0}^j g_i \text{ i.e. } \sum_{i=0}^j d_i \setminus \sum_{i=0}^j t_i \sum_{i=0}^j g_i - \sum_{i=0}^j g_i \sum_{i=0}^j c_i$$

Hence $TG \equiv CG(mod D)$.

Proof of theorem 9.

It is easy to check that $TX + CY = D$ is equivalent to:

$$\sum_{i=0}^j t_i \sum_{i=0}^j x_i + \sum_{i=0}^j c_i \sum_{i=0}^j y_i = \sum_{i=0}^j d_i ; 0 \leq j \leq 8$$

The previous eight Diophantine equations are solvable if and only if:

$$\gcd\left(\sum_{i=0}^j t_i, \sum_{i=0}^j c_i\right) \mid \sum_{i=0}^j d_i; 0 \leq j \leq 8$$

And the proof holds.

Proof of theorem 10.

$T^2 + C^2 = D^2$ implies that:

$$\left\{ \begin{array}{l} t_0^2 + c_0^2 = d_0^2 \\ \left(\sum_{i=0}^1 t_i\right)^2 + \left(\sum_{i=0}^1 c_i\right)^2 = \left(\sum_{i=0}^1 d_i\right)^2 \\ \left(\sum_{i=0}^2 t_i\right)^2 + \left(\sum_{i=0}^2 c_i\right)^2 = \left(\sum_{i=0}^2 d_i\right)^2 \\ \left(\sum_{i=0}^3 t_i\right)^2 + \left(\sum_{i=0}^3 c_i\right)^2 = \left(\sum_{i=0}^3 d_i\right)^2 \\ \left(\sum_{i=0}^4 t_i\right)^2 + \left(\sum_{i=0}^4 c_i\right)^2 = \left(\sum_{i=0}^4 d_i\right)^2 \\ \left(\sum_{i=0}^5 t_i\right)^2 + \left(\sum_{i=0}^5 c_i\right)^2 = \left(\sum_{i=0}^5 d_i\right)^2 \\ \left(\sum_{i=0}^6 t_i\right)^2 + \left(\sum_{i=0}^6 c_i\right)^2 = \left(\sum_{i=0}^6 d_i\right)^2 \\ \left(\sum_{i=0}^7 t_i\right)^2 + \left(\sum_{i=0}^7 c_i\right)^2 = \left(\sum_{i=0}^7 d_i\right)^2 \\ \left(\sum_{i=0}^8 t_i\right)^2 + \left(\sum_{i=0}^8 c_i\right)^2 = \left(\sum_{i=0}^8 d_i\right)^2 \end{array} \right.$$

Which implies the proof.

Proof of Theorem 11

$T^2 + C^2 + D^2 = L^2$ implies that:

$$\left. \begin{aligned}
 & t_0^2 + c_0^2 + d_0^2 = l_0^2 \\
 & \left(\sum_{i=0}^1 t_i \right)^2 + \left(\sum_{i=0}^1 c_i \right)^2 + \left(\sum_{i=0}^1 d_i \right)^2 = \left(\sum_{i=0}^1 l_i \right)^2 \\
 & \left(\sum_{i=0}^2 t_i \right)^2 + \left(\sum_{i=0}^2 c_i \right)^2 + \left(\sum_{i=0}^2 d_i \right)^2 = \left(\sum_{i=0}^2 l_i \right)^2 \\
 & \left(\sum_{i=0}^3 t_i \right)^2 + \left(\sum_{i=0}^3 c_i \right)^2 + \left(\sum_{i=0}^3 d_i \right)^2 = \left(\sum_{i=0}^3 l_i \right)^2 \\
 & \left(\sum_{i=0}^4 t_i \right)^2 + \left(\sum_{i=0}^4 c_i \right)^2 + \left(\sum_{i=0}^4 d_i \right)^2 = \left(\sum_{i=0}^4 l_i \right)^2 \\
 & \left(\sum_{i=0}^5 t_i \right)^2 + \left(\sum_{i=0}^5 c_i \right)^2 + \left(\sum_{i=0}^5 d_i \right)^2 = \left(\sum_{i=0}^5 l_i \right)^2 \\
 & \left(\sum_{i=0}^6 t_i \right)^2 + \left(\sum_{i=0}^6 c_i \right)^2 + \left(\sum_{i=0}^6 d_i \right)^2 = \left(\sum_{i=0}^6 l_i \right)^2 \\
 & \left(\sum_{i=0}^7 t_i \right)^2 + \left(\sum_{i=0}^7 c_i \right)^2 + \left(\sum_{i=0}^7 d_i \right)^2 = \left(\sum_{i=0}^7 l_i \right)^2 \\
 & \left(\sum_{i=0}^8 t_i \right)^2 + \left(\sum_{i=0}^8 c_i \right)^2 + \left(\sum_{i=0}^8 d_i \right)^2 = \left(\sum_{i=0}^8 l_i \right)^2
 \end{aligned} \right\}$$

Which implies the proof.

Definition.

Let $S = s_0 + \sum_{i=0}^7 s_i P_i \in 7 - SP_Z$, hence we say that $S > 0$ if and only if $s_0 > 0, \sum_{i=0}^k s_i > 0 ; 1 \leq k \leq 8$

If $J = j_0 + \sum_{i=1}^7 j_i P_i \in 7 - SP_Z$, we say that $S \geq J$ if and only if $s_0 \geq j_0, \sum_{i=0}^k s_i \geq \sum_{i=0}^k j_i ; 1 \leq k \leq 8$.

Definition.

Let $T = t_0 + \sum_{i=1}^8 t_i P_i, C = c_0 + \sum_{i=1}^8 c_i P_i \geq 0$, hence:

$$\begin{aligned}
 T^C = & t_0^{c_0} + P_1 \left[\left(\sum_{i=0}^1 t_i \right)^{\sum_{i=0}^1 c_i} - t_0^{c_0} \right] + P_2 \left[\left(\sum_{i=0}^2 t_i \right)^{\sum_{i=0}^2 c_i} - \left(\sum_{i=0}^1 t_i \right)^{\sum_{i=0}^1 c_i} \right] \\
 & + P_3 \left[\left(\sum_{i=0}^3 t_i \right)^{\sum_{i=0}^3 c_i} - \left(\sum_{i=0}^2 t_i \right)^{\sum_{i=0}^2 c_i} \right] + P_4 \left[\left(\sum_{i=0}^4 t_i \right)^{\sum_{i=0}^4 c_i} - \left(\sum_{i=0}^3 t_i \right)^{\sum_{i=0}^3 c_i} \right] \\
 & + P_5 \left[\left(\sum_{i=0}^5 t_i \right)^{\sum_{i=0}^5 c_i} - \left(\sum_{i=0}^4 t_i \right)^{\sum_{i=0}^4 c_i} \right] + P_6 \left[\left(\sum_{i=0}^6 t_i \right)^{\sum_{i=0}^6 c_i} - \left(\sum_{i=0}^5 t_i \right)^{\sum_{i=0}^5 c_i} \right] \\
 & + P_7 \left[\left(\sum_{i=0}^7 t_i \right)^{\sum_{i=0}^7 c_i} - \left(\sum_{i=0}^6 t_i \right)^{\sum_{i=0}^6 c_i} \right] + P_8 \left[\left(\sum_{i=0}^8 t_i \right)^{\sum_{i=0}^8 c_i} - \left(\sum_{i=0}^7 t_i \right)^{\sum_{i=0}^7 c_i} \right]
 \end{aligned}$$

Definition.

Let $T = t_0 + \sum_{i=1}^8 t_i P_i > 0$, then:

$$\begin{aligned} \varphi(T) = & \varphi(t_0) + P_1 \left[\varphi \left(\sum_{i=0}^1 t_i \right) - \varphi(t_0) \right] + P_2 \left[\varphi \left(\sum_{i=0}^2 t_i \right) - \varphi \left(\sum_{i=0}^1 t_i \right) \right] + P_3 \left[\varphi \left(\sum_{i=0}^3 t_i \right) - \varphi \left(\sum_{i=0}^2 t_i \right) \right] \\ & + P_4 \left[\varphi \left(\sum_{i=0}^4 t_i \right) - \varphi \left(\sum_{i=0}^3 t_i \right) \right] + P_5 \left[\varphi \left(\sum_{i=0}^5 t_i \right) - \varphi \left(\sum_{i=0}^4 t_i \right) \right] \\ & + P_6 \left[\varphi \left(\sum_{i=0}^6 t_i \right) - \varphi \left(\sum_{i=0}^5 t_i \right) \right] + P_7 \left[\varphi \left(\sum_{i=0}^7 t_i \right) - \varphi \left(\sum_{i=0}^6 t_i \right) \right] \\ & + P_8 \left[\varphi \left(\sum_{i=0}^8 t_i \right) - \varphi \left(\sum_{i=0}^7 t_i \right) \right] \end{aligned}$$

Where φ is Euler's function on Z .

Theorem 12.

Let $T = t_0 + \sum_{i=1}^8 t_i P_i, C = c_0 + \sum_{i=1}^8 c_i P_i \in 8 - SP_Z, gcd(T, C) = 1$ and $T, C > 0$, hence:
 $T^{\varphi(C)} \equiv 1(mod C)$

Proof.

$gcd(t_0, c_0) = 1$, hence $t_0^{\varphi(c_0)} \equiv 1(mod c_0)$.

$gcd(\sum_{i=0}^1 t_i, \sum_{i=0}^1 c_i) = 1$, hence $(\sum_{i=0}^1 t_i)^{\varphi(\sum_{i=0}^1 c_i)} \equiv 1(mod \sum_{i=0}^1 c_i)$

By a similar argument, we get:

$$\begin{aligned} \left(\sum_{i=0}^2 t_i \right)^{\varphi(\sum_{i=0}^2 c_i)} & \equiv 1 \left(mod \sum_{i=0}^2 c_i \right), \left(\sum_{i=0}^3 t_i \right)^{\varphi(\sum_{i=0}^3 c_i)} \equiv 1 \left(mod \sum_{i=0}^3 c_i \right) \\ \left(\sum_{i=0}^4 t_i \right)^{\varphi(\sum_{i=0}^4 c_i)} & \equiv 1 \left(mod \sum_{i=0}^4 c_i \right), \left(\sum_{i=0}^5 t_i \right)^{\varphi(\sum_{i=0}^5 c_i)} \equiv 1 \left(mod \sum_{i=0}^5 c_i \right), \left(\sum_{i=0}^6 t_i \right)^{\varphi(\sum_{i=0}^6 c_i)} \\ & \equiv 1 \left(mod \sum_{i=0}^6 c_i \right), \left(\sum_{i=0}^7 t_i \right)^{\varphi(\sum_{i=0}^7 c_i)} \equiv 1 \left(mod \sum_{i=0}^7 c_i \right), \left(\sum_{i=0}^8 t_i \right)^{\varphi(\sum_{i=0}^8 c_i)} \equiv 1 \left(mod \sum_{i=0}^8 c_i \right) \end{aligned}$$

This implies

$$T^{\varphi(C)} \equiv 1(mod Y).$$

5. Conclusion

In this paper, we have presented the foundations of symbolic 7-plithogenic and 8-plithogenic number theory. We have discussed the central concepts such as symbolic 7-plithogenic/8-plithogenic congruencies, and some types of linear and non-linear Diophantine equations.

Also, we have proved that the Euler's phi-theorem is still true in the case of the symbolic 7-plithogenic and 8-plithogenic number theory.

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